Linear Independence, Basis, and Dimension

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ACK : Linear Algebra and Its Applications - Gilbert Strang

The goal of this section is to explain and use four ideas:

1. Linear independence or dependence.
2. Spanning a subspace.
3. Basis for a subspace (a set of vectors).
4. Dimension of a subspace (a number).

Linear Independence

Given a set of vectors $v_1, \dots, v_k$, we look at their combinations

$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$. The trivial combination, with all weights $ci = 0$, obviously produces the zero vector: $0v_1 + \cdots + 0v_k = 0$. The question is whether this is the *only way* to produce zero. If so, the vectors are independent.

If any other combination of the vectors gives zero, they are *dependent.*

2E Suppose $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ only happens when $c_1 = \cdots = c_k = 0$. Then the vectors $v_1, \dots, v_k$ are **linearly independent**. If any c's are nonzero, the v's are **linearly dependent**. One vector is a combination of the others.

Linear dependence is easy to visualize in three-dimensional space, when all vectors go out from the origin. **Two vectors are dependent if they lie on the same line**. ***Three vectors are dependent if they lie in the same plane***. A random choice of three vectors, without any special accident, should produce linear independence (not in a plane). Four vectors are always linearly dependent in $\boldsymbol{R}^3$.

Example 1. If $v_1 =$ zero vector, then the set is linearly dependent. We may choose $c_1 = 3$ and all other $c_i = 0$; this is a nontrivial combination that produces zero.

Example 2. The columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent, since the second column is three times the first. The combination of columns with weights $-3, 1, 0, 0$ gives a column of zeros.

The rows are also linearly dependent; row 3 is two times row 2 minus five times row 1. (This is the same as the combination of b_1, b_2, b_3, that had to vanish on the right-hand side in order for $Ax = b$ to be consistent. Unless $b_3 - 2b_2 + 5b_1 = 0$, the third equation would not become $0 = 0$.)

Example 3. The columns of this triangular matrix are linearly *independent*:

$$\textbf{No zeros on the diagonal} \qquad A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}.$$

Look for a combination of the columns that makes zero:

$$\text{Solve } Ac = 0 \qquad c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We have to show that c_1, c_2, c_3 are all forced to be zero. The last equation gives $c_3 = 0$. Then the next equation gives $c_2 = 0$, and substituting into the first equation forces $c_1 = 0$. The only combination to produce the zero vector is the trivial combination. **The null space of A contains only the zero vector $c_1 = c_2 = c_3 = 0$.**

> ***The columns of A are independent exactly when $N(A) = \{\textbf{zero vector}\}$.***

A similar reasoning applies to the rows of A, which are also independent. Suppose

$$c_1(3,4,2) + c_2(0,1,5) + c_3(0,0,2) = (0,0,0).$$

From the first components we find $3c_1 = 0$ or $c_1 = 0$. Then the second components give $c_2 = 0$, and finally $c_3 = 0$.

The nonzero rows of any echelon matrix U must be independent. Furthermore, if we pick out *the columns that contain the pivots*, they also are linearly independent. In our earlier example, with

Two independent rows
Two independent columns

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the pivot columns 1 and 3 are independent. No set of three columns is independent, and certainly not all four. It is true that columns 1 and 4 are also independent, but if that last 1 were changed to 0 they would be dependent. *It is the columns with pivots that are guaranteed to be independent.*

The general rule is:

2F The r nonzero rows of an echelon matrix U and a reduced matrix R are linearly independent. So are the r columns that contain pivots.

Example 4. The columns of the n by n identity matrix are independent:

$$I = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

These columns $e_1, \ldots, e_n$ represent unit vectors in the coordinate directions; in $\mathbf{R}^4$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Most sets of four vectors in $\boldsymbol{R}^4$ are independent. Those e’s might be the safest.

To check any set of vectors $v_1, \dots, v_n$ for independence, put them in the columns of A. Then solve the system $Ac = 0$; the vectors are dependent if there is a solution other than $c = 0$. With no free variables (*rank* n), there is no null space except $c = 0$; the vectors are independent. If the rank is less than n, at least one free variable can be nonzero and the columns are dependent.

One case has special importance. Let the n vectors have m components, so that A is an $m\ by\ n$ matrix. Suppose now that $n > m$. There are too many columns to be independents There cannot be n pivots, since there are not enough rows to hold them. The rank will be less than n. Every system $Ac = 0$ with more unknowns than equations has solutions $c \neq 0$.

2G A set of n vectors in $\boldsymbol{R}^m$ must be linearly dependent if $n > m$.

The reader will recognize this as a disguised form of **2C**: Every $m\ by\ n$ system $Ax = 0$ has nonzero solutions if $n > m$.

Example 5. These three columns in $\mathbf{R}^2$ cannot be independent:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}.$$

To find the combination of the columns producing zero we solve $Ac = 0$:

$$A \rightarrow U =$$ [redacted] .

[redacted]

Spanning a Subspace

Now we define what it means for a set of vectors to span a space. The column space of A is spanned by the columns. Their combinations produce the whole space:

2H If a vector space $\boldsymbol{V}$ consists of all linear combinations of $w_1, \ldots, w_\ell$, then these vectors span the space. Every vector v in $\boldsymbol{V}$ is some combination of the w's:

Every v comes from w's $\quad v = c_1 w_1 + \cdots + c_\ell w_\ell \quad$ for some coefficients c_i .

It is permitted that a different combination of w's could give the same vector v. The c's need not be unique, because the spanning set might be excessively large—it could include the zero vector, or even all vectors.

Example 6. The vectors $w_1 = (1,0,0), w_2 = (0,1,0), and\ w_3 = (-2,0,0)$ span a plane (the $x - y$ plane) in $\boldsymbol{R}^3$. The first two vectors also span this plane, whereas w_1 and w_3 span only a line.

Spanning a Subspace (contd.)

Example 7. The column space of A is exactly **the space that is spanned by its columns**. The row space is spanned by the rows. The definition is made to order. Multiplying A by any x gives a combination of the columns; it is a vector Ax in the column space.

The coordinate vectors $e_1, \ldots, e_n$ coming from the identity matrix span $\boldsymbol{R}^n$. Every vector $b = (b_1, \ldots, b_n)$ is a combination of those columns. In this example the weights are the components bi themselves: $b = b_1 e_1 + \cdots + b_n e_n$. But the columns of other matrices also span $\boldsymbol{R}^n$!

Basis for a Vector Space

To decide if b is a combination of the columns, we try to solve $Ax = b$. To decide if the columns are independent, we solve $Ax = 0$. **Spanning involves the column space, and independence involves the null space.** The coordinate vectors $e_1, \dots, e_n$ span R^n and they are linearly independent. Roughly speaking, **no vectors in that set are wasted**. This leads to the crucial idea of a **basis**.

2I A basis for $\boldsymbol{V}$ is a sequence of vectors having two properties at once:

1. The vectors are linearly independent (not too many vectors).
2. They span the space $\boldsymbol{V}$ (not too few vectors).

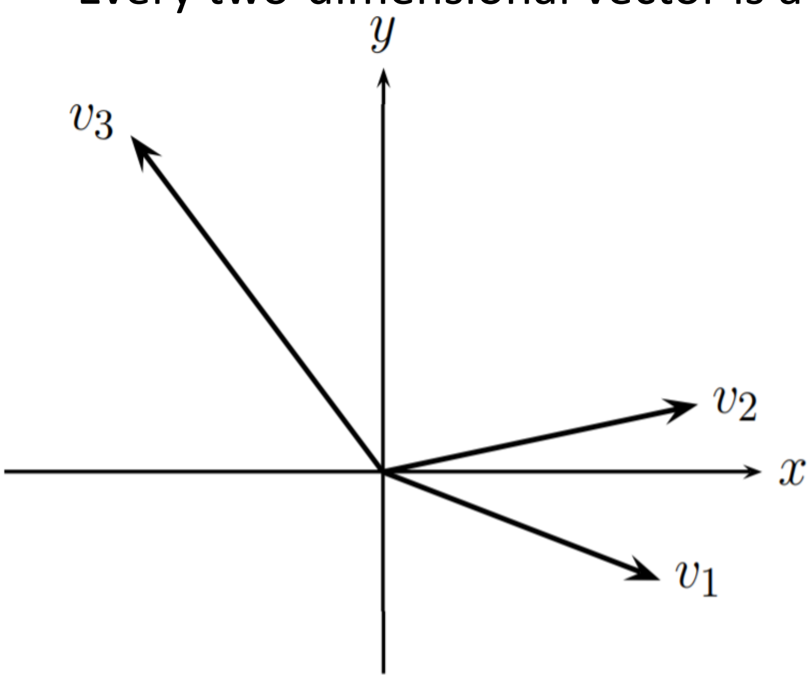
This combination of properties is absolutely fundamental to linear algebra. It means that every vector in the space is a combination of the basis vectors, because they span. It also means that the combination is unique: If $v = a_1 v_1 + \cdots + a_k v_k$ and also $v = b_1 v_1 + \cdots + b\_k\ v_k$, then subtraction gives $0 = \sum (a_i - b_i) v_i$. Now independence plays its part; every coefficient $a_i - b_i$ must be zero. Therefore $a_i = b_i$. **There is one and only one way to write v as a combination of the basis vectors.**

Basis for a Vector Space (contd.)

We had better say at once that the coordinate vectors $e_1, \dots, e_n$ are not the only basis for $\boldsymbol{R}^n$. Some things in linear algebra are unique, but not this. A vector space has **infinitely many different bases**. Whenever a square matrix is invertible, its columns are independent—and they are a basis for $\boldsymbol{R}^n$. The two columns of this non-singular matrix are a basis for $\boldsymbol{R}^2$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Every two-dimensional vector is a combination of those (independent!) columns.



Example 8. The $x - y$ plane in Figure 2.4 is just $\boldsymbol{R}^2$. The vector v_1 by itself is linearly independent, but it fails to span $\boldsymbol{R}^2$. The three vectors v_1, v_2, v_3 certainly span $\boldsymbol{R}^2$, but are not independent. *Any two* of these vectors, say v_1 and v_2, have both properties—they span, and they are independent. So they form a basis. Notice again that *a vector space does not have a unique basis*.

Figure 2.4: A spanning set v_1, v_2, v_3. Bases v_1, v_2 and v_1, v_3 and v_2, v_3.

Basis for a Vector Space (contd.)

Example 9. These four columns span the column space of U, but they are not independent:

$$\textbf{Echelon matrix} \qquad U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are many possibilities for a basis, but we propose a specific choice: **The columns that contain pivots** (in this case the first and third, which correspond to the basic variables) **are a basis for the column space.** These columns are independent, and it is easy to see that they span the space. In fact, the column space of U is just the $x - y$ plane within $\boldsymbol{R}^3$. $C(U)$ is not the same as the column space $C(A)$ before elimination—but the number of independent columns didn't change.

To summarize: *The columns of any matrix span its column space.* If they are independent, they are a basis for the column space—whether the matrix is square or rectangular. If we are asking the columns to be a basis for the whole space $\boldsymbol{R}^n$, then the matrix must be square and invertible.

Dimension of a Vector Space

A space has infinitely many different bases, but there is something common to all of these choices. The **number of basis vectors** is a property of the space itself:

2J Any two bases for a vector space $\boldsymbol{V}$ contain the same number of vectors. This number, which is shared by all bases and expresses the number of *"degrees of freedom"* of the space, is the ***dimension*** of $\boldsymbol{V}$.

We have to prove this fact: All possible bases contain the same number of vectors. The $x - y$ plane in Figure 2.4 has two vectors in every basis; its dimension is 2. In three dimensions we need three vectors, along the $x - y - z$ axes or in three other (linearly independent!) directions. **The dimension of the space $\boldsymbol{R}^n$ is** n.

The column space of U in Example 9 had dimension 2; it was a "two-dimensional subspace of $\boldsymbol{R}^3$." The zero matrix is rather exceptional, because its column space contains only the zero vector. By convention, the empty set is a basis for that space, and its dimension is zero.

Here is our first big theorem in linear algebra:

2K If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are both bases for the same vector space, then $m = n$. The number of vectors is the same.

Proof of theorem: 2K

Proof. Suppose there are more w's than v's ($n > m$). We will arrive at a contradiction. Since the v's form a basis, they must span the space. *Every w_j can be written as a combination of the v's*: If $w_1 = a_{11}v_1 + \cdots + a_{m1}v_m$, this is the first column of a matrix multiplication VA:

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} \begin{bmatrix} a_{11} & & \\ \vdots & & \\ a_{m1} & & \end{bmatrix} = VA.$$

We don't know each a_{ij}, but we know the shape of A (it is m by n). The second vector w_2 is also a combination of the v's. The coefficients in that combination fill the second column of A. The key is that A has a row for every v and a column for every w. A is a short, wide matrix, since $n > m$. There is a nonzero solution to $Ax = 0$. Then $VAx = 0$ which is $Wx = 0$. A combination of the w's gives zero! The w's could not be a basis—so we cannot have $n > m$.

If $m > n$ we exchange the v's and w's and repeat the same steps. The only way to avoid a contradiction is to have $m = n$. This completes the proof that $m = n$. To repeat: The dimension of a space is the number of vectors in every basis.

This proof was used earlier to show that every set of $m+1$ vectors in $\boldsymbol{R}$^m must be dependent. The v's and w's need not be column vectors—the proof was all about the matrix A of coefficients. In fact we can see this general result: *In a subspace of dimension k, no set of more than k vectors can be independent, and no set of more than k vectors can span the space.*

There are other "*dual*" theorems, of which we mention only one. We can start with a set of vectors that is too small or too big, and end up with a basis:

2L Any linearly independent set in $\boldsymbol{V}$ can be extended to a basis, by adding more vectors if necessary.

Any spanning set in $\boldsymbol{V}$ can be reduced to a basis, by discarding vectors if necessary.

The point is that a basis is a **maximal independent set**. It cannot be made larger without losing independence. A basis is also a **minimal spanning set**. It cannot be made smaller and still span the space.

You must notice that the word “dimensional” is used in two different ways. We speak about a four-dimensional **vector**, meaning a vector in $\boldsymbol{R}^4$. Now we have defined a four dimensional **subspace**; an example is the set of vectors in $\boldsymbol{R}^6$ whose first and last components are zero. The members of this four-dimensional subspace are six-dimensional vectors like $(0,5,1,3,4,0)$.

One final note about the language of linear algebra. We never use the terms “*basis of a matrix*” or “*rank of a space*” or “*dimension of a basis*.” These phrases have no meaning. It is *the dimension of the column space that equals the rank of the matrix,* as we prove in the coming section