Linear Algebra and Random Processes (CS6015)

DEPT. OF COMPUTER SCIENCE AND ENGINEERING
Indian Institute of Technology Madras

TUTORIAL 3
(Time allowed: FIFTY minutes)

NOTE: Attempt **ALL** questions. Total Marks : **25**

1. Describe the nullspaces of A and B in two ways.

(a) Give the equations for the plane (or line)

(b) Give all vectors x that satisfy those equations as combinations of the special solutions.

$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$ (4 marks)

Solution :

A: $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \implies -x_1 + 3x_2 + 5x_3 = 0$; an equation of a plane.

In another way, all vectors of the form $x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

B: $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \implies -x_1 + 3x_2 + 5x_3 = 0,\ -3x_3 = 0 \therefore -x_1 + 3x_2 = 0$; an equation of a line.

In another way, all vectors of the form $x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

2. Describe the column space and nullspace of A and the complete solution to $Ax = b$.

$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ (5 marks)

Solution :

$$\mathbf{1} \quad \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b_1} \\ 2 & 5 & 7 & 6 & \mathbf{b_2} \\ 2 & 3 & 5 & 2 & \mathbf{b_3} \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b_1} \\ 0 & 1 & 1 & 2 & \mathbf{b_2 - b_1} \\ 0 & -1 & -1 & -2 & \mathbf{b_3 - b_1} \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b_1} \\ 0 & 1 & 1 & 2 & \mathbf{b_2 - b_1} \\ 0 & 0 & 0 & 0 & \mathbf{b_3 + b_2 - 2b_1} \end{bmatrix}$$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!).

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Considering $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; Ax $= 0 \implies$

$$2x_1 + 4x_2 + 6x_3 + 4x_4 = 0 \qquad (1)$$
$$x_2 + x_3 + 2x_4 = 0 \qquad (2)$$

From Equation 2

$$x_2 = -x_3 - 2x_4 \qquad (3)$$

Substituting Equation 3 in Equation 1

$$x_1 = -x_3 + 2x_4 \qquad (4)$$

$\therefore$ The nullspace is $\begin{bmatrix} -x_3 + 2x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

The nullspace contains all combinations of $\boldsymbol{s}_1 = (-1, -1, 1, 0)$ and $\boldsymbol{s}_2 = (2, -2, 0, 1)$;
$\boldsymbol{x}_{complete} = \boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$;

Alternatively substitute $(x_3, x_4) = (0, 1)$ and $= (x_3, x_4) = (1, 0)$ in Equation 3 and 4 to obtain s_1 and s_2.

$$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ gives the particular solution $x_p = (4, -1, 0, 0)$.

3. Show that the vectors $a_1 = (1, 0, -1)$; $a_2 = (1, 2, 1)$; $a_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$ (2 marks)

Solution :
Show the vectors are linearly independent by showing that matrix whose rows are a_i's is invertible

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

And any set of three linearly independent vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$. Hence the set of vectors is indeed a basis for $\mathbb{R}^3$.

4. Consider a vector $x = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4), (x_4, x_3, x_1, x_2), and so on. Those 24 vectors, including x itself, span a subspace S. Find examples of particular vectors x so that the dimension of S is:
(a) 0, (b) 1, (c) 3, (d) 4. Justify your answer. (4 marks)

Solution :

(a) A subspace of dimension 0 in $\mathbb{R}^4$ is the origin. So the vector that spans it is (0,0,0,0).

(b) A subspace of dimension 1 implies it has only one vector in its basis. This is possible only if the vector does not change on rearrangement. One such vector is (1,1,1,1).

(c) Consider a non-zero vector (x_1, x_2, x_3, x_4) such that $x_1 + x_2 + x_3 + x_4 = 0$, i.e., its dot product with (1,1,1,1) is 0. This vector along with its permutations will span a 3-dimensional subspace that is perpendicular to the line through (1,1,1,1) and origin.

(d) Take the standard basis for $\mathbb{R}^4$, i.e., (1,0,0,0) and its permutations since any vector space is a subspace of itself.

5. Let V and W be subspaces of $\mathbb{R}^n$ with $V \cap W = \{0\}$. Show that $dim(V) + dim(W) \leq n$.

Hint: Consider $v_1, ..., v_k$ to be a basis for V , and $w_1, ..., w_l$ to be a basis for W, and show that $v_1, ..., v_k, w_1, ..., w_l$ are linearly independent. (3 marks)

Solution :

Let $v_1, ..., v_k$ be a basis for V, and let $w_1, ..., w_l$ be a basis for W. I claim that $v_1, ..., v_k, w_1, ..., w_l$ are linearly independent. To see this, suppose $a_1v_1 + ... + a_kv_k + b_1w_1 + ... + b_lw_l = 0$. We need to show that $a_1 = ... = a_k = b_1 = ... = b_l = 0$. To simplify the notation, let $v = a_1v_1 + ... + a_kv_k$ and $w = b_1w_1 + ... + b_lw_l$. Then $v + w = 0$, so $v = -w$. Now v is in V and $-w$ is in W, so v and w are both in the intersection of V and W. But this intersection was assumed to be $\{0\}$. So $v = w = 0$. Since $v_1, ..., v_k$ are independent, $v = 0$ implies $a_1 = ... = a_k = 0$. Since $w_1, ..., w_l$ are independent, $w = 0$ implies $b_1 = ... = b_l = 0$.

So we have $k+l$ linearly independent vectors in $\mathbb{R}^n$. Therefore $k+l$ is less than or equal to n. Since $k = dim(V)$ and $l = dim(W)$, this proves what we wanted.

6. Which of the following functions T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations?

(a) $T(x_1; x_2) = (1 + x_1; x_2)$
(b) $T(x_1; x_2) = (x_2; x_1)$
(c) $T(x_1; x_2) = (x_1^2; x_2)$
(d) $T(x_1; x_2) = (sinx_1; x_2)$

(2 marks)

Solution :

(a) T is not a linear transformation because $T(0,0) = (1,0)$. If T were a linear transformation then it must always be that $T(0,0) = (0,0)$.

(b) T is a linear transformation. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$. Then $T(c\alpha+\beta) = T((cx_1+y_1, cx_2+y_2)) = (cx_2+y_2, cx_1+y_1) = c(x_2, x_1) + (y_2, y_1) = cT(\alpha) + T(\beta)$.

(c) T is not a linear transformation. If T were a linear transformation then we'd have $(1,0) = T((-1,0)) = T(-1 \cdot (1,0)) = -1 \cdot T(1,0) = -1 \cdot (1,0) = (-1,0)$ which is a contradiction, $(1,0) \neq (-1,0)$.

(d) T is not a linear transformation. If T were a linear transformation then $(0,0) = T(\pi,0) = T(2(\pi/2,0)) = 2T((\pi/2,0)) = 2(\sin(\pi/2),0) = 2(1,0) = (2,0)$ which is a contradiction, $(0,0) \neq (2,0)$.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation of the 2-dimensional vector space $\mathbb{R}^2$ (the x-y-plane) to itself which is the reflection across a line $y = mx$ for some $m \in \mathbb{R}$.

Find the matrix representation of the linear transformation T with respect to the standard basis $B = \{e_1, e_2\}$ of $\mathbb{R}^2$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (5 marks)

Solution :

Let A be the matrix representation of T with respect to the standard basis B.

Observe that each vector on the line $y = mx$ does not move under the linear transformation T.
Since the vector $\begin{bmatrix} 1 \\ m \end{bmatrix}$ is on the line $y = mx$, it follows that

$$A \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}. \qquad (*)$$

Note that if $m \neq 0$ then the line $y = \frac{-1}{m} x$ is perpendicular to the line $y = mx$ at the origin.
If $m = 0$, then the line $x = 0$ is perpendicular to the line $y = 0$ at the origin.
In either case the vector $\begin{bmatrix} -m \\ 1 \end{bmatrix}$ is on the perpendicular line.

Thus, by the reflection across the line $y = mx$, this vector is mapped to $\begin{bmatrix} m \\ -1 \end{bmatrix}$.
That is, we have

$$A \begin{bmatrix} -m \\ 1 \end{bmatrix} = \begin{bmatrix} m \\ -1 \end{bmatrix}. \qquad (**)$$

It follows from (*) and (**) that

$$A \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} 1 \\ m \end{bmatrix} & A \begin{bmatrix} -m \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}.$$

The determinant of the matrix $\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}$ is $1 + m^2 \neq 0$, hence it is invertible.
(Note that since column vectors are nonzero orthogonal vectors, we knew it is invertible.)

The inverse matrix is

$$\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}^{-1} = \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned} A &= \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix} \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix} \cdot \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} \\ &= \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}. \end{aligned}$$