Linear Algebra and Random Processes (CS6015)

DEPT. OF COMPUTER SCIENCE AND ENGINEERING
Indian Institute of Technology Madras

TUTORIAL 7
(Time allowed: FIFTY minutes)

NOTE: Attempt **ALL** questions. Total Marks : **30**

1. (a) Suppose that X is uniform on [0, 1]. Compute the pdf and cdf of X.

(b) If Y = 2X + 5, compute the pdf and cdf of Y. (3 marks)

Solution:

34. **(a)** We have $f_X(x) = 1$ for $0 \leq x \leq 1$. The cdf of X is

$$F_X(x) = \int_0^x f_X(t)dt = \int_0^x 1dt = x.$$

(b) Since X is between 0 and 1 we have Y is between 5 and 7. Now for $5 \leq y \leq 7$, we have

$$F_Y(y) = P(Y \leq y) = P(2X + 5 \leq y) = P(X \leq \frac{y-5}{2}) = F_X(\frac{y-5}{2}) = \frac{y-5}{2}.$$

Differentiating $P(Y \leq y)$ with respect to y, we get the probability density function of Y, for $5 \leq y \leq 7$,

$$f_Y(y) = \frac{1}{2}.$$

2. Let X be a random variable with range [0, 1] and cdf
$F(X) = 2x^2 - x^4 \quad 0 \leq x \leq 1$

(a) Compute $P(1/4 \leq X \leq 3/4)$.

(b) What is the pdf of X? (3 marks)

Solution:

45. **(a)** $P(1/4 \leq X \leq 3/4) = F(3/4) - F(1/4) = \boxed{11/16 = .6875.}$

(b) $f(x) = F'(x) = 4x - 4x^3$ in [0,1].

3. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & otherwise \end{cases}$$

Find $P(X \leq 2/3 | X > 1/3)$ (2 marks)

Solution:

CONTINUED

$$P(X \leq \frac{2}{3} | X > \frac{1}{3}) = \frac{P(\frac{1}{3} < X \leq \frac{2}{3})}{P(X > \frac{1}{3})} = \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} 4x^3 dx}{\int_{\frac{1}{3}}^{1} 4x^3 dx} = \frac{3}{16}.$$

4. Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x, y \geq 0 \\ 0 & otherwise \end{cases}$$

(a) Are X and Y independent? Justify.

(b) Find $E[Y|X > 2]$.

(c) Find $P(X > Y)$. (4 marks)

Solution:

1. We can write

$$f_{X,Y}(x,y) = f_X(x) f_Y(y),$$

where

$$f_X(x) = 2e^{-2x}u(x), \qquad f_Y(y) = 3e^{-3y}u(y).$$

Thus, X and Y are independent.

2. Since X and Y are independent, we have $E[Y|X > 2] = E[Y]$. Note that $Y \sim Exponential(3)$, thus $EY = \frac{1}{3}$.

3. We have

$$P(X > Y) = \int_0^{\infty} \int_y^{\infty} 6e^{-(2x+3y)} dx dy = \int_0^{\infty} 3e^{-5y} dy = \frac{3}{5}.$$

5. Show that the mean of exponential distribution (with parameter λ) is $1/\lambda$. (2 marks)

Solution:

$$\begin{aligned} E[X] &= \int_0^{\infty} x\lambda e^{-\lambda x} dx \\ &= \lambda \left[\frac{-xe^{-\lambda x}}{\lambda} \Big|_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right] \\ &= \lambda \left[0 + \frac{1}{\lambda} \frac{-e^{-\lambda x}}{\lambda} \Big|_0^{\infty} \right] \\ &= \lambda \frac{1}{\lambda^2} = \frac{1}{\lambda}. \end{aligned}$$

6. The properties of gamma function are as follows:

For any positive real number α:

1. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$;
2. $\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$, for $\lambda > 0$;
3. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$;
4. $\Gamma(n) = (n-1)!$, for $n = 1, 2, 3, \cdots$;
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Use the above properties to answer the following questions:

(a) Find $\Gamma(7/2)$.

(b) Find the value of the following integral:
$I = \int_0^\infty x^6 e^{-5x} dx$

(c) Show that the gamma pdf integrates to 1 i.e.,
$\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = 1$ (5 marks)

Solution:

1. To find $\Gamma(\frac{7}{2})$, we can write

$$\begin{aligned}
\Gamma(\frac{7}{2}) &= \frac{5}{2} \cdot \Gamma(\frac{5}{2}) && \text{(using Property 3)} \\
&= \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma(\frac{3}{2}) && \text{(using Property 3)} \\
&= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\frac{1}{2}) && \text{(using Property 3)} \\
&= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} && \text{(using Property 5)} \\
&= \frac{15}{8}\sqrt{\pi}.
\end{aligned}$$

2. Using Property 2 with $\alpha = 7$ and $\lambda = 5$, we obtain

$$\begin{aligned}
I &= \int_0^\infty x^6 e^{-5x} dx \\
&= \frac{\Gamma(7)}{5^7} \\
&= \frac{6!}{5^7} && \text{(using Property 4)} \\
&\approx 0.0092
\end{aligned}$$

We can write

$$\begin{aligned}
\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\lambda^\alpha} && \text{(using Property 2 of the gamma function)} \\
&= 1.
\end{aligned}$$

7. Suppose a sample $x_1, ..., x_n$ is modelled by a Poisson distribution with parameter denoted by λ, so that
$f_X(x;\theta) = f_X(x;\lambda) = \frac{\lambda^x}{x!}e^{-\lambda} \quad x = 0, 1, 2, ...$
for some $\lambda > 0$. Estimate λ by maximum likelihood. (3 marks)

Solution:

STEP 1 Calculate the likelihood function $L(\lambda)$.

$$L(\lambda) = \prod_{i=1}^{n} f_X(x_i;\lambda) = \prod_{i=1}^{n} \left\{\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}\right\} = \frac{\lambda^{x_1+...+x_n}}{x_1!....x_n!}e^{-n\lambda}$$

STEP 2 Calculate the log-likelihood $\log L(\lambda)$.

$$\log L(\lambda) = \sum_{i=1}^{n} x_i \log\lambda - n\lambda - \sum_{i=1}^{n} \log(x_i!)$$

STEP 3 Differentiate $\log L(\lambda)$ with respect to λ, and equate the derivative to zero to find the m.l.e..

$$\frac{d}{d\lambda}\{\log L(\lambda)\} = \sum_{i=1}^{n} \frac{x_i}{\lambda} - n = 0 \Rightarrow \widehat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

Thus the maximum likelihood estimate of λ is $\widehat{\lambda} = \bar{x}$

8. The CEO of light bulbs manufacturing company claims that an average light bulb lasts 300 days. A researcher randomly selects 15 bulbs for testing. The sampled bulbs last an average of 290 days, with a standard deviation of 50 days. If the CEO's claim were true, what is the probability that 15 randomly selected bulbs would have an average life of no more than 290 days? (3 marks)

Solution:

The traditional approach requires you to compute the t statistic, based on data presented in the problem description.

The first thing we need to do is compute the t statistic, based on the following equation:

Where $\bar{x}$ is the sample mean, μ is the population mean, s is the standard deviation of the sample, and n is the sample size.

Using the formula: $t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{N}}}$

$$t = \frac{290-300}{\frac{50}{\sqrt{15}}}$$

$$= \frac{-10}{12.909945} = 0.7745966$$

Since we will work with the raw data, we select "Sample mean" from the Random Variable dropdown box.

- The degrees of freedom are equal to 15 − 1 = 14.
- Assuming the CEO's claim is true, the population mean equals 300.
- The sample mean equals 290.
- The standard deviation of the sample is 50.

The cumulative probability: 0.226. Hence, if the true bulb life were 300 days, there is a 22.6% chance that the average bulb life for 15 randomly selected bulbs would be less than or equal to 290 days

9. Many casinos use card-dealing machines to deal cards at random. Occasionally, the machine is tested to ensure an equal likelihood of dealing for each suit. To conduct the test, 1,500 cards are

CONTINUED

dealt from the machine, while the number of cards in each suit is counted. Theoretically, 375 cards should be dealt from each suit. As you can see from the results in our table, this is not the case:

	Spades	Diamonds	Clubs	Hearts
Observed	402	358	273	467
Expected	375	375	375	375

Use **chi square** to determine if the discrepancies are significant so that measures would need to be taken to ensure that the game is fair. (3 marks)

Solution:

$$x^2 = \frac{(402-375)^2}{375} + \frac{(358-375)^2}{375} + \frac{(273-375)^2}{375} + \frac{(467-375)^2}{375}$$
$$= 1.944 + 0.7707 + 27.744 + 22.5707$$
$$= 53.0294$$

10. After choosing the parameters of the Beta distribution as $\alpha = 3.8$ and $\beta = 91.2$ so as to represent the priors about the probability of producing a defective item, the plant manager now wants to update her priors by observing new data. She decides to inspect a production lot of 100 items, and she finds that 3 of the items in the lot are defective. How should she change the parameters of the Beta distribution in order to take this new information into account? (2 marks)

Solution:

Under the hypothesis that the items are produced independently of each other, the result of the inspection is a binomial random variable with parameters $n = 100$ and $p = X$. But updating a Beta distribution based on the outcome of a binomial random variable gives as a result another Beta distribution. Moreover, the two parameters α_1 and β_1 of the updated Beta distribution are

$$\alpha_1 = \alpha + 3 = 3.8 + 3 = 6.8$$
$$\beta_1 = \beta + (100 - 3) = 91.2 + 97 = 188.2$$