Linear Algebra and Random Processes (CS6015)

DEPT. OF COMPUTER SCIENCE AND ENGINEERING
Indian Institute of Technology Madras

TUTORIAL 5
(Time allowed: FIFTY minutes)

NOTE: Attempt **ALL** questions. Total Marks : **25**

1. Which of the matrices A_1, A_2, A_3, A_4 is positive-definite? Justify
$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$ $A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$ $A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ (6 marks)

Solution :

Each of the following tests is a necessary and sufficient condition for the real symmetric matrix A to be ***positive definite***:

(I) $x^{\mathrm{T}}Ax > 0$ for all nonzero real vectors x.

(II) All the eigenvalues of A satisfy $\lambda_i > 0$.

(III) All the upper left submatrices A_k have positive determinants.

(IV) All the pivots (without row exchanges) satisfy $d_k > 0$.

Only $A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues since $101 > 102$.

2. Which classes of matrices does P belong to: invertible, Hermitian, unitary? What are the eigenvalues of P?
$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}$ (3 marks)

Solution :
$|P| \neq 0$. Hence invertible.
$P \neq P^H$. Hence not Hermitian.
$PP^H = I$. Hence Unitary.

This P is invertible and unitary. The eigenvalues of P are the roots of $\lambda^3 = -i$, which are i and $ie^{2\pi i/3}$ and $ie^{4\pi i/3}$.

3. Prove that similar matrices A and B, have the same eigen values. (2 marks)

Solution :

The proof is quick, since $B = M^{-1}AM$ gives $A = MBM^{-1}$. Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$MBM^{-1}\boldsymbol{x} = \lambda\boldsymbol{x} \quad \textbf{\textit{means that}} \quad B(M^{-1}\boldsymbol{x}) = \lambda(M^{-1}\boldsymbol{x}).$$

The eigenvalue of B is the same λ. The eigenvector has changed to $M^{-1}x$.

4. Suppose A has orthogonal columns $w_1, w_2, ..., w_n$ of lengths $\sigma_1, \sigma_2, ...\sigma_n$. What are U, Σ, and V in the SVD? (3 marks)

CONTINUED

Solution :

If A has orthogonal columns $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ of lengths $\sigma_1, \ldots, \sigma_n$, then $A^{\mathrm{T}}A$ will be diagonal with entries $\sigma_1^2, \ldots, \sigma_n^2$. So the σ's are definitely the singular values of A (as expected). The eigenvalues of that diagonal matrix $A^{\mathrm{T}}A$ are the columns of I, so $V = I$ in the SVD. Then the $\boldsymbol{u}_i$ are $A\boldsymbol{v}_i/\sigma_i$ which is the unit vector $\boldsymbol{w}_i/\sigma_i$.

The SVD of this A with orthogonal columns is $A = U\Sigma V^{\mathrm{T}} = (A\Sigma^{-1})(\Sigma)(I)$.

5. Construct the singular value decomposition (SVD) of A $= \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ (3 marks)

Solution :

Solution: Compute $\mathbf{AA}^T$, find its eigenvalues (it is generally preferred to put them into decreasing order) and then find corresponding unit eigenvectors:

$$\mathbf{AA}^T = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \det\left(\mathbf{AA}^T - \lambda\mathbf{I}\right) = \det\begin{bmatrix} 8-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(8-\lambda)(2-\lambda) = 0 \Rightarrow \lambda_1 = 8, \lambda_2 = 2$$

Their corresponding unit eigenvectors are:

$$\mathbf{AA}^T\underline{u}_1 = \lambda_1\underline{u}_1 \Rightarrow \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 8\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} \Rightarrow \begin{matrix} 8u_{11} = 8u_{11} \Rightarrow u_{11} = 1 \\ 2u_{12} = 8u_{12} \Rightarrow u_{12} = 0 \end{matrix} \Rightarrow \underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{AA}^T\underline{u}_2 = \lambda_2\underline{u}_2 \Rightarrow \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 2\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} \Rightarrow \begin{matrix} 8u_{21} = 2u_{21} \Rightarrow u_{21} = 0 \\ 2u_{22} = 2u_{22} \Rightarrow u_{22} = 1 \end{matrix} \Rightarrow \underline{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The matrix $\mathbf{U}$ is then:

$$\mathbf{U} = [\ \underline{u}_1 \quad \underline{u}_2\ ] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of the $\mathbf{A}^T\mathbf{A}$ are the same as the eigenvalues of the $\mathbf{AA}^T$. The eigenvectors of the $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ are:

$$\mathbf{A}^T\mathbf{A}\underline{v}_1 = \lambda_1\underline{v}_1 \Rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 8\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \Rightarrow \begin{matrix} 5v_{11} + 3v_{12} = 8v_{11} \Rightarrow v_{11} = v_{12} \\ 3v_{11} + 5v_{12} = 8v_{12} \Rightarrow v_{12} = v_{11} \end{matrix}$$

Choice of v_{11} will define v_{12} and vice versa. In general v_{11} and v_{12} can be any numbers, but since vector $\underline{v}_1$ should have length of 1, the v_{11} and v_{12} are chosen as follows:

$$\|\underline{v}_1\| = 1 \Rightarrow \sqrt{v_{11}^2 + v_{12}^2} = 1 \Rightarrow v_{11} = v_{12} = \frac{1}{\sqrt{2}} \Rightarrow \underline{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Unit eigenvector $\underline{v}_2$ is:

$$\mathbf{A}^T\mathbf{A}\underline{v}_2 = \lambda_2\underline{v}_2 \Rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 2\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \Rightarrow \begin{matrix} 5v_{21} + 3v_{22} = 2v_{21} \Rightarrow v_{21} = 0 \\ 3v_{21} + 5v_{22} = 2v_{22} \Rightarrow v_{22} = 1 \end{matrix}$$

$$v_{21} = -v_{22} \Rightarrow \underline{v}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix $\mathbf{V}$ is then:

$$\mathbf{V} = [\ \underline{v}_1 \quad \underline{v}_2\ ] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \mathbf{V}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The matrix $\mathbf{S}$ is:

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

6. You are given the quadratic polynomial $f(x, y, z)$:

$$f(x, y, z) = 2x^2 - 2xy - 4xz + y^2 + 2yz + 3z^2 - 2x + 2z$$

(a) Write $f(x, y, z)$ in the form $f(x, y, z) = x^T Ax - b^T x$ where $x = (x, y, z)^T$, A is a real symmetric matrix, and b is some constant vector. **Solution :**
We have

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}.$$

(b) Find the point (x, y, z) where $f(x, y, z)$ is at an extremum. **Solution :**

We compute the partial derivatives to find the extremum point:

$$\frac{\partial f}{\partial x} = 4x - 2y - 4z - 2 = 0$$
$$\frac{\partial f}{\partial y} = -2x + 2y + 2z = 0$$
$$\frac{\partial f}{\partial x} = -4x + 2y + 6z + 2 = 0$$

The equation is just $2A\mathbf{x} = \mathbf{b}$. The solution to the the equations above is $x = 1, y = 1, z = 0$. So the extreme point is $(1, 1, 0)$.

(c) Is this point a minimum, maximum, or a saddle point of some kind? **Solution :**

We look for the pivots of A:

$$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since A is positive definite, the extreme point is a minimum.

(4 marks)

7. Toss a fair coin 3 times. Let H_1 = 'heads on first toss' and A = 'two heads total'. Are H_1 and A independent? (2 marks)

Solution :

We know that $P(A) = 3/8$. Since this is not 0 we can check if the formula in Equation 5 holds. Now, $H_1 = \{$HHH, HHT, HTH, HTT$\}$ contains exactly two outcomes (HHT, HTH) from A, so we have $P(A|H_1) = 2/4$. Since $P(A|H_1) \neq P(A)$ these events are not independent.

8. How many permutations of the letters ABCDEFGH contain the string ABC? (2 marks)

Solution :
Think of the letters ABC as glued together. Thus we really have six objects, namely, the superletter ABC, and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters ABCDEFGH in which ABC occurs as a block.