Vector Spaces

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For the concept of a ***vector space***, we start immediately with the most important spaces. They are denoted by $\mathbf{R}^1, \mathbf{R}^2, \mathbf{R}^3, \ldots$; the space $\mathbf{R}^n$ consists of *all column vectors with n components*. (We write $\mathbf{R}$ because the components are real numbers.) $\mathbf{R}^2$ is represented by the usual x-y plane; the two components of the vector become the x and y coordinates of the corresponding point. The three components of a vector in $\mathbf{R}^3$ give a point in three-dimensional space. The one-dimensional space $\mathbf{R}^1$ is a line.

A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers. Addition and multiplication must produce vectors in the space, and they must satisfy the eight conditions.

Vector Space

A real vector space V is a *set* of elements together with two operations, addition and scalar multiplication, satisfying the following properties:

Let *x, y*, and *z* be vectors in V, and let c_1 and c_2 be scalars.

1. $x+y=y+x$.
2. $x+(y+z)=(x+y)+z$.
3. There is a unique "zero vector" such that $x+0=x$ for all x.
4. For each x there is a unique vector $-x$ such that $x+(-x)=0$.
5. $1x=x$.
6. $(c_1c_2)x=c_1(c_2x)$.
7. $c(x+y)=cx+cy$.
8. $(c_1+c_2)x=c_1x+c_2x$.

Geometrically, think of the usual three-dimensional R^3 and choose any plane through the origin. That plane is a vector space in its own right.

If we multiply a vector in the plane by 3, or -3, or any other scalar, we get a vector in the same plane.

If we add two vectors in the plane, their sum stays in the plane. This plane through (0;0;0) illustrates one of the most fundamental ideas in linear algebra; it is a subspace of the original space R^3.

Definition. A *subspace* of a vector space is a nonempty subset that satisfies the requirements for a vector space: ***Linear combinations stay in the subspace***.

(i) If we add any vectors x and y in the subspace, $x+y$ is *in the subspace*.

(ii) If we multiply any vector x in the subspace by any scalar c, cx is *in the subspace*.

A sub*space* is a subset that is "closed" under addition and scalar multiplication. Those operations follow the rules of the host space, keeping us *inside the subspace*.

The eight required properties are satisfied in the larger space and will automatically be satisfied in every subspace. Notice in particular that *the zero vector will belong to every subspace*.

The Column Space of A

We now come to the key examples, the **column space** and the **nullspace** of a matrix A. *The column space contains all linear combinations of the columns of* A. It is a subspace of $\mathbf{R}^m$. We illustrate by a system of $m = 3$ equations in $n = 2$ unknowns:

$$\textbf{Combination of columns equals } b \qquad \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad (1)$$

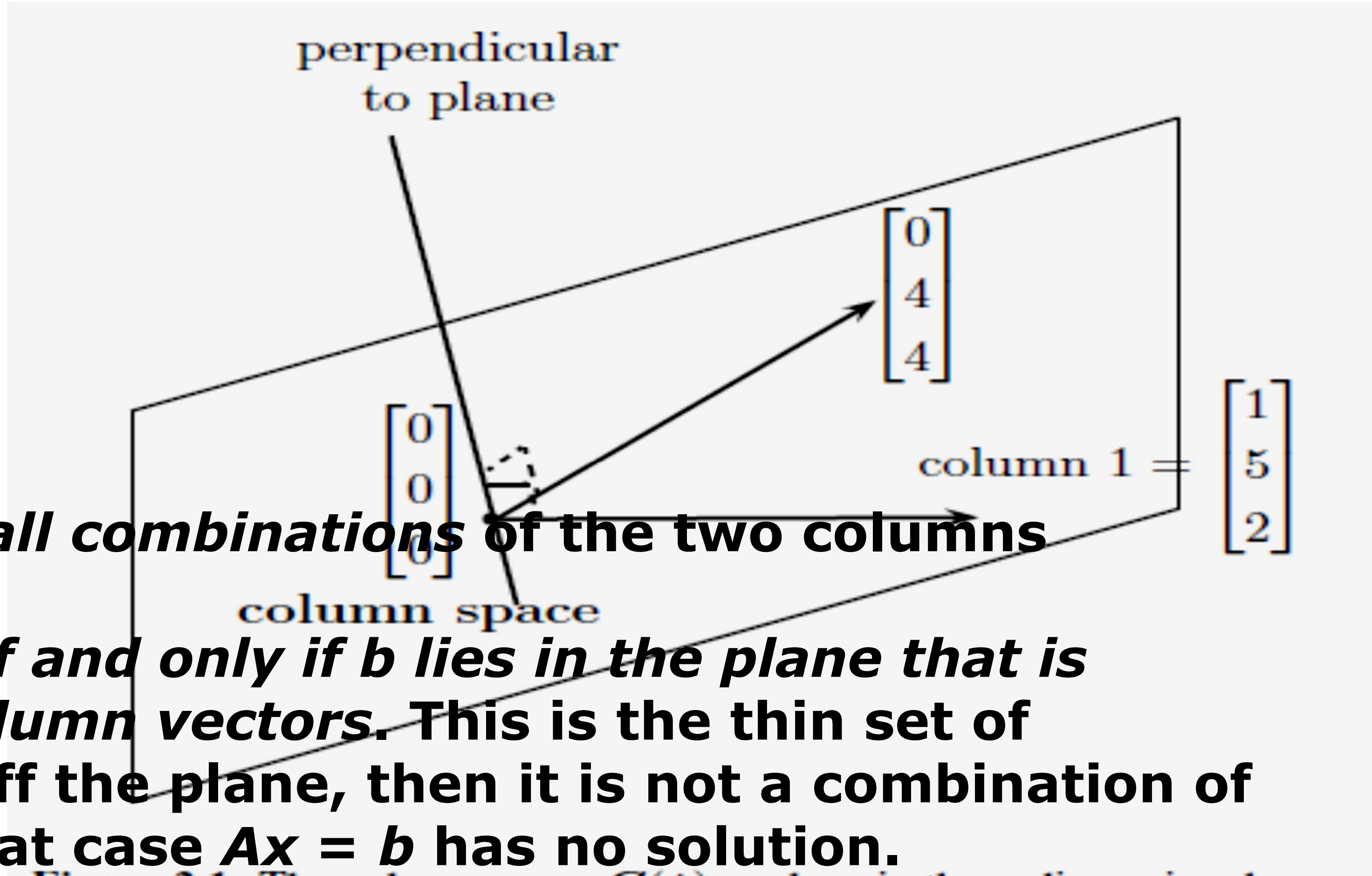

Figure 2.1: The column space $C(A)$, a plane in three-dimensional space.

We can describe *all combinations* of the two columns geometrically:
***Ax = b can be solved if and only if b lies in the plane that is spanned by the two column vectors*. This is the thin set of attainable *b*. If *b* lies off the plane, then it is not a combination of the two columns. In that case *Ax = b* has no solution.**

What is important is that this plane is <u>not just a subset</u> of R^3 , it is a subspace. It is the *column space* of *A*, consisting of *all combinations of the columns*.

Any 5 by 5 matrix A, that is ***nonsingular*** *will have the whole of* R^5 *as its column space*.

For such a matrix we can solve *Ax* = *b* by Gaussian elimination; there are five pivots (to be discussed later).

Therefore every *b* is in *C*(*A*) for a non=singular matrix.

For every n by n matrix, column space is R^n.

Allowing singular matrices, and rectangular matrices of any shape, C(A) can be somewhere between the zero space and the whole space R^m.

The Nullspace of A

If there are more unknowns than equations, $n > m$.)

***The solutions to Ax* = 0 *form a vector space—the nullspace of A*.**

Defn. -

The *nullspace* of a matrix consists of all vectors *x* such that *Ax* = 0. It is denoted by *N*(*A*). It is a subspace of R^n, just as the column space was a subspace of R^m.

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The nullspace contains only the vector (0;0).

This matrix has "independent columns.

Now, B **has the same column space as *A*.**

Larger nullspace $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$

The 3^{RD} column lies in the plane of ??
it is the sum of 1^{ST} two column vectors.

But the nullspace of *B* contains the vector (1, 1, -1) and automatically contains any multiple (*c, c, -c*):

Nullspace is a line $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c & c & -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The nullspace of B is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $C(A)$ and $N(A)$: all attainable right-hand sides b and all solutions to $Ax = 0$.

The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Will read about RANK-space later