# Linear Algebra and Random Processes (CS6015)

**DEPT. OF COMPUTER SCIENCE AND ENGINEERING**
**Indian Institute of Technology Madras**

**TUTORIAL 6**
**(Time allowed: FIFTY minutes)**

**NOTE:** Attempt **ALL** questions. Total Marks : **40**

**1.** If $A$, $B$, $C$ are independent events and $P(C) > 0$, then are $A$ and $B$ conditionally independent given $C$? If so, prove. (2 marks)

**Solution:** True. Since $A, C$ are independent $\mathbb{P}(A \mid C) = \mathbb{P}(A)$. Similarly, $\mathbb{P}(B \mid C) = \mathbb{P}(B)$.

$$\mathbb{P}(A \cap B \mid C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)}{\mathbb{P}(C)} = \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A \mid C)\mathbb{P}(B \mid C).$$

**2.** A telephone operator on an average handles 5 calls every 3 minutes. What is the probability that there will be

(a) no calls in the next minute?
(b) atleast 2 calls?
(c) atmost 2 calls?
(d) exactly 2 calls in the next minute? (4 marks)

**Solution:**
If we denote X as the number of calls in a minute, then X has the poisson distribution with $\lambda = 5/3$. So

a) $Pr(\text{no calls in the next minute}) = Pr(X = 0) = e^{-5/3} = .189$.

b) $Pr(\text{at least 2 calls}) = Pr(X \geq 2) = 1 - [Pr(X = 0) + Pr(X = 1)] = 1 - \left[.189 + \left(e^{-5/3}\right)(5/3)\right] = .4963$. Note that Table C cannot be used here since $\lambda t = 5/3$.

c) $Pr(\text{at most 2 calls}) = Pr(X \leq 2) = Pr(X = 0) + Pr(X = 1) + Pr(X = 2) = e^{-5/3}\left[1 + (5/3) + (5/3)^2/2\right] = .1888756(4.055555) = .766$.

d) $Pr(\text{exactly 2 calls}) = Pr(X = 2) = .262$

**3.** Ten percent of computer parts produced by a certain supplier are defective. What is the probability that a sample of 10 parts contains more than 3 defective ones? (3 marks)

**Solution:**

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We need to find $P(X > 3)$, where $X$ is the number of defective parts in a sample of 10 parts. This $X$ is the number of "successes" in 10 trials, therefore, it has Binomialdistribution with parameters $n = 10$ and $p = 0.1$.

From the Table of Binomial distribution,

$$P(X > 3) = 1 - F(3) = 1 - 0.9872 = 0.0128$$

Or, by the formula of Binomial PMF,

$$P(X > 3) = 1 - P(0) - P(1) - P(2) - P(3)$$

$$= 1 - (0.9)^{10} - (10)(0.1)(0.9)^9 - (\frac{10 \cdot 9}{2})(0.1)^2(0.9)^8 - (\frac{10 \cdot 9 \cdot 8}{6})(0.1)^3(0.9)^7 = 0.0128$$

**4.** Let X be a discrete random variable with the following PMF

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\mathrm{E}(X)$.

(b) Find $\mathrm{var}(X)$.

(c) If $Y = (X - 2)^2$ find $\mathrm{E}(Y)$. (3 marks)

**Solution:**

a.
$$\begin{aligned} EX &= \textstyle\sum_{x_k \in R_X} x_k P_X(x_k) \\ &= 0(0.1) + 1(0.4) + 2(0.3) + 3(0.2) \\ &= 1.6 \end{aligned}$$

b. We can use Var$(X) = EX^2 - (EX)^2 = EX^2 - (1.6)^2$ . Thus we need to find $EX^2$. Using LOTUS, we have

$$EX^2 = 0^2(0.1) + 1^2(0.4) + 2^2(0.3) + 3^2(0.2) = 3.4$$

Thus, we have

$$\mathrm{Var}(X) = (3.4) - (1.6)^2 = 0.84$$

c. Again, using LOTUS, we have

$$E(X-2)^2 = (0-2)^2(0.1) + (1-2)^2(0.4) + (2-2)^2(0.3) + (3-2)^2(0.2) = 1.$$

**5.** Let A and B be two events. Suppose the probability that neither $A$ or $B$ occurs is 2/3. What is the probability that one or both occur? (2 marks)

**Solution:**

We are given $P(A^c \cap B^c) = 2/3$ and asked to find $P(A \cup B)$.

$A^c \cap B^c = (A \cup B)^c \Rightarrow P(A \cup B) = 1 - P(A^c \cap B^c) = \boxed{1/3.}$

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**6.** Data is taken on the height and shoe size of a sample of few students. Height is coded by 3 values: 1 (short), 2 (average), 3 (tall) and shoe size is coded by 3 values 1 (small), 2 (average), 3 (large). The joint counts are given in the following table.

| height or shoe size | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 234 | 225 | 84 |
| 2 | 180 | 453 | 161 |
| 3 | 39 | 192 | 157 |

Let X be the coded shoe size and Y the height of a random person in the sample.

(a) Find the joint and marginal pmf of X and Y.

(b) Are X and Y independent? (5 marks)

**Solution:** The joint distribution is found by dividing each entry in the data table by the total number of people in the sample. Adding up all the entries we get 1725. So the joint probability table with marginals is

| $Y \setminus X$ | 1 | 2 | 3 | |
|---|---|---|---|---|
| 1 | $\frac{234}{1725}$ | $\frac{225}{1725}$ | $\frac{84}{1725}$ | $\frac{543}{1725}$ |
| 2 | $\frac{180}{1725}$ | $\frac{453}{1725}$ | $\frac{161}{1725}$ | $\frac{794}{1725}$ |
| 3 | $\frac{39}{1725}$ | $\frac{192}{1725}$ | $\frac{157}{1725}$ | $\frac{388}{1725}$ |
| | $\frac{453}{1725}$ | $\frac{839}{1725}$ | $\frac{433}{1725}$ | 1 |

The marginal distribution of $X$ is at the right and of $Y$ is at the bottom.

**(b)** $X$ and $Y$ are dependent because, for example,

$$P(X = 1 \text{ and } Y = 1) = \frac{234}{1725} \approx 0.136$$

is not equal to

$$P(X = 1)P(Y = 1) = \frac{453}{1725} \cdot \frac{543}{1725} \approx 0.083.$$

**7.** If X and Y are independent, prove that $var(XY) = \sigma_x^2\sigma_y^2 + \sigma_x^2\mu_y^2 + \sigma_y^2\mu_x^2$ where $\mu$ is the mean and $\sigma^2$ is the variance. (3 marks)

**Solution:**

$$\begin{aligned}
Var(XY) &= E[(XY)^2] - \{E[XY]\}^2 && (\because Var(A) = E[A^2] - \{E[A]\}^2) \\
&= E[X^2Y^2] - \{E[X]E[Y]\}^2 && (\because \text{X and Y are independent}) \\
&= E[X^2]E[Y^2] - E[X]^2E[Y]^2 \\
&= (\sigma_x^2 + \mu_x^2)(\sigma_y^2 + \mu_y^2) - \mu_x^2\mu_y^2 && (\because E[A^2] = Var(A) + \{E[A]\}^2) \\
&= \sigma_x^2\sigma_y^2 + \sigma_x^2\mu_y^2 + \sigma_y^2\mu_x^2 + \mu_x^2\mu_y^2 - \mu_x^2\mu_y^2 \\
&= \sigma_x^2\sigma_y^2 + \sigma_x^2\mu_y^2 + \sigma_y^2\mu_x^2
\end{aligned}$$

**8.** A lost tourist arrives at a point with 3 roads. The first road brings him back to the same point after 1 hour of walk. The second road brings him back to the same point after 6 hours of travel. The last road leads to the city after 2 hours of walk. There are no signs on the roads. Assuming that the

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tourist chooses a road equally likely at all times. What is the mean time until the tourist arrives to the city. (3 marks)

**Solution:** Let T be the time it takes to reach the city, and $D_i$ be the event that the tourist chooses road i for $i = 1, 2, 3$. To compute $E[T]$ we will condition on the road tourist picks at the first time:

$$\begin{aligned} E[T] &= E[T|D_1]P(D_1) + E[T|D_2]P(D_2) + E[T|D_3]P(D_3) \\ &= \frac{1}{3}(E[T|D_1] + E[T|D_2] + E[T|D_3]) \\ &= \frac{1}{3}(1 + E[T] + 6 + E[T] + 2) \qquad (2.1) \\ & \qquad (2.2) \end{aligned}$$

where in 2.1 we use the fact that after each return to the point, the tourist *does not keep memory of its past chooses.*

Rearranging the terms in the last equation, we obtain

$$E[T] = 9$$

**9.** The length of time X, needed by students in a particular course to complete a 1 hour exam is a random variable with PDF given by:

$$f(x) = \begin{cases} k(x^2 + x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

For the random variable X, find the cumulative distribution function. (3 marks)

**Solution:**

The CDF, $F(x)$, is area function of the PDF, obtained by integrating the PDF from negative infinity to an arbitrary value $x$.

If $x$ is in the interval $(-\infty, 0)$, then

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{x} 0dt \\ &= 0 \end{aligned}$$

If $x$ is in the interval $[0, 1]$, then

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt \\ &= 0 + \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right) \\ &= \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right) \end{aligned}$$

If $x$ is in the interval $(1, \infty)$ then

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt \\ &= 0 + \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1 + 0 \\ &= \frac{6}{5} \cdot \frac{5}{6} \\ &= 1 \end{aligned}$$

Note that the PDF $f$ is equal to zero for $x > 1$. The CDF is therefore given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right) & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x > 1. \end{cases}$$

**10.** Let $X \sim exp(\lambda)$. Find $E(X^2)$ using the expectation formula. (2 marks)

**Solution:**

$$E(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx = \left[-x^2 e^{-\lambda x} - \frac{2x}{\lambda} e^{-\lambda x} - \frac{2}{\lambda^2} e^{-\lambda x}\right]_0^{\infty} = \boxed{\frac{2}{\lambda^2}}.$$

**11.** Find the expressions of first and second derivative of univariate Gaussian function. Also, draw the plots of all the three functions (Gaussian, first derivative of the Gaussian, second derivative of the Gaussian). (5 marks)

**Solution:**

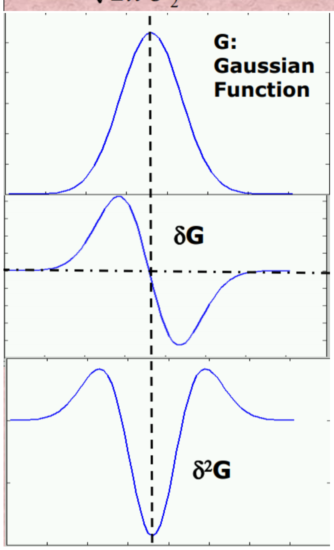
$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$\nabla G(x) =$$

$$\frac{-x}{\sqrt{2\pi}\sigma_2^3} \exp\left(-\frac{x^2}{2\sigma_2^2}\right)$$

$$\nabla^2 G(x) =$$

$$\frac{-\left[\left(\frac{x}{\sigma}\right)^2 - 1\right]}{\sqrt{2\pi}\sigma_2^3} \exp\left(-\frac{x^2}{2\sigma_2^2}\right)$$




**12.** Consider the standard bivariate normal distribution

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right).$$

The covariance between X and Y is given as :

$$cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \rho$$

which can be split into

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$$\int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}y^2} \left(\int_{-\infty}^{\infty} x g(x,y) dx\right) dy$$

Prove that $g(x,y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} exp(-\frac{1}{2}\frac{(x-\rho y)^2}{(1-\rho^2)})$ (5 marks)

**Solution:**

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}$$

To show :- $g(x,y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2}\frac{(x-\rho y)^2}{(1-\rho^2)}}$

From the given data,

$$\frac{xy\, f(x,y)}{\frac{1}{\sqrt{2\pi}} y e^{-y^2/2} x g(x,y)} = 1$$

$$\frac{\cancel{x}\cancel{y} \frac{1}{\cancel{2\pi}\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}}{\frac{1}{\cancel{\sqrt{2\pi}}} \cancel{y} e^{-y^2/2} \cancel{x} \frac{1}{\cancel{\sqrt{2\pi(1-\rho^2)}}} e^{-\frac{1}{2}\frac{(x-\rho y)^2}{(1-\rho^2)}}}$$

$$e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)} = e^{-\frac{y^2}{2} - \frac{1}{2}\frac{(x-\rho y)^2}{(1-\rho^2)}}$$

Taking log on both sides

$$\frac{-1}{2(1-\rho^2)}(x^2+y^2-2\rho xy) = -\frac{y^2}{2} - \frac{1}{2(1-\rho^2)}(x-\rho y)^2$$

$$\frac{-\cancel{1}}{\cancel{2(1-\rho^2)}}(x^2+y^2-2\rho xy) = -\frac{(1-\rho^2)y^2 + (x-\rho y)^2}{\cancel{2(1-\rho^2)}}$$

$$x^2+y^2-\cancel{2\rho xy} = y^2 - \cancel{\rho^2 y^2} + x^2 + \cancel{\rho^2 y^2} - \cancel{2\rho xy}$$

$$x^2+y^2 = x^2+y^2$$

LHS = RHS

Hence proved.