# Solving $Ax = 0$ and $Ax = b$

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**ACK : Linear Algebra and Its Applications - Gilbert Strang**

# Introduction

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$ and it was $x = -A^{-1}b$. That solution was found by elimination (not by computing $A^{-1}$ ). A rectangular matrix brings new possibilities - $U$ may not have a full set of pivots. This section goes onward from $U$ to a reduced form $R$ $-$ the simplest matrix that elimination can give. $R$ reveals all solutions immediately.

For an invertible matrix, the null space contains only $x = 0$ (multiply $Ax = 0$ by $A^{-1}$ ). The column space is the whole space ($Ax = b$ has a solution for every $b$). The new questions appear when the null space contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector $x_n$ in the null space can be added to a particular solution $x_p$. The solutions to all linear equations have this form, $x = x_p + x_n$:

**Complete solution** $\quad Ax_p = b$ **and** $Ax_n = 0$ **produce** $A(x_p + x_n) = b$**.**

2. When the column space doesn't contain every $b$ in $\boldsymbol{R}^m$, we need the conditions on $b$ that make $Ax = b$ solvable.

Approach:

First write down all solutions to $Ax = 0$.

Then find the conditions for b to lie in the column space (so that $Ax = b$ is solvable).

The $1\ by\ 1$ system $0x = b$, one equation and one unknown, shows two possibilities:

- $0x = b$ has no solution unless $b = 0$. The column space of the $1\ by\ 1$ zero matrix contains only $b = 0$.
- $0x = 0$ has infinitely many solutions. The null space contains all $x$. A particular solution is $x_p = 0$, and the complete solution is:

$$\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n} = \boldsymbol{0} + (\boldsymbol{any\ x}).$$

The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible:

$y + z = b_1$ and $2y + 2z = b_2$ usually have no solution.
There is no solution unless $b_2 = 2b_1$.

The column space of $A$ contains only those $b$'s, the multiples of $(1,2)$.

When $b_2 = 2b_1$ there are infinitely many solutions. A particular solution to $y + z = 2$ and $2y + 2z = 4$ is $x_p = (1, 1)$.

The nullspace of *A* contains - ??

(-1, 1) and all its multiples $x_n = (-c, c)$.

The null space of $A$ in figure below contains $(-1, 1)$ and all its multiples $x_n = (-c, c)$:

**Complete solution** $\begin{aligned} y + z &= 2 \\ 2y + 2z &= 4 \end{aligned}$ is solved by $x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-c \\ 1+c \end{bmatrix}$.

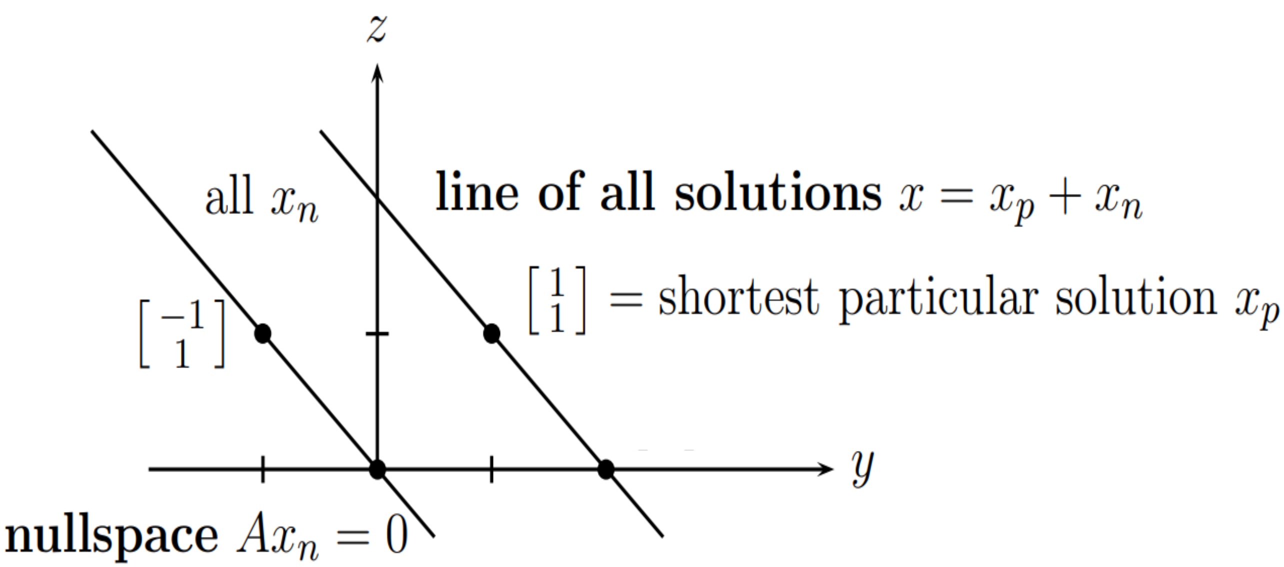



**Fig 2.2:** The parallel lines of solutions to $Ax_n = 0$ and $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

# Echelon Form $U$ and Row Reduced Form $R$

We start by simplifying this $3$ $by$ $4$ matrix, first to $U$ and then further to $R$:

**Basic example** $$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}.$$

The pivot $a_{11} = 1$ is nonzero. The usual elementary operations

The candidate for the second pivot has become zero: *unacceptable*. We look below that zero for a nonzero entry—intending to carry out a row exchange. In this case the *entry below it is also zero*. If $A$ were square, this would signal that the matrix was singular. With a rectangular matrix, we must expect trouble anyway, and there is no reason to stop.

All we can do is to *go on to the next column*, where the pivot entry is 3. Subtracting twice the second row from the third, we arrive at $U$:

**Echelon matrix** $U$ $$U = \begin{bmatrix} \mathbf{1} & 3 & 3 & 2 \\ 0 & 0 & \mathbf{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

# Echelon Form $U$ and Row Reduced Form $R$ (contd.)

Strictly speaking, we proceed to the fourth column. A zero is in the third pivot position, and nothing can be done. $U$ is upper triangular, but its pivots are not on the main diagonal. The nonzero entries of $U$ have a "staircase pattern," or **echelon form**. For the $5$ *by* $8$ case in Figure 2.3, the starred entries may or may not be zero.

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Figure 2.3:** The entries of a 5 by 8 echelon matrix $U$ and its reduced form $R$.

We can always reach this echelon form $U$, with zeros below the pivots:

1. The pivots are the first nonzero entries in their rows.
2. Below each pivot is a column of zeros, obtained by elimination.
3. Each pivot lies to the right of the pivot in the row above. This produces the staircase pattern, and zero rows come last.

# Echelon Form $U$ and Row Reduced Form $R$ (contd.)

Since we started with $A$ and ended with $U$, the reader is certain to ask: Do we have $A = LU$ as before? There is no reason why not, since the elimination steps have not changed. Each step still subtracts a multiple of one row from a row beneath it. The inverse of each step adds back the multiple that was subtracted. These inverses come in the right order to put the multipliers directly into $L$:

$$\textbf{Lower triangular} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = LU.$$

Note that $L$ is square. It has the same number of rows as $A$ and $U$. The only operation not required by our example, but needed in general, is row exchange by a permutation matrix $P$. Since we keep going to the next column when no pivots are available, there is no need to assume that $A$ is nonsingular. Here is $PA = LU$ for all matrices:

**2B** For any $m\ by\ n$ matrix $A$ there is a permutation $P$, a lower triangular $L$ with unit diagonal, and an $m\ by\ n$ echelon matrix $U$, such that $PA = LU$.

# Echelon Form $U$ and Row Reduced Form $R$ (contd.)

***Now comes*** $R$. We can go further than $U$, to make the matrix even simpler. Divide the second row by its pivot 3, so that ***all pivots are*** 1. Then use the pivot row to produce ***zero above the pivot***. This time we subtract a row from a higher row. The final result (the best form we can get) is the ***reduced row echelon form*** $R$:

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = R.$$

This matrix $R$ is the final result of elimination on $A$.

What is the row reduced form of a square invertible matrix? In that case $R$ is the identity matrix. There is a full set of pivots, all equal to 1, with zeros above and below.

So $rref(A) = I$, when $A$ is invertible.

For a 5 $by$ 8 matrix with four pivots, Figure 2.3 shows the reduced form $R$. **It still contains an identity matrix, in the four pivot rows and four pivot columns.** From $R$ we will quickly find the null space of $A$. $Rx = 0$ has the same solutions as $Ux = 0$ and $Ax = 0$.

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & \mathbf{0} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & * & * & * & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Figure 2.3:** The entries of a 5 by 8 echelon matrix $U$ and its reduced form $R$.

# Pivot Variables and Free Variables

Our goal is to read off all the solutions to $Rx = 0$. The pivots are crucial:

**Nullspace of $R$ (pivot columns in boldface)**

$$Rx = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ \mathbf{0} & 0 & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ v \\ \boldsymbol{w} \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The unknowns $u, v, w, y$ go into two groups. One group contains the **pivot variables**, those that correspond to **columns with pivots**. The first and third columns contain the pivots, so $u$ and $w$ are the pivot variables. The other group is made up of the **free variables**, corresponding to **columns without pivots**. These are the second and fourth columns, so $v$ and $y$ are free variables.

To find the most general solution to $Rx = 0$ (or, equivalently, to $Ax = 0$) we may assign arbitrary values to the free variables. Suppose we call these values simply $v$ and $y$. The pivot variables are completely determined in terms of $v$ and $y$:

$$Rx = 0 \qquad \begin{aligned} \boldsymbol{u} + 3v - y &= 0 \quad \text{yields} \quad \boldsymbol{u} = -3v + y \\ \boldsymbol{w} + y &= 0 \quad \text{yields} \quad \boldsymbol{w} = \phantom{-3v} - y \end{aligned} \qquad (1)$$

There is a "*double infinity*" of solutions, with v and y free and independent. The complete solution is a combination of two **special solutions**:

**Nullspace contains all combinations of special solutions**

$$x = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = v \begin{bmatrix} \mathbf{-3} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + y \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{-1} \\ \mathbf{1} \end{bmatrix}. \qquad (2)$$

Relook at the complete solution to $Rx = 0$ and $Ax = 0$. The special solution $(-3, 1, 0, 0)$ has free variables $v = 1, y = 0$. The other special solution $(1, 0, -1, 1)$ has $v = 0$ and $y = 1$.

*All solutions are linear combinations of these two*. The best way to find all solutions to $Ax = 0$ is from the special solutions:

1. After reaching $Rx = 0$, identify the pivot variables and free variables.
2. Give one free variable the value 1, set the other free variables to 0, and solve $Rx = 0$ for the pivot variables. This $x$ is a special solution.
3. Every free variable produces its own "special solution" by step 2. The combinations of special solutions form the null space—all solutions to $Ax = 0$.

Within the four-dimensional space of all possible vectors x, the solutions to $Ax = 0$ form a two-dimensional subspace—the null space of $A$. In the example, $N(A)$ is generated by the special vectors $(-3, 1, 0, 0)$ and $(1, 0, -1, 1)$. The combinations of these two vectors produce the whole null space.

The special solutions are especially easy from $R$. The numbers [3 and 0] and [−1 and 1] lie in the "*non-pivot columns*" of $R$. **Reverse their signs to find the pivot variables (**not free**) in the special solutions.** Two special solutions from equation (2) are put into a null space matrix $N$**:**

$$\begin{bmatrix} \mathbf{1} & \boxed{\mathbf{3}} & \mathbf{0} & \boxed{\mathbf{-1}} \\ \mathbf{0} & \boxed{\mathbf{0}} & \mathbf{1} & \boxed{\mathbf{1}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = R.$$

**Nullspace matrix (columns are special solutions)**

$$N = \begin{bmatrix} \mathbf{-3} & \mathbf{1} \\ 1 & 0 \\ \mathbf{0} & \mathbf{-1} \\ 0 & 1 \end{bmatrix} \begin{matrix} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \end{matrix}$$

The free variables have values 1 and 0. When the free columns moved to the right-hand side of equation (2), their coefficients 3 and 0 and −1 and 1 switched sign. That determined the pivot variables in the special solutions (the columns of $N$).

This is the place to recognize one extremely important theorem. Suppose a matrix has more columns than rows, $n > m$. Since m rows can hold at most $m$ pivots, **there must be at least** $n - m$ **free variables**. There will be even more free variables if some rows of $R$ reduce to zero; but no matter what, at least one variable must be free. This free variable can be assigned any value, leading to the following conclusion:

| |
|---|
| **2C** If $Ax = 0$ has more unknowns than equations ($n > m$), it has at least one special solution: There are more solutions than the trivial $x = 0$. |

There must be infinitely many solutions, since any multiple $cx$ will also satisfy $A(cx) = 0$. The null space contains the line through $x$. And if there are additional free variables, the null space becomes more than just a line in $n$-dimensional space. *The null space has the same "dimension" as the number of free variables and special solutions.*

This central idea—the **dimension** of a subspace—is made precise in the next section. **We count the free variables for the null space. We count the pivot variables for the column space!**

# Solving $\boldsymbol{Ax = b}$, $\boldsymbol{Ux = c}$, and $\boldsymbol{Rx = d}$

The case $b \neq 0$ is quite different from $b = 0$. The row operations on $A$ must act also on the right-hand side (on $b$). We begin with letters $(b1, b2, b3)$ to find the solvability condition—for $b$ to lie in the column space. Then we choose:

$b = (1, 5, 5) \quad < - why\,??\,;$ and find all solutions $x$.

For the original example $Ax = b = (b1, b2, b3)$, apply to both sides the operations that led from $A$ to $U$. The result is an upper triangular system $Ux = c$:

**Basic example**

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$Ux = c \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \qquad (3)$$

The vector c on the right-hand side, which appeared after the forward elimination steps, is just $L^{-1}b$ as in the previous chapter. Start now with $Ux = c$.

$$Ux = c \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}. \tag{3}$$

It is not clear that these equations have a solution. The third equation is very much in doubt, because its left-hand side is zero.

**The equations are inconsistent unless** $b_3 - 2b_2 + 5b_1 = 0$. Even though there are more unknowns than equations, there may be no solution.

We know another way of answering the same question: $Ax = b$ can be solved if and only if $b$ lies in the column space of $A$. This subspace comes from the four columns of $A$ (not of $U$!):

**Columns of** $A$ **"span" the column space** $\qquad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}.$

**Columns of $A$ "span" the column space** $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}3\\6\\-3\end{bmatrix}, \begin{bmatrix}3\\9\\3\end{bmatrix}, \begin{bmatrix}2\\7\\4\end{bmatrix}.$

Even though there are four vectors, their combinations only fill out a plane in three dimensional space. Column 2 is three times column 1. The fourth column equals the third minus the first.

*These dependent columns, the second and fourth, are exactly the ones without pivots.*

The column space $C(A)$ can be described in two different ways. On the one hand, it is *the plane generated by columns 1 and 3*. The other columns lie in that plane, and contribute nothing new. Equivalently, it is the plane of all vectors b that satisfy $b_3 - 2b_2 + 5b_1 = 0$; this is the constraint if the system is to be solvable.

**Every column satisfies this constraint, so it is forced on b!** Geometrically, we shall see that the vector $(5, -2, 1)$ is perpendicular to each column.

If $b$ belongs to the column space, the solutions of $Ax = b$ are easy to find. The last equation in $Ux = c$ is $0 = 0$. To the free variables $v$ and $y$, we may assign any values, as before. The pivot variables $u$ and $w$ are still determined by back-substitution. For a specific example with $b_3 - 2b_2 + 5b_1 = 0$, choose $b = (1, 5, 5)$:

$$Ax = b \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

$$Ax = b \qquad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

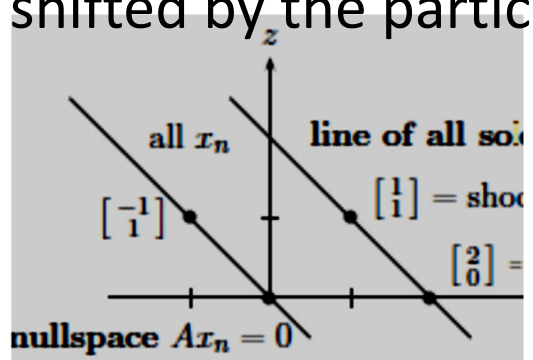
Forward elimination produces $U$ on the left and $c$ on the right:

This has all solutions to $Ax = 0$, plus the new $x_p = (-2,0,1,0)$. That $x_p$ is a **particular solution** to $Ax = b$. The last two terms with $v$ and $y$ yield more solutions (because they satisfy $Ax = 0$). **Every solution to $Ax = b$ is the sum of one particular solution and a solution to** $Ax = 0$:

$$x_{complete} = x_{particular} + x_{nullspace}$$

The particular solution in equation (4) comes from solving the equation *with all free variables set to zero*. That is the only new part, since the null space is already computed. When you multiply the above equation by A, you get $Ax_{complete} = b + 0$.

Geometrically, the solutions again fill a two-dimensional surface—but it is not a subspace. It does not contain $x = 0$. It is parallel to the null space we had before, shifted by the particular solution $x_p$ as in Figure 2.2



Equation (4) is a good way to write the answer:

**Complete solution**

$x = x_p + x_n$

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}. \quad (4)$$

1. Reduce $Ax = b$ to $Ux = c$.
2. With free variables = 0, find a particular solution to $Ax_p = b$ and $Ux_p = c$.
3. Find the special solutions to $Ax = 0$ (or $Ux = 0$ or $Rx = 0$). Each free variable, in turn, is 1. Then $x = x_p +$ (any combination $x_n$ of special solutions).

When the equation was $Ax = 0$, the particular solution was the zero vector! It fits the pattern, but $x_{particular} = 0$ was not written in equation (2). Now $x_p$ is added to the null space solutions, as in equation (4).

**Question:** How does the reduced form $R$ make this solution even clearer? You will see it in our example. Subtract equation 2 from equation 1, and then divide equation 2 by its pivot. On the left-hand side, this produces $R$, as before. On the right-hand side, these operations change $c = (1,3,0)$ to a new vector $d = (-2,1,0)$:

$$\textbf{Reduced equation} \quad Rx = d \qquad \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \tag{5}$$

Our particular solution $x_p$, (one choice out of many) has free variables $v = y = 0$. Columns 2 and 4 can be ignored. Then we immediately have $u = -2$ and $w = 1$, exactly as in equation (4). **The entries of $d$ go directly into $x_p$.** This is because the identity matrix is sitting in the pivot columns of $R$!

Thus, elimination reveals the pivot variables and free variables. **If there are $r$ pivots, there are $r$ pivot variables and $n - r$ free variables.** That important number r will be given a name—it is the **rank of the matrix**.

**2D** Suppose elimination reduces $Ax = b$ to $Ux = c$ and $Rx = d$, with $r$ pivot rows and $r$ pivot columns. **The rank of those matrices is** $r$. The last $m - r$ rows of $U$ and $R$ are zero, so there is a solution only if the last $m$ −r entries of $c$ and $d$ are also zero.

The complete solution is $x = x_p + x_n$. One particular solution $x_p$ has all free variables zero. Its pivot variables are the first $r$ entries of $d$, so $Rx_p = d$.

The null space solutions $x\_n$ are combinations of $n - r$ special solutions, with one free variable equal to 1. The pivot variables in that special solution can be found in the corresponding column of $R$ (with sign reversed).

Thus, rank $r$ is crucial. It counts the pivot rows in the "*row space*" and the pivot columns in the column space. There are $n - r$ special solutions in the null space. There are $m - r$ solvability conditions on $b$ or $c$ or $d$.

# Another Worked Example

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The 3 by 4 matrix $A$ has rank 2:

$$Ax = b \quad \text{is} \quad \begin{array}{rcrcrcrcl} \mathbf{1}x_1 & + & \mathbf{2}x_2 & + & \mathbf{3}x_3 & + & \mathbf{5}x_4 & = & b_1 \\ \mathbf{2}x_1 & + & \mathbf{4}x_2 & + & \mathbf{8}x_3 & + & \mathbf{12}x_4 & = & b_2 \\ \mathbf{3}x_1 & + & \mathbf{6}x_2 & + & \mathbf{7}x_3 & + & \mathbf{13}x_4 & = & b_3 \end{array} \tag{6}$$

1. Reduce $[A \;\; b]$ to $[U \;\; c]$, to reach a triangular system $Ux = c$.
2. Find the condition on $b_1, b_2, b_3$ to have a solution.
3. Describe the column space of $A$: Which plane in $\mathbf{R}^3$?
4. Describe the nullspace of $A$: Which special solutions in $\mathbf{R}^4$?
5. Find a particular solution to $Ax = (0, 6, -6)$ and the complete $x_p + x_n$.
6. Reduce $[U \;\; c]$ to $[R \;\; d]$: Special solutions from $R$ and $x_p$ from $d$.

**Solution:** (Notice how the right-hand side is included as an extra

1. The multipliers in elimination are 2 and 3 and $-1$, taking $[A \;\; b]$ to $[U \;\; c]$.

## Solution (contd.)

5. Choose $b = (0, 6, -6)$, which has $b_3 + b_2 - 5b_1 = 0$. Elimination takes $Ax = b$ to $Ux = c = (0, 6, 0)$. Back-substitute with free variables $= 0$: