# Linear Algebra and Random Processes (CS6015)

**DEPT. OF COMPUTER SCIENCE AND ENGINEERING**
**Indian Institute of Technology Madras**

**TUTORIAL 4**
**(Time allowed: FIFTY minutes)**

**NOTE:** Attempt **ALL** questions. Total Marks : **27**

**1.** Suppose $q_1, q_2, q_3$ are orthonormal vectors in $\mathbb{R}^3$. Find the possible values for the following 3 by 3 determinants:

(a) $det([q_1 \quad q_2 \quad q_3])$
**Solution :**

The determinant of any square matrix with orthonormal columns ("orthogonal matrix") is $\pm 1$.

(b) $det([q_1 + q_2 \quad q_2 + q_3 \quad q_3 + q_1])$
**Solution :**

$$\begin{aligned}
\det\begin{bmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{bmatrix} &= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3+q_1 \end{bmatrix} \\
&= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & 2q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}
\end{aligned}$$

Again, whatever $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is, this determinant will be twice that, or $\pm 2$.

(3 marks)

**2.** What matrix $P$ projects every point of $\mathbb{R}^3$ onto the line of intersection of the planes $x + y + t = 0$ and $x - t = 0$? (4 marks)

**Solution :**

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First, we need to find a vector on the line of intersection. Note that any such vector is necessarily a solution of the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve this equation, we'll do Gaussian elimination on the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Subtract row 1 from row 2 to get

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix},$$

Then add row 2 to row 1 and multiply row 2 by $-1$:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

This corresponds to the system of equations

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3. \end{aligned}$$

Therefore, solutions to the matrix equation are of the form

$$x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

meaning that the vector $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in the desired line of intersection.

Now, just recall that the projection matrix $P$ is given by

$$P = \frac{\mathbf{a}\,\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$$

The denominator is simply

$$\mathbf{a}^T\mathbf{a} = [1\ -2\ 1] \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 + 4 + 1 = 6.$$

The numerator is

$$\mathbf{a}\,\mathbf{a}^T = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} [1\ -2\ 1] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

Therefore,

$$P = \frac{\mathbf{a}\,\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

**3.** (a) Find orthonormal vectors $q_1$, $q_2$, $q_3$ such that $q_1$, $q_2$ span the column space of A $= \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$

**Solution :**

Gram-Schmidt chooses $\boldsymbol{q}_1 = \boldsymbol{a}/||\boldsymbol{a}|| = \frac{1}{3}(1,2,-2)$ and $\boldsymbol{q}_2 = \frac{1}{3}(2,1,2)$. Then $\boldsymbol{q}_3 = \frac{1}{3}(2,-2,-1)$.

(b) Solve $Ax = (1,2,7)$ by least squares.

**Solution :**

$\hat{x} = (A^TA)^{-1}A^T \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A^TA = \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix}$

$(A^TA)^{-1} = \frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$(A^TA)^{-1}A^T = \frac{1}{9}\begin{bmatrix} 3 & 3 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

$(A^TA)^{-1}A^T \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(6 marks)

**4.** Find the determinant of the following matrix by Gaussian elimination:

$\text{A} = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$ (3 marks)

**Solution :**

$\text{A} = \begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{vmatrix} = \begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2-1 & 0 & 1-t^2 \end{vmatrix} = (1-t^2)\begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ -1 & 0 & 1 \end{vmatrix}$

$= (1-t^2)\begin{vmatrix} 1-t^2 & 0 & 0 \\ t & 1 & t \\ -1 & 0 & 1 \end{vmatrix} = (1-t^2)^2\begin{vmatrix} 1 & 0 & 0 \\ t & 1 & t \\ -1 & 0 & 1 \end{vmatrix}$

The determinant is $1 - 2t^2 + t^4 = (1-t^2)^2$

**5.** A 3 by 3 matrix $B$ is known to have eigenvalues $0, 1, 2$. Find

(a) the rank of $B$

**Solution :**

rank = 2

(b) the determinant of $B^TB$

**Solution :**

The matrix $B$ and its transpose $B^T$ has the same eigenvalues. The product of the $n$ eigenvalues of $B$ is the same as the determinant of $B$. $det(B^TB) = 0$

(c) the eigenvalues of $(B^2+I)^{-1}$.

Let the eigenvalues for the matrix $B$ be $\lambda$, and $x$ be the eigenvectors.

$Bx = \lambda x$ and $Ix = x$

$B^2x = \lambda Bx \implies B^2x = \lambda^2 x$

$(B^2+I)x = (\lambda^2+1)Bx \implies (B^2+I)^{-1}x = \frac{1}{(\lambda^2+1)}Bx$

The eigenvalues of $(B^2+I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$

(3 marks)

**6.** Describe all matrices $S$ that diagonalize this matrix $A$ (find all eigenvectors):

$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ (3 marks)

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**Solution :**
$|A - \lambda I| = 0 \implies \lambda_1 = 4, \lambda_2 = 2$
Using the eigenvalues we obtain the eigenvectors $(2, 1)$ and $(0, 1)$. The columns of $S$ are nonzero multiples of $(2, 1)$ and $(0, 1)$: either order

**7.** Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of matrix $A$. Likewise, let $\mu_1, \mu_2, ..., \mu_n$ be the eigenvalues of matrix $B$.(Note: Both $A$ and $B$ are $n \times n$ matrices.) If the eigenvectors of $A$ and $B$ are the same, are the eigenvalues of $(AB)^T$ and $(AB)^{-1}$ related? If so, how? (3 marks)

**Hint :** Use the properties of eigenvalues
**Solution :**

The characteristic polynomial $p_X(t) = det(X - sI)$ of $X$ is the same as the characteristic polynomial $p_{X^T}(s) = det(X^T - sI)$ of the transpose $X^T$. Therefore, the eigenvalues of $X$ and $X^T$ are the same.

If $t$ is an eigenvalue of $Y$, then $\frac{1}{t}$ is an eigenvalue of the matrix $Y^{-1}$.

Consider $\lambda$ to be an eigenvalue of $A$ and $\mu$ to be an eigenvalue of $B$ and $\mathbf{x}$ be the corresponding eigenvector.

$$\begin{aligned}(AB)\mathbf{x} &= A(B\mathbf{x})\\ &= A(\mu\mathbf{x})\\ &= \mu(A\mathbf{x})\\ &= \mu(\lambda\mathbf{x})\\ &= (\lambda\mu)\mathbf{x}.\end{aligned}$$

Therefore, the eigenvalues of $(AB)^T$ are $\lambda_1\mu_1, \lambda_2\mu_2, ..., \lambda_n\mu_n$ and that of $(AB)^{-1}$ is $\frac{1}{\lambda_1\mu_1}, \frac{1}{\lambda_2\mu_2}, ..., \frac{1}{\lambda_n\mu_n}$

**8.** Find all cofactors of $A$ and put them into the cofactor matrix $C$. Find $det(A)$. A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$ (2 marks)

**Solution :**

C = $\begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$

det(A) = 1(0) + 2(42) + 3(-35) = -21.