Linear Algebra and Random Processes (CS6015)

DEPT. OF COMPUTER SCIENCE AND ENGINEERING
Indian Institute of Technology Madras

TUTORIAL 2
(Time allowed: FIFTY minutes)

NOTE: Attempt **ALL** questions. Total Marks : **25**

1. Prove that a square matrix A can have at most one inverse. (2 marks)

Solution :
Suppose B and C are both inverses of A i.e. $AB = BA = I$ and $AC = CA = I$.
Then $B = BI = B(AC) = (BA)C = IC = C$. Thus $B = C$, and A can have at most one inverse

2. Compute L and U for the symmetric matrix A such that $A = LU$.
$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$ Find four conditions on a, b, c, d to get $A = LU$ with four pivots. (3 marks)

Solution :

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}. \text{ Need } \begin{matrix} a \neq 0 \\ b \neq a \\ c \neq b \\ d \neq c \end{matrix}$$

3. Find a 3 by 3 permutation matrix with $P^3 = I$ (but $P \neq I$). (2 marks)

Solution :

A cyclic $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or its transpose will have $P^3 = I : (1,2,3) \to (2,3,1) \to (3,1,2) \to (1,2,3)$.

4. Find the transpose of a block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Under what conditions on A, B, C, D is the block matrix symmetric. (2 marks)

Hint: A block matrix is a matrix that is defined using smaller matrices, called blocks which fit together to form a rectangle

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Solution :

$$M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}; M^{\mathrm{T}} = M \text{ needs } A^{\mathrm{T}} = A \text{ and } B^{\mathrm{T}} = C \text{ and } D^{\mathrm{T}} = D.$$

5. Discover whether $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is invertible, and find A^{-1} if it exists using Gauss-Jordan method. (5 marks)

Solution :

The inverse exists since the matrix produces 4 non-zero pivots.

We row-reduce the augmented matrix as follows:

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/4 \end{array}\right]$$

Thus the A does have an inverse and

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}.$$

6. Let W_1 and W_2 be subspaces of a vector space V. Prove that the intersection $W_1 \cap W_2$ is a subspace of V. (2 marks)

Solution :

Nonempty: since W_1 and W_2 are subspaces, they both contain the 0 vector, so 0 is in the intersection of W_1 and W_2, so this intersection is nonempty.

Closure under addition: suppose x and y are in the intersection of W_1 and W_2. Then x and y are in both W_1 and W_2. Since W_1 is a subspace, and x and y are in W_1, it follows that $x + y$ is in W_1.

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Likewise, since W_2 is a subspace, $x+y$ is in W_2. Since $x+y$ is in both W_1 and in W_2, it follows that $x+y$ is in the intersection of W_1 and W_2.

Closure under scalar multiplication: same idea, but simpler.

7. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \leq b_2 \leq b_3$

(3 marks)

Solution :
The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of v and w (e) the plane with $b1 + b2 + b3 = 0$.

8. Describe the column spaces (lines or planes) of the following matrices. (2 marks)

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution :
The column space of A is the line of vectors $(x, 2x, 0)$. The column space of B is the xy plane = all vectors $(x, y, 0)$.

9. For which right-hand sides (find a condition on b_1 , b_2 , b_3) are these systems solvable? Solve using Gaussian elimination.

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(4 marks)

Solution :

(a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + b_3$ in equation 3: Solution only if $b_3 = -b_1$.