BASICS OF PROBABILITY

CHAPTER 1

CS6015-LINEAR ALGEBRA AND RANDOM PROCESSES

Additive rule

- $\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

- If two events are mutually exclusive, then

$$P(A \cap B) = 0$$

Hence

$$P(A \cup B) = P(A) + P(B)$$

https://www.hccfl.edu/media/733405/mgf1106-l7-16-section%2010-3.ppt

Example

- Saskatoon and Moncton are two of the cities competing for the World university games. (There are also many others). The organizers are narrowing the competition to the **final 5 cities.**
  There is a 20% chance that Saskatoon will be amongst the **final 5**. There is a 35% chance that Moncton will be amongst the **final 5** and an 8% chance that both Saskatoon and Moncton will be amongst the **final 5**. What is the probability that Saskatoon or Moncton will be amongst the **final 5.**

https://view.officeapps.live.com/op/view.aspx?src=https://math.usask.ca/~laverty/S241/Stats%20241%20current%20lectures/03%20S241%20Probability%20Rules.ppt

Solution

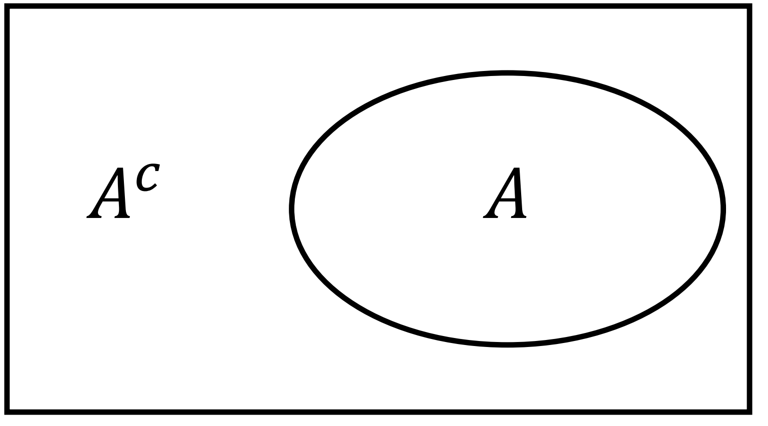
- Let *A* = the event that Saskatoon is amongst the **final 5**.
- Let *B* = the event that Moncton is amongst the **final 5**.
- Given $P(A) = 0.20$, $P(B) = 0.35$, and $P(A \cap B) = 0.08$
- **To find**: $P(A \cup B)$

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$= 0.20 + 0.35 - 0.08 = 0.47$$

Rule for complements

- $P(A^c) = 1 - P(A)$
- A and A^c are mutually exclusive.
- Sample space $S = A \cup A^c$
- Thus $P(S) = P(A) + P(A^c)$=1
- So, $\boldsymbol{P(A^c) = 1 - P(A)}$



Multiplicative rule

- $P(A \cap B) = P(A)P(B|A)$ if $P(A) \neq 0$

- $P(A \cap B) = P(B)P(A|B)$ if $P(B) \neq 0$

- If A and B are independent events,

$$P(A \cap B) = P(A)P(B)$$

Example

- An urn contains 6 red marbles and 4 black marbles. Two marbles are drawn ***without replacement*** from the urn. What is the probability that both the marbles are black?

Solution: Let A = the event that the first marble is black; and let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the urn, 4 of which are black.
- Therefore, $P(A) =$
- After the first selection, there are 9 marbles in the urn, 3 of which are black.
- Therefore, $P(B|A) =$
- Therefore, based on the multiplicative rule:

$$\boldsymbol{P(A \cap B) =}$$

Techniques for counting

Rule 1

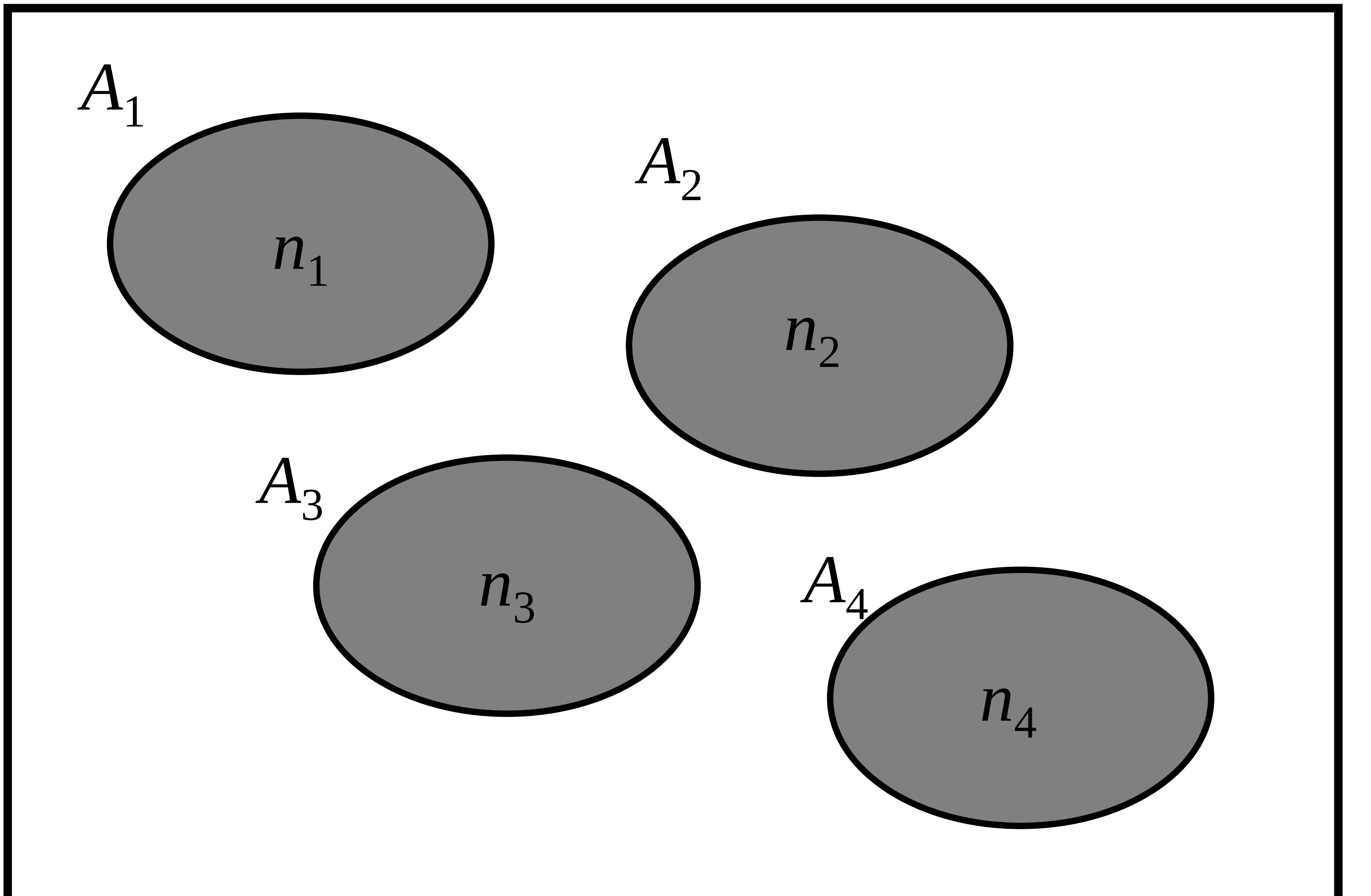
- Suppose we have sets $A_1, A_2, A_3, \ldots$ and that any pair is mutually exclusive (i.e. $A_1 \cap A_2 = \phi$ and likewise).

- Let $n_i = n(A_i)$ be the number of elements in A_i.

- Let $A = A_1 \cup A_2 \cup A_3 \cup \ \ldots$

- Then $\boldsymbol{N} = n(A) =$ the number of elements in A

$$= \boldsymbol{n_1} + \boldsymbol{n_2} + \boldsymbol{n_3} + \ \ldots$$


A_1
n_1
A_2
n_2
A_3
n_3
A_4
n_4

Rule 2

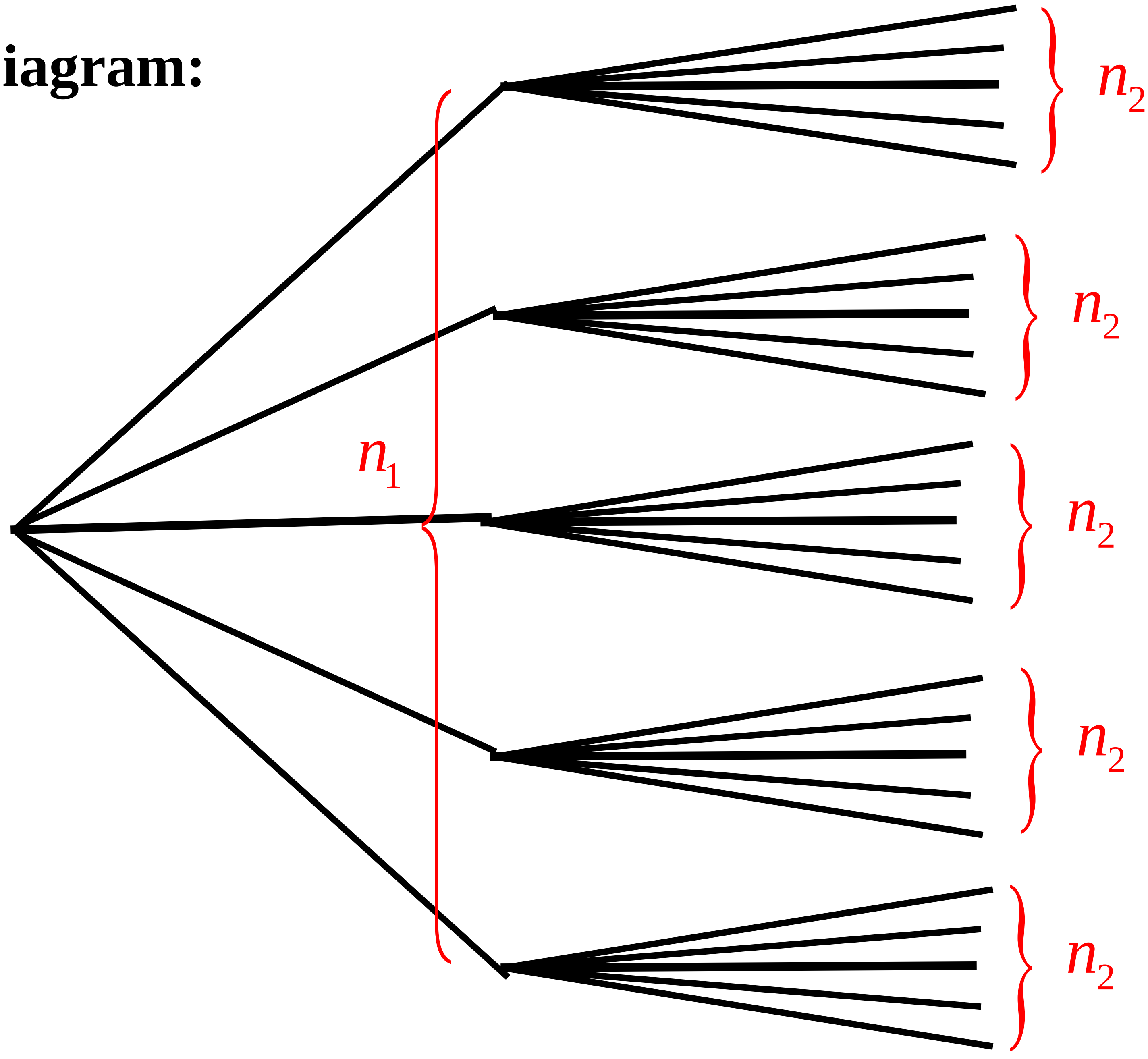
Suppose we carry out two operations in sequence.

Let

- $n_1 =$ the number of ways the first operation can be performed
- $n_2 =$ the number of ways the second operation can be performed once the first operation has been completed.

Then $\boldsymbol{N} = \boldsymbol{n_1}\,\boldsymbol{n_2}$ = the number of ways the two operations can be performed in sequence.

Diagram:



Example

1. We have a committee of 10 people. We choose from this committee, a chairman and a vice chairman. How may ways can this be done?

Solution:

Let n_1 $=$ the number of ways the chairman can be chosen = 10.

Let n_2 $=$ the number of ways the vice-chairman can be chosen once the chair has been chosen = 9.

Then $\boldsymbol{N = n_1 n_2 = (10)(9) = 90}$

Permutations

A **Permutation** is an arrangement of items in a particular order.

ORDER MATTERS!

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Permutations

The number of ways to arrange the letters ABC:

___ ___ ___

Number of choices for first blank? 3 ___ ___

Number of choices for second blank? 3 2 ___

Number of choices for third blank? 3 2 1

3*2*1 = 6   3! = 3*2*1 = 6

ABC ACB BAC BCA CAB CBA

How many ways can you order *n* objects

Ordering *n* objects is equivalent to performing *n* operations in sequence.

1. Choosing the first object in the sequence $(n_1 = n)$
2. Choosing the 2nd object in the sequence $(n_2 = n - 1)$.

   …

k. Choosing the k^{th} object in the sequence $(n_k = n - k + 1)$

   …

n. Choosing the n^{th} object in the sequence $(n_n = 1)$

The total number of ways this can be done is:

$$\mathrm{N} = n(n - 1)\ldots(n - k + 1)\ldots(3)(2)(1) = n!$$

How many ways can you choose *k* objects from *n* objects in a specific order

This operation is equivalent to performing *k* operations in sequence.

1. Choosing the first object in the sequence $(n_1 = n)$
2. Choosing the 2^{nd} object in the sequence $(n_2 = n - 1)$.

   …

k. Choosing the k^{th} object in the sequence $(n_k = n - k + 1)$

The total number of ways this can be done is:

$$N = n(n - 1) \ldots (n - k + 1) = n!/(n - k)!$$

This number is denoted by the symbol

$$_n\boldsymbol{P}_{\boldsymbol{k}} = \frac{\boldsymbol{n}!}{(\boldsymbol{n} - \boldsymbol{k})!}$$

Example: We have a committee of *n* = 10 people and we want to choose a **chairperson**, **a vice-chairperson** and a **treasurer**

Solution: Essentially we want to select 3 persons from the committee of 10 in a specific order. (Permutations of size 3 from a group of 10).

$$ {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10(9)(8) = 720 $$

Combinations

A **Combination** is an arrangement of items in which order does not matter.

ORDER DOES NOT MATTER!

Combinations

To find the number of combinations of n items chosen r at a time, use the formula

$$ {}_{n}C_{r} = \frac{n!}{(n-r)!\,r!} $$

Combinations

To play a particular card game, each player is dealt five cards from a standard deck of 52 cards. How many different hands are possible?

$$ {}_{52}C_5 = \frac{52!}{5!\,(52-5)!} = \frac{52!}{5!\,47!} $$

$$ = \frac{52(51)(50)(49)(48)}{5(4)(3)(2)(1)} = 2{,}598{,}960 $$

Combinations

A basketball team consists of two centers, five forwards, and four guards. In how many ways can the coach select a starting line up of one center, two forwards, and two guards?

Center:

$${}_2C_1 = \frac{2!}{1!1!} = 2$$

Forwards:

$${}_5C_2 = \frac{5!}{2!3!} = \frac{5*4}{2*1} = 10$$

Guards:

$${}_4C_2 = \frac{4!}{2!2!} = \frac{4*3}{2*1} = 6$$

$${}_2C_1 * {}_5C_2 * {}_4C_2$$

Thus, the number of ways to select the starting line up is 2*10*6 = **120**.