RANDOM VARIABLES AND THEIR DISTRIBUTIONS

CHAPTER-2

CS6015-LINEAR ALGEBRA AND RANDOM PROCESSES

- **<u>Random variable definition</u>** : A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that $\{w \in \Omega \,|X(w) \leq x\} \in \mathcal{F}$ for each $x \in \mathbb{R}$. Such a function is said to be $\boldsymbol{\mathcal{F}}$-measurable.
- We shall always use upper-case letters, such as X, Y, and Z, to represent generic random variables, whilst lowercase letters, such as x, y, and z, will be used to represent possible numerical values of these variables.
- Every random variable has a **distribution function.**
  - **<u>Distribution function definition</u> :** The distribution function of a random variable X is the function $F : \mathbb{R} \rightarrow [0, 1]$ given by $\boldsymbol{F(x) = P(X \leq x)}$; the Prob. that X ($w$) <= x.
- Events written as $\{w \in \Omega \,|X(w) \leq x\}$ are commonly abbreviated to

$\{w : X\,(w) \leq x\}$ or $\{X \leq x\}$.

$$\mathrm{F(x)} = P(A(x)) \qquad (2)$$

where $A(x) \subseteq \Omega$ is given by $A(x) = \{\omega \in \Omega : X(\omega) \leq x\}$.

Example

- A fair coin is tossed twice: $\Omega = \{HH, HT, TH, TT\}$. For $w \in \Omega$, let $X(w)$ be the number of heads, so that

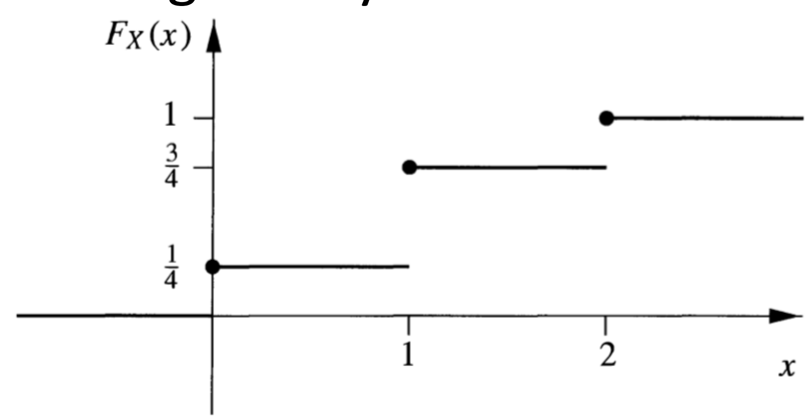
$$X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 0.$$

- Now suppose that a gambler wagers his fortune of £1 on the result of this experiment. He gambles cumulatively so that his fortune is doubled each time a head appears, and is annihilated on the appearance of a tail. His subsequent fortune W is a random variable given by :

$$W(HH) = 4, W(HT) = W(TH) = W(TT) = 0.$$

- A typical distribution function F_X of X is given by :

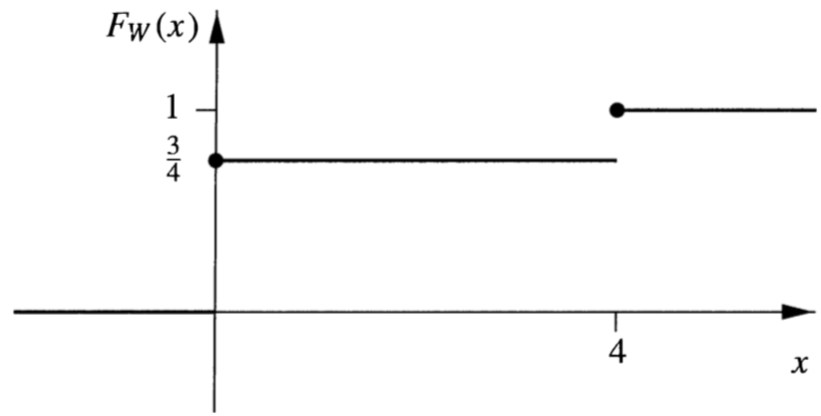
$$F_X(x) = \begin{cases} 0 \; if \; x < 0 \\ 1/4 \; if \; 0 \leq x < 1 \\ 3/4 \; if \; 1 \leq x < 2 \\ 1 \; if \; x \geq 2 \end{cases}$$



The distribution function of a random variable X tells us about the values taken by X and their relative likelihoods,
rather than about the sample space and the collection of events.

- The distribution function F_W of W is given by

$$F_W(x) = \begin{cases} 0 \ if \ x < 0 \\ 3/4 \ \ if \ 0 \leq x < 4 \\ 1 \ if \ x \geq 4 \end{cases}$$



<u>Lemma</u> :

1. A distribution function F has the f

properties :

$$\boxed{\lim_{x\to-\infty} F(x) = 0 \, , \lim_{x\to\infty} F(x) = 1}$$

<u>Proof</u> : **Part 1** : Let $B_n = \{w \in \Omega | X(w) \leq -n\} = \{X \leq -n\}$

The sequence $B_1, B_2, \ldots$ is decreasing with the empty set as limit.

i.e., $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$

$B = \bigcap_{i} B_i = \phi$

$P(B) = \lim_{n\to\infty} P(B_n)$

(From chapter 1 we know that if B_1, B_2 ...is a decreasing sequence of events, so that $B_1 \supseteq\ B_2 \supseteq \cdots$and B is written for their limit, then:

$$\mathrm{B} = \bigcap_{i=1}^{\infty} B_i = \lim_{i\to\infty} B_i$$

Then, $P(B) = \lim_{i\to\infty} P(B_i)\ \ )$

$P(B_n) = F(-n)$

So, $\boldsymbol{P(B) = 0.}$ Hence $\boldsymbol{\lim_{x\to-\infty} F(x) = 0}$

- **Part 2** :

Let $A_n = \{X \leq n\}$
The sequence $A_1, A_2, \ldots$ is increasing.
i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$

$$A = \bigcup_i A_i = \Omega$$

$$P(A) = \lim_{n\to\infty} P(A_n) = 1$$

But $P(A) = F(n) = 1$.

Hence $\boldsymbol{\lim_{x\to\infty} F(x) = 1}$

Lemma :

$\boxed{2.\ \textbf{If}\ \boldsymbol{x \leq y, F(x) \leq F(y)}}$

Proof :

Let $A(x) = \{X \leq x\}, \quad A(x, y) = \{x < X \leq y\}$

Then $A(y) = A(x) \cup A(x, y)$ is a disjoint union.

So, $P(A(y)) = P\big(A(x)\big) + P\big(A(x, y)\big)$

Giving, $\boldsymbol{F(y)} = F(x) + P(x < X \leq y) \geq \boldsymbol{F(x)}$

2.1) *F is right-continuous, that is, F(x + h) → F(x)*

Before going to the next lemma, visit:

$$\boldsymbol{P}\Big(\bigcup_{\boldsymbol{i=1}}^{\infty} \boldsymbol{A_i}\Big) = \lim_{\boldsymbol{n\to\infty}} \boldsymbol{P}\Big(\bigcup_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{A_i}\Big)$$

if $A_1, A_2, \ldots, A_n$ are disjoint events, then $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i);$

Proof :

Let $B_1 = A_1, B_2 = A_2 \backslash \mathrm{A}_1, B_3 = A_3 \backslash (A_2 A_1), \ldots$

$A \setminus B$	Difference	A, but not B

$$B_i \cap B_j = \phi$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

$$B_i \cap B_j = \phi$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

if $A_1, A_2, \ldots, A_n$ are disjoint events, then $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$;

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$
$$= \lim_{n\to\infty} P(\cup_{i=1}^{n} A_i)$$

Thus, $\boldsymbol{P(\cup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} P(\cup_{i=1}^{n} A_i)}$

- **Constant R.V** : The simplest random variable takes a constant value on the whole domain Ω. Let $c \in \mathbb{R}$ and define $X : \Omega \to \mathbb{R}$ by

$$X(w) = c \text{ for all } w \in \Omega.$$

$$F(x) = \begin{cases} 0 \; if \; x < c \\ 1 \; if \; x \geq c \end{cases}$$ the step function

More generally, we call X *constant (almost surely)* if there exists $c \in \mathbb{R}$ such that P(X = c) = 1.

- **Bernoulli R.V** : Let $X : \Omega \to \mathbb{R}$ be given by $X(H) = 1, X(T) = 0$. Then X is the simplest non-trivial random variable, having two possible values, 0 and 1. Its distribution function (*Bern(P)*) $F(x) = P(X \leq x)$ is:

$$F(x) = \begin{cases} 0 & if \; x < 0 \\ 1 - p & if \; 0 \leq x < 1 \\ 1 & if \; x \geq 1 \end{cases}$$

Indicator functions

- Let A be an event and let $I_A\colon \Omega \to \mathbb{R}$ be the *indicator function* of A; that is,

$$I_A(w) = \begin{cases} 1 \; if \; w \in A \\ 0 \; if \; w \in A^c \end{cases}$$

- Then I_A is a Bernoulli random variable taking the values 1 and 0 with probabilities $P(A)$ and $\mathrm{P}(\mathrm{A}^\mathrm{c})$ respectively.

Properties of Distribution function

Lemma :

Let F be the distribution function of X. Then,

- $P(X > x) = 1 - F(x)$
- $P(x < X \leq y) = F(y) - F(x)$
- $P(X = x) = F(x) - \lim_{y \uparrow x} F(y)$

The law of averages

- The **law of averages** is the law that a particular outcome or event is inevitable or certain, simply because it is statistically possible. This notion can lead to the gambler's fallacy when one becomes convinced that a particular outcome must come soon simply because it has not occurred recently.

- In **gambler's fallacy** the gambler believes that a particular outcome is more likely because it has not happened recently, or (conversely) that because a particular outcome has recently occurred, it will be less likely in the immediate future.

Example

- A common example of how the law of averages can mislead involves the tossing of a fair coin (a coin equally likely to come up heads or tails on any given toss).
- If someone tosses a fair coin and gets several heads in a row, that person might think that the next toss is more likely to come up tails than heads in order to "even things out."
- But the true probabilities of the two outcomes are still equal for the next coin toss and any coin toss that might follow.
- **Past results have no effect whatsoever: Each toss is an independent event.**

- The **law of large numbers** is often confused with the **law of averages**, and many texts use the two terms interchangeably. However, the law of averages, strictly defined, is not a law at all, but a **logic error** that is sometimes referred to as the **gambler's fallacy**.
- The law of averages is not a mathematical principle, whereas the law of large numbers is.
- In probability theory, **the law of large numbers** is a theorem that describes the result of performing the same experiment a large number of times.
- According to the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed.

Discrete and Continuous R.V.s (just the definitions)

- The random variable X is called **discrete** if it takes values in some countable subset $\{x_1, x_2, \dots\}$ only, of $\mathbb{R}$. The discrete random variable X has **(probability) mass function (PMF)** $f: \mathbb{R} \rightarrow [0, 1]$ given by :

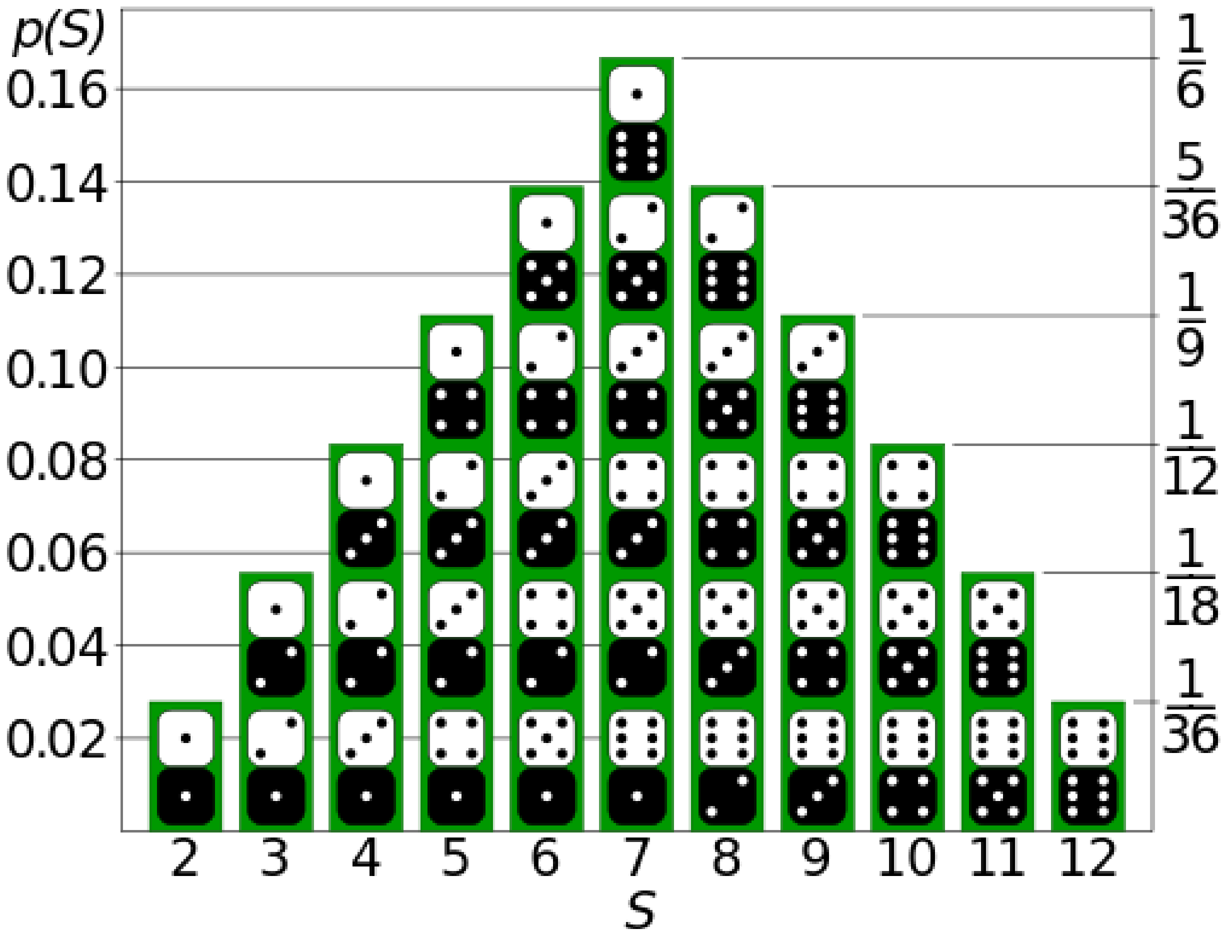
$$\boldsymbol{f(x) = P(X = x)}.$$

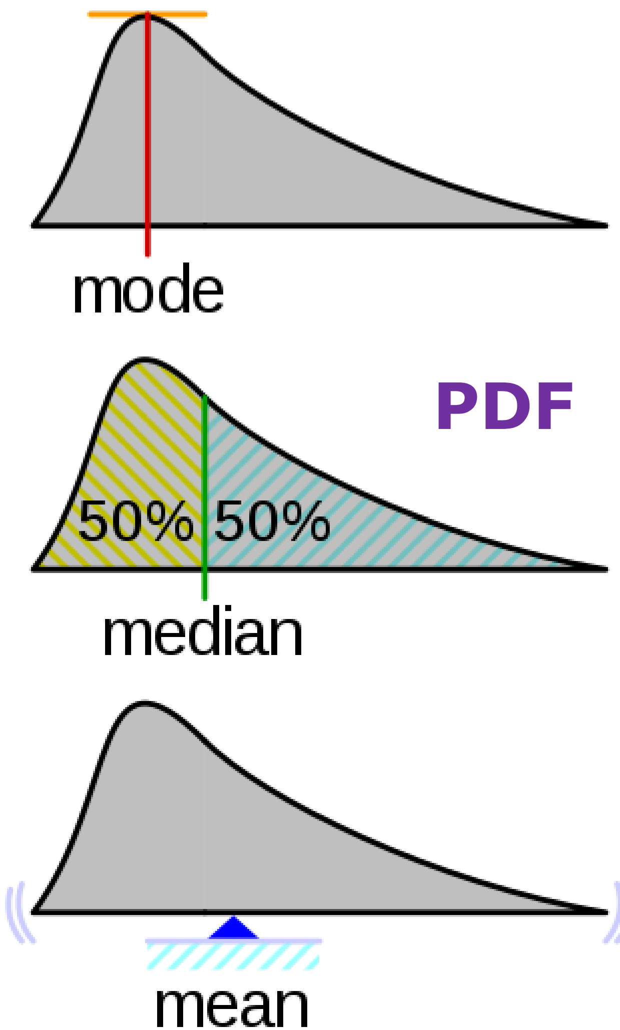
- The random variable X is called **continuous** if its distribution function (CDF) can be expressed as:

$$\boldsymbol{F(x) = \int_{-\infty}^{x} f(u)du \quad x \in \mathbb{R}} \qquad \boldsymbol{f = {}^{\delta F}/_{\delta x}}$$

for some integrable function $f: \mathbb{R} \rightarrow [0, \infty)$ called the **(probability) density function (PDF)** of X.

If the sample space is the set of possible numbers rolled on two dice, and the random variable of interest is the sum *S* of the numbers on the two dice, then *S* is a discrete random variable whose distribution is described by the **probability mass function (PMF)** plotted as the height of picture columns here. < Src: WIKI >



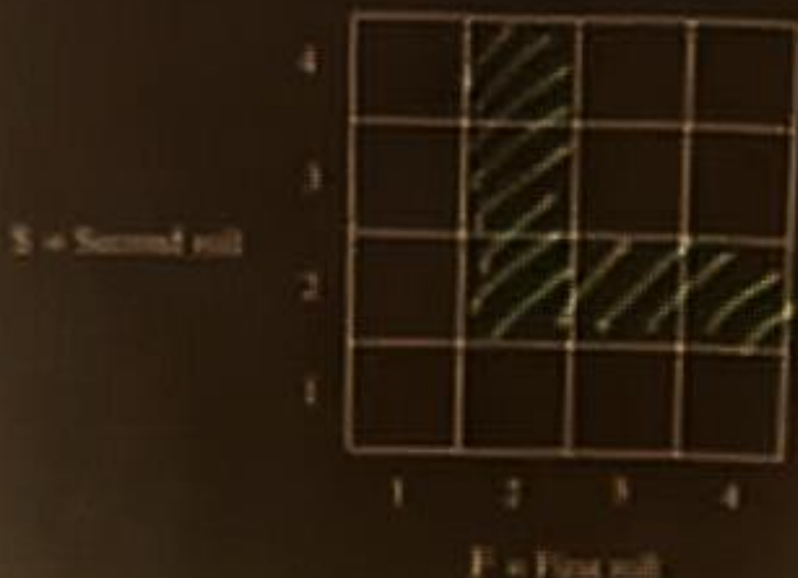


Example: Two independent rolls of a fair tetrahedral die

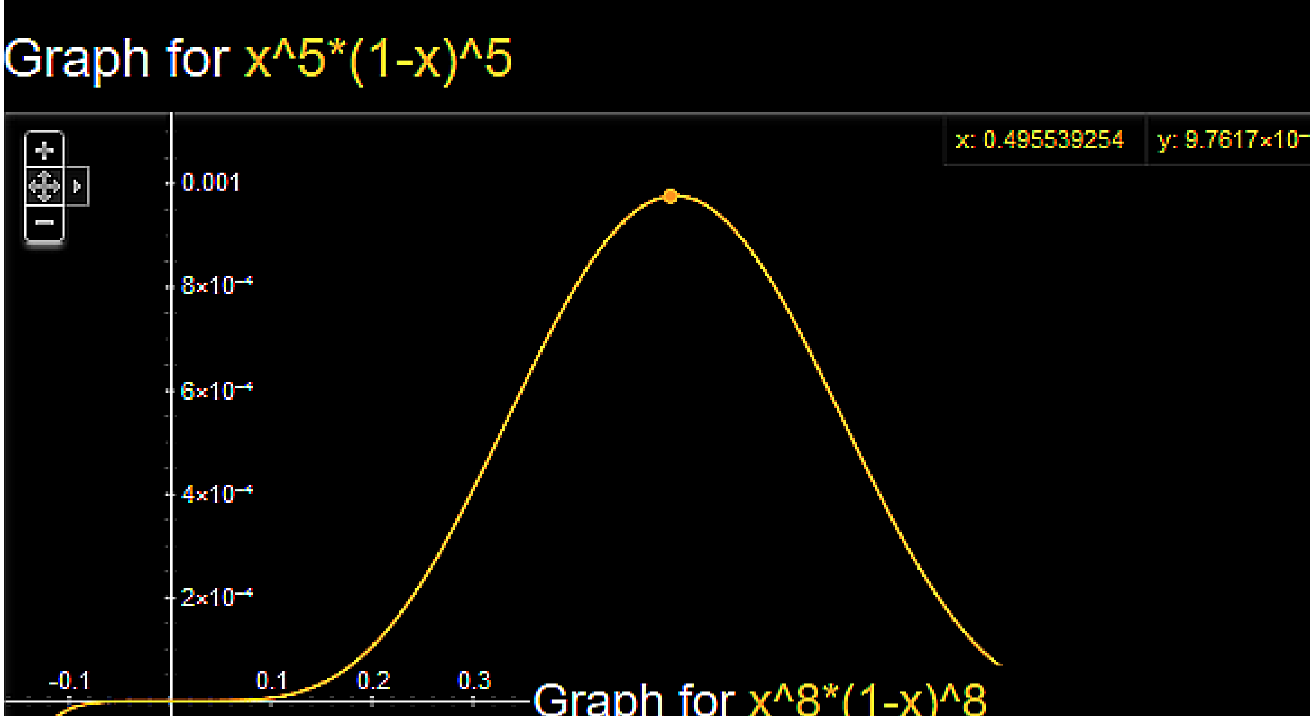
F: outcome of first throw 2

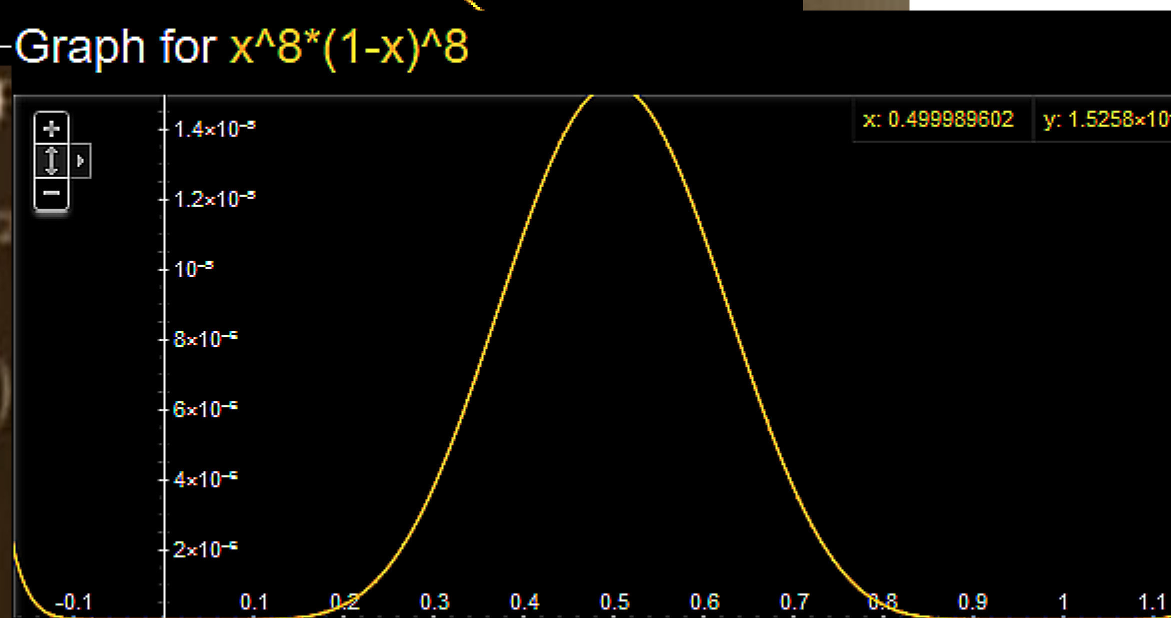
S: outcome of second throw 3

$X = \min(F, S)$ 2



$p_X(2) = 5 \cdot \frac{1}{16}$





In general:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n$$

- **Distribution function definition :** The **distribution function (CDF)** of a random variable X is the function $F : \mathbb{R} \to [0,1]$ given by $\boldsymbol{F(x) = P(X \leq x)}$; the Prob. that X ($w$) <= x.

(probability) mass function (PMF) $f : \mathbb{R} \to [0,1]$ of discrete x, is given by

$$\boldsymbol{f(x) = P(X = x)}.$$

$$\boldsymbol{f = {}^{\delta F}/_{\delta x}}$$

$$\boldsymbol{F(x) = \int_{-\infty}^{x} f(u)du} \quad \boldsymbol{x} \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \to [0,\infty)$ called the **(probability) density function (PDF)** of continuous X.

(3) Example. Discrete variables. The variables X and W of Example (2.1.1) take values in the sets $\{0, 1, 2\}$ and $\{0, 4\}$ respectively; they are both discrete. ●

Continuous variables.

$$X(\omega) = \omega, \quad Y(\omega) = \omega^2.$$

Notice that Y is a function of X in that $Y = X^2$. The distribution functions of X and Y are

$$F_X(x) = \begin{cases} 0 & x \le 0, \\ x/(2\pi) & 0 \le x < 2\pi, \\ 1 & x \ge 2\pi, \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y \le 0, \\ \sqrt{y}/(2\pi) & 0 \le y < 4\pi^2, \\ 1 & y \ge 4\pi^2. \end{cases}$$

To see this, let $0 \le x < 2\pi$ and $0 \le y < 4\pi^2$. Then

$$\begin{aligned} F_X(x) &= \mathbb{P}\big(\{\omega \in \Omega : 0 \le X(\omega) \le x\}\big) \\ &= \mathbb{P}\big(\{\omega \in \Omega : 0 \le \omega \le x\}\big) = x/(2\pi), \\ F_Y(y) &= \mathbb{P}\big(\{\omega : Y(\omega) \le y\}\big) \\ &= \mathbb{P}\big(\{\omega : \omega^2 \le y\}\big) = \mathbb{P}\big(\{\omega : 0 \le \omega \le \sqrt{y}\}\big) = \mathbb{P}(X \le \sqrt{y}) \\ &= \sqrt{y}/(2\pi). \end{aligned}$$

The random variables X and Y are continuous because

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du, \qquad F_Y(y) = \int_{-\infty}^{y} f_Y(u)\,du,$$

where

$$f_X(u) = \begin{cases} 1/(2\pi) & \text{if } 0 \le u \le 2\pi, \\ 0 & \text{otherwise,} \end{cases}$$
$$f_Y(u) = \begin{cases} u^{-\frac{1}{2}}/(4\pi) & \text{if } 0 \le u \le 4\pi^2, \\ 0 & \text{otherwise.} \end{cases}$$

Random Vectors

- Suppose that X and Y are random variables on the probability space (Ω, F, P). Their distribution functions, F_X and F_Y, contain information about their associated probabilities.
- But how may we encapsulate information about their properties *relative to each other?*
- The key is to think of X and Y as being the components of a **'random vector'** (X, Y) taking values in $\mathbb{R}^2$, rather than being unrelated random variables each taking values in $\mathbb{R}$.

Example: Coin Tossing

- Suppose that we toss a coin n times, and set X_i equal to 0 or 1 depending on whether the i_{th} toss results in a tail or a head.

- We think of the vector $\boldsymbol{X} = (X_1, X_2, \dots., X_n)$ as describing the result of this composite experiment. The total number of heads is the sum of the entries in $\boldsymbol{X}$.

Joint Distribution Function

- An individual random variable X has a distribution function F_X defined by $F_X(x) = P(X \leq x)\ for\ x \in \mathbb{R}$.
- The corresponding **'joint' distribution function** of a random vector $(X_1, X_2, \dots., X_n)$ is the quantity $P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n)$, a function of n real variables $x_1, x_2, \dots, x_n$.
- In order to aid the notation, we introduce an ordering of vectors of real numbers: for vectors

  $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ we write $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for each $i = 1,2, \dots, n$.

Definition and Properties of Joint Distribution Function

- The joint distribution function of a random vector $\boldsymbol{X} = (X_1, X_2, \ldots.., X_n)$ on the probability space (Ω, F, P) is the function $F_{\boldsymbol{X}} : \mathbb{R}^{\mathrm{n}} \rightarrow [0,1]$ given by $F_{\boldsymbol{X}}(\boldsymbol{x}) = P(\boldsymbol{X} \leq \boldsymbol{x})\ for\ \boldsymbol{x} \in \mathbb{R}^{\mathrm{n}}$.

<u>Lemma</u> :

- Joint distribution function $F_{X,Y}$ of random vector (X, Y) have properties similar to those of ordinary distribution functions which are as follows:

1. $\lim_{x,y \to -\infty} F_{X,Y}(x, y) = 0\ and\ \lim_{x,y \to \infty} F_{X,Y}(x, y) = 1$
2. $If\ (x_1, y_1) \leq (x_2, y_2)\ then\ F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$
3. $F_{X,Y}$ is continuous from above, in that

   $$F_{X,Y}(x + u, y + v) \rightarrow F_{X,Y}(x, y) \text{ as } u, v \downarrow 0.$$

(6) $$\lim_{y\to\infty} F_{X,Y}(x,y) = F_X(x) \; (= \mathbb{P}(X \le x))$$

and similarly

(7) $$\lim_{x\to\infty} F_{X,Y}(x,y) = F_Y(y) \; (= \mathbb{P}(Y \le y)).$$

- **Note**: The individual distribution functions of ***X*** and ***Y*** can be recaptured from a knowledge of their joint distribution function.
- The converse is false : it is not generally possible to calculate $F_{X,Y}$ from a knowledge of F_X and F_Y alone.
- The functions F_X and F_Y are called the **'marginal'** distribution functions of $F_{X,Y}$.

Example

- A schoolteacher asks each member of his or her class to flip a fair coin twice and to record the outcomes.
- The diligent pupil D does this and records a pair (X_D, Y_D) of outcomes. The lazy pupil L flips the coin only once and writes down the result twice, recording thus a pair $(X_L, Y_L)$where $X_L = \mathrm{Y_L}$.
- Clearly X_D, Y_D, X_L, Y_L are random variables with the same distribution functions. However, the pairs (X_D, Y_D) and (X_L, Y_L) have different *joint* distribution functions.
- In particular, $P(X_D = Y_D = heads) = \frac{1}{4}$ since only one of the four possible pairs of outcomes contains heads only, whereas $P(X_L = Y_L = heads) = \frac{1}{2}$.

- The random variables X and Y on the probability space (Ω, F, P) are called **(jointly) discrete** if the vector (X, Y) takes values in some countable subset of $\mathbb{R}^2$ only. The jointly discrete random variables X, Y have **joint (probability) mass function** $f : \mathbb{R}^2 \to [0,1]$ given by $f(x, y) = P(X = x, Y = y)$.

- The random variables X and Y on the probability space (Ω, F, P) are called **(jointly) continuous** if their joint distribution function can be expressed as

$$F_{X,Y}(x, y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(u, v)\,du\,dv \quad x, y \in \mathbb{R}$$

for some integrable function $f : \mathbb{R}^2 \to [0, \infty)$ called the **joint (probability) density function** of the pair (X, Y).

Monte Carlo Simulation (**MCS**)

- '**Monte Carlo simulation**' is used to describe a method for propagating uncertainties in model inputs into uncertainties in model outputs (results).
- Hence, it is a type of simulation that explicitly and quantitatively represents uncertainties.
- Monte Carlo simulation relies on the process of explicitly representing uncertainties by specifying inputs as probability distributions. If the inputs describing a system are uncertain, the prediction of future performance is necessarily uncertain.
- That is, the result of any analysis based on inputs represented by probability distributions is itself a probability distribution.

- Compared to deterministic analysis, the Monte Carlo method provides a superior simulation of risk. **It gives an idea of not only what outcome to expect but also the probability of occurrence of that outcome.**

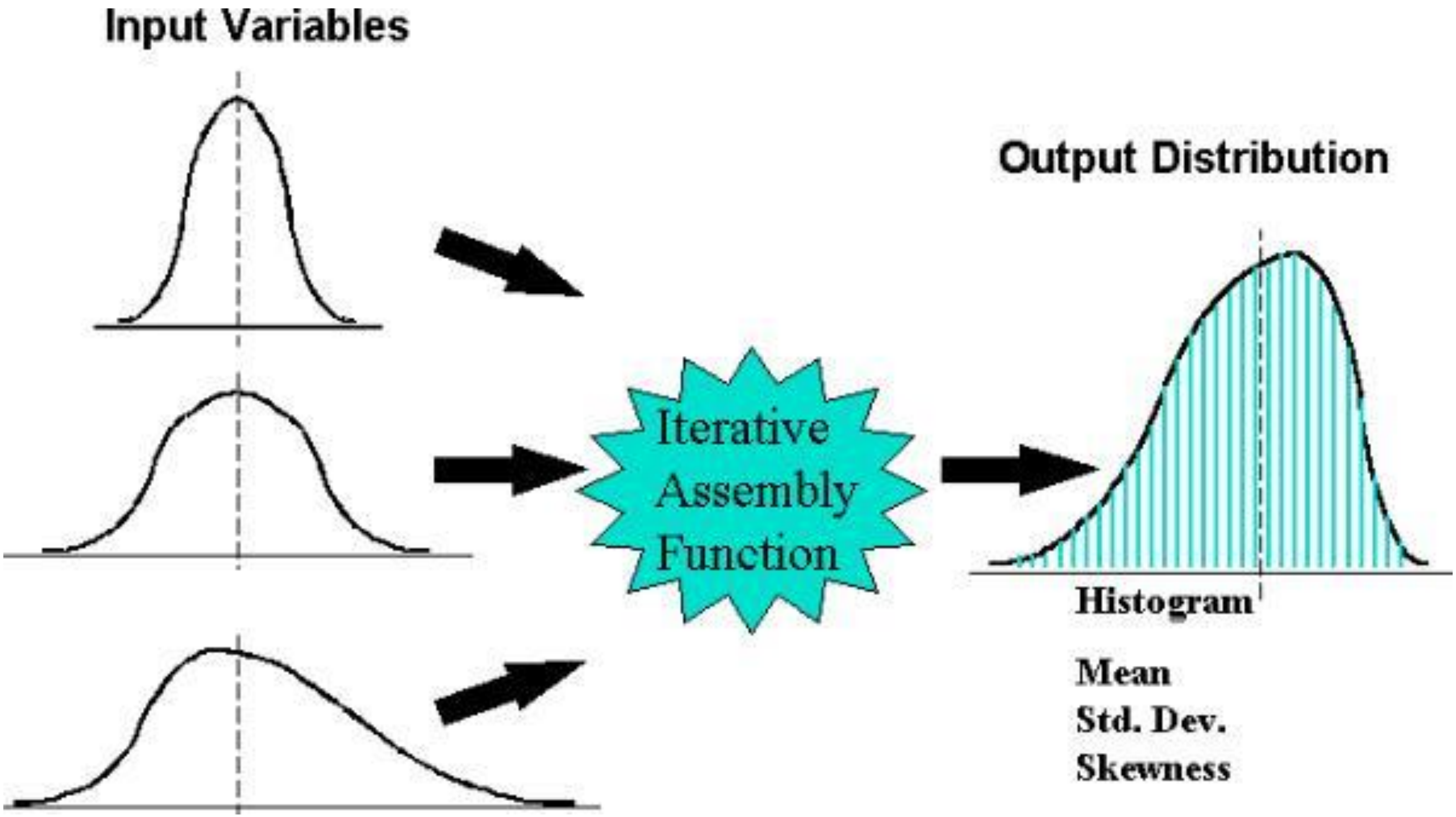
- **Different explanation** : When you develop a forecasting model – any model that plans ahead for the future – you make certain assumptions.
- Because these are projections into the future, the best you can do is estimate the expected value. Based on historical data, or expertise in the field, or past experience, you can draw an estimate. While this estimate is useful for developing a model, it contains some inherent uncertainty and risk, because it's an estimate of an unknown value.

In telecommunications, when planning a wireless network, design must be proved to work for *a wide variety of scenarios that depend mainly on the number of users, their locations and the services they want to use*. Monte Carlo methods are typically used to generate these users and their states. The network performance is then evaluated and, if results are not satisfactory, the network design goes through an optimization process.

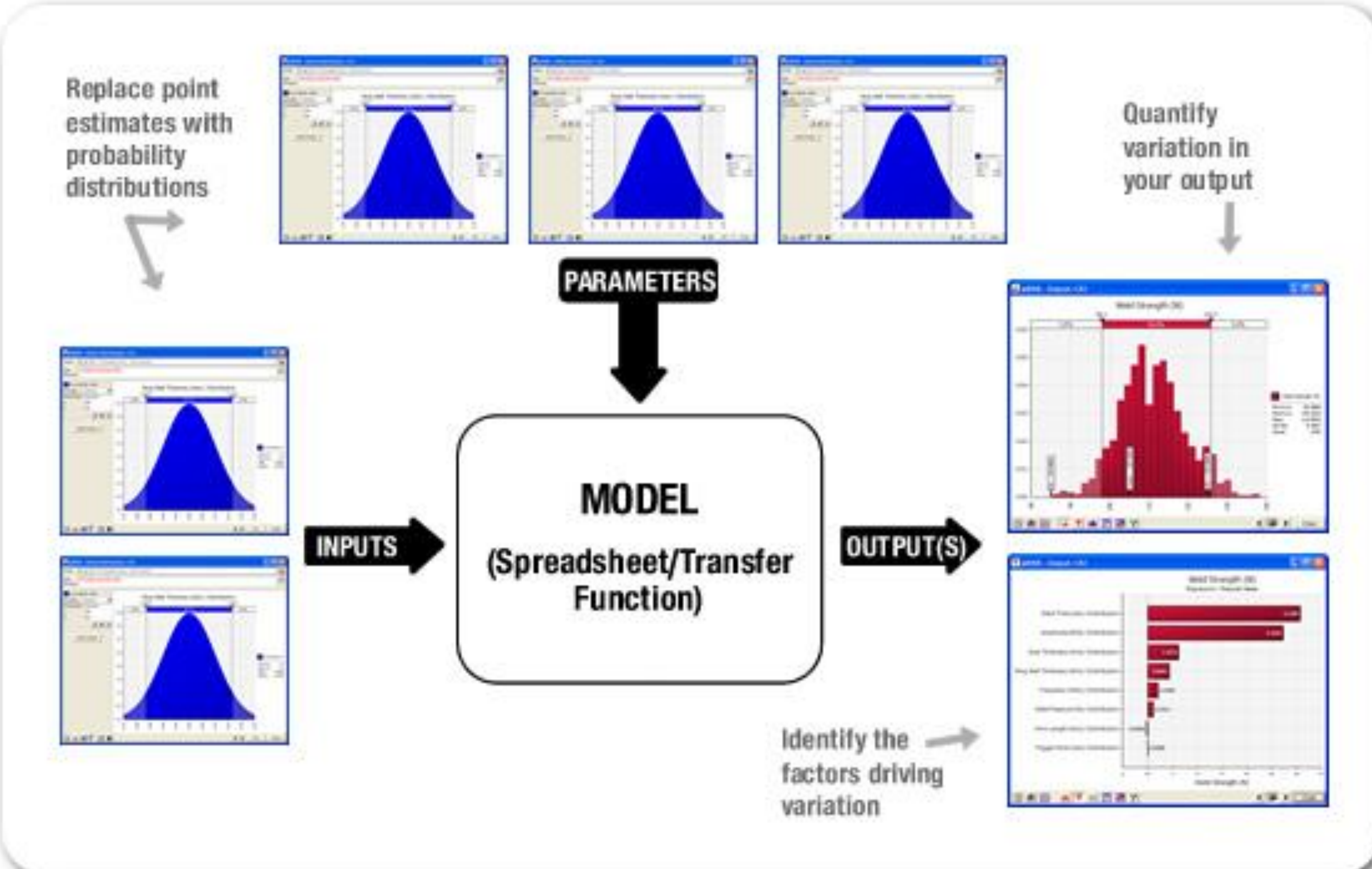
In autonomous robotics, Monte Carlo localization can determine the position of a robot. It is often applied to *stochastic filters* such as the Kalman filter or particle filter that forms the heart of the SLAM (simultaneous localization and mapping) algorithm.

Path tracing, occasionally referred to as Monte Carlo ray tracing, renders a 3D scene *by randomly tracing samples of possible light paths*. Repeated sampling of any given pixel will eventually cause the average of the samples to converge on the correct solution of the rendering equation, making it one of the most physically accurate 3D graphics rendering methods.

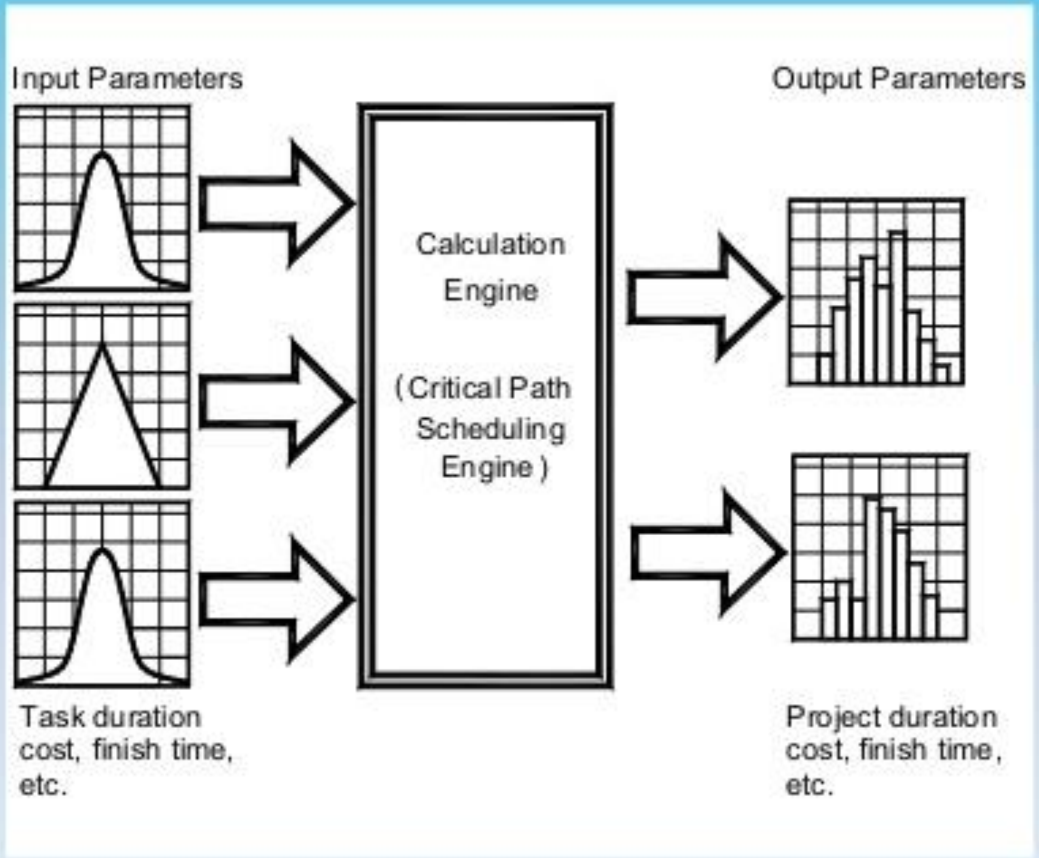
Monte Carlo methods have been developed into a technique called Monte-Carlo tree search that is useful for searching for the best move in a game. *Possible moves are organized in a search tree and a large number of random simulations are used* to estimate the long-term potential of each move. A black box simulator represents the opponent's moves.



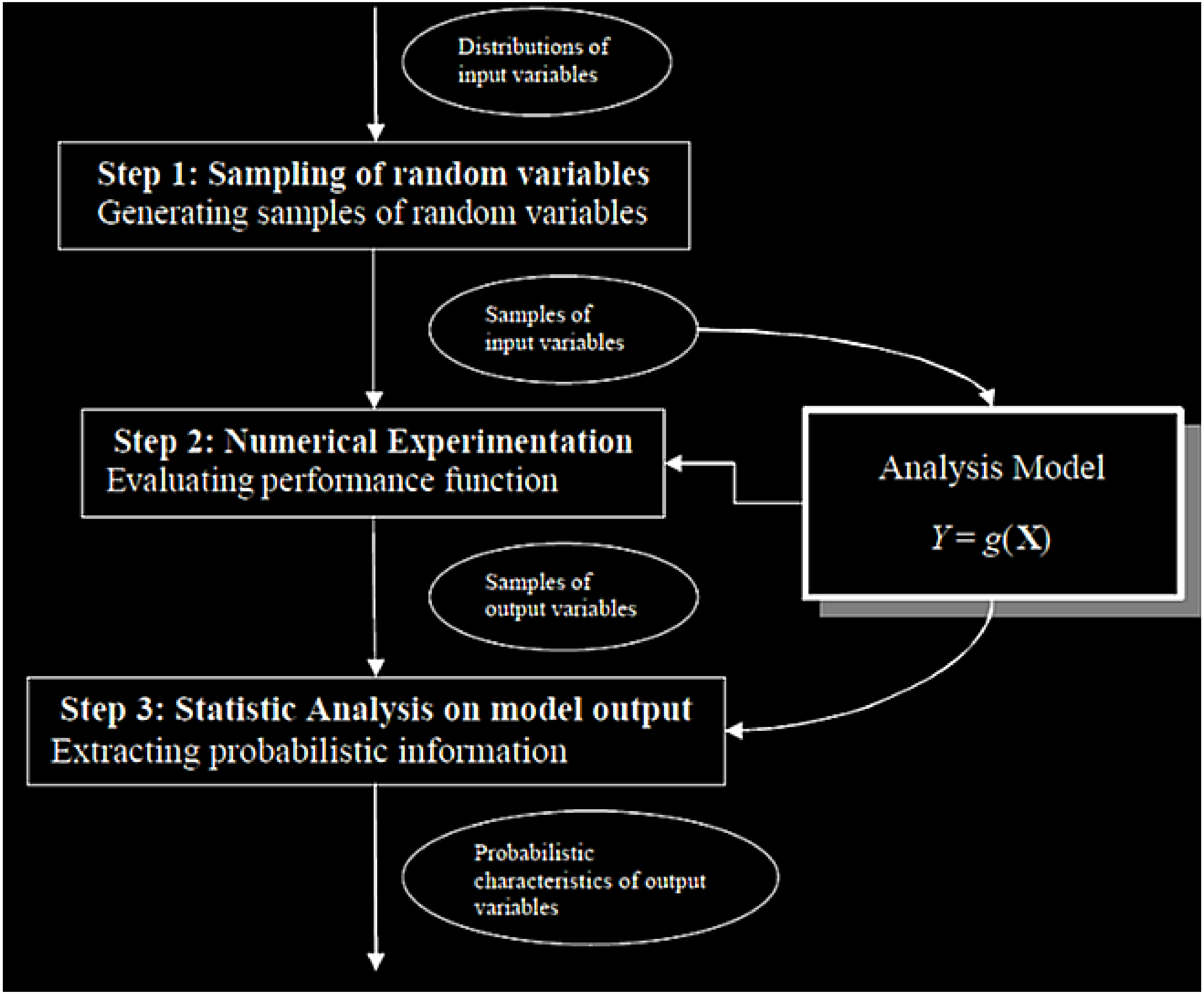
Input Variables
Iterative
Assembly
Function
Output Distribution
Histogram
Mean
Std. Dev.
Skewness


Replace point estimates with probability distributions
PARAMETERS
Quantify variation in your output
INPUTS
MODEL
(Spreadsheet/Transfer Function)
OUTPUT(S)
Identify the factors driving variation

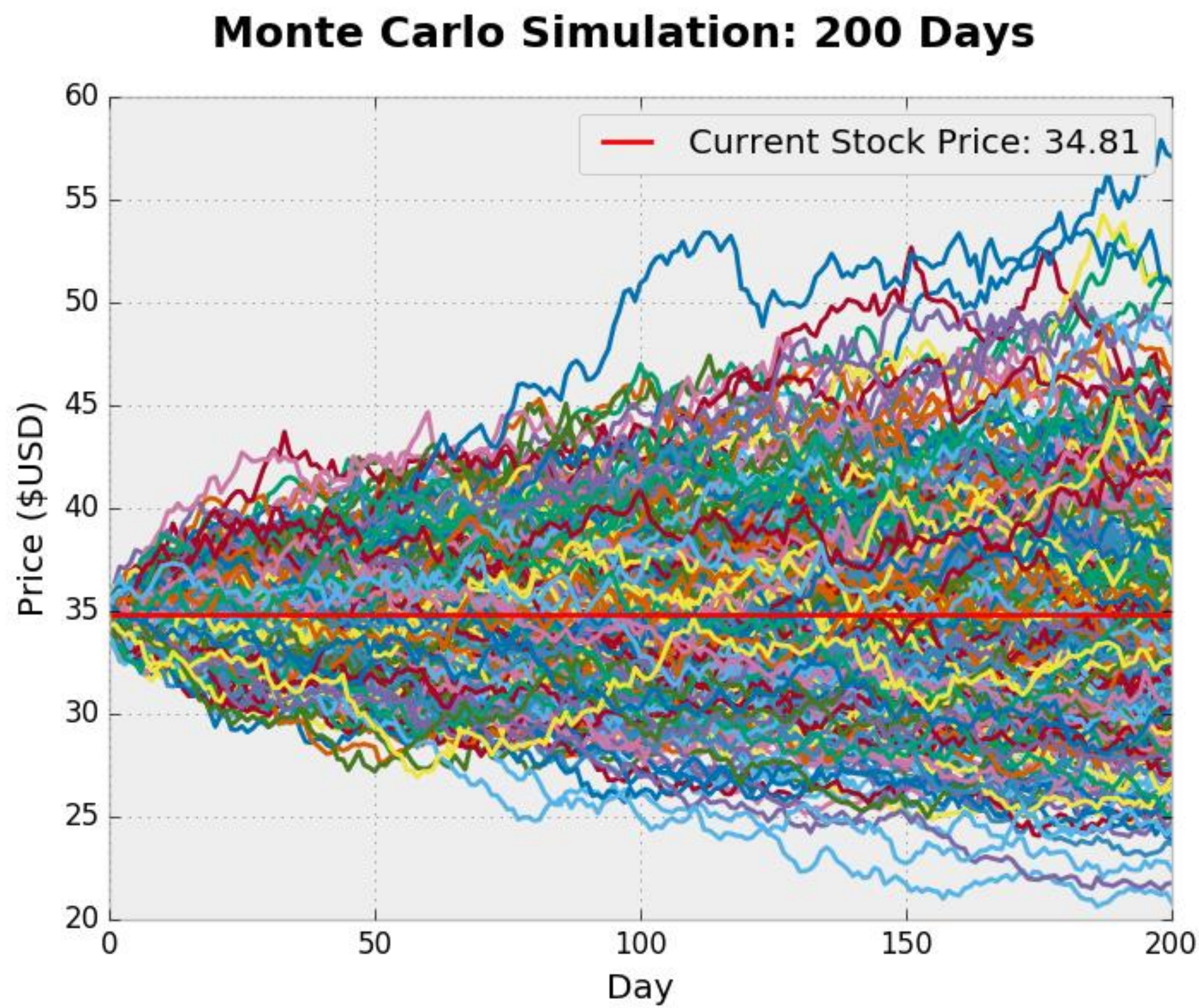
Monte Carlo Simulations



Monte Carlo simulations use distributions as inputs, which are also the results



Distributions of input variables
Step 1: Sampling of random variables
Generating samples of random variables
Samples of input variables
Step 2: Numerical Experimentation
Evaluating performance function
Analysis Model
$Y = g(\mathbf{X})$
Samples of output variables
Step 3: Statistic Analysis on model output
Extracting probabilistic information
Probabilistic characteristics of output variables


Monte Carlo Simulation: 200 Days
Current Stock Price: 34.81
Price ($USD)
Day
60
55
50
45
40
35
30
25
20
0
50
100
150
200

- In some cases, it's possible to estimate a range of values. In a construction project, you might estimate the time it will take to complete a particular job; based on some expert knowledge, you can also estimate the absolute maximum time it might take, in the worst possible case, and the absolute minimum time, in the best possible case.
- The key feature of a Monte Carlo simulation is that it can tell you – based on how you create the ranges of estimates – **how likely the resulting outcomes are**.
- **Example: A dam.** It is proposed to build a dam in order to regulate the water supply, and in particular to prevent seasonal flooding downstream. How high should the dam be?
- Dams are expensive to construct, and some compromise between cost and risk is necessary.

- It is decided to build a dam which is just high enough to ensure that the chance of a flood of some given extent within ten years is less than 10^{-2},say.
- No one knows exactly how high such a dam need be, and a young probabilist proposes the following scheme.
- Through examination of existing records of rainfall and water demand we may arrive at an acceptable model for the pattern of supply and demand.
- This model includes, for example, estimates for the distributions of rainfall on successive days over long periods.
- With the aid of a computer, the 'real world' situation is simulated many times in order to study the likely consequences of building dams of various heights.
- In this way we may arrive at an accurate estimate of the height required.

Example

- A dentist schedules all his/her patients for 30 minutes appointments.
- Some of the patients take more or less than 30 minutes depending on the type of dental work to be done.
- The following summary shows the categories of work, their probabilities and the time actually needed to complete the work:

Category	Time required	No. of patients
Filling	45 min	40
Crown	60 min	15
Cleaning	15 min	15
Extracting	45 min	10
Checkup	15 min	20

- Simulate the dentist’s clinic for 4 hours and find out the average waiting time for the patients as well as the idleness of the doctor. Assume that all the patients show up at the clinic at exactly their scheduled arrival time starting at 8:00 a.m.

- Use the following random numbers for handling the above problem:

40, 82, 11, 34, 25, 66, 17, 79

Steps:

- Find the probability distribution
- Cumulative distribution
- Setting random number intervals
- Generating random numbers
- Find the solution based on the above details

- Keep repeating above several times to get different distributions of the solution space.

Category	Time required	No. of patients
Filling	45 min	40
Crown	60 min	15
Cleaning	15 min	15
Extracting	45 min	10
Checkup	15 min	20

Category	Probability	Cumulative Probability	Random No. Interval
Filling	0.40	0.40	0-39
Crown	0.15	0.55	40-54
Cleaning	0.15	0.70	55-69
Extracting	0.10	0.80	70-79
Checkup	0.20	1.00	80-99

Patient	Scheduled arrival	Random Number	Category	Service time needed
1	8:00	40	Crown	60 min
2	8:30	82	Checkup	15 min
3	9:00	11	Filling	45 min
4	9:30	34	Filling	45 min
5	10:00	25	Filling	45 min
6	10:30	66	Cleaning	15 min
7	11:00	17	Filling	45 min
8	11:30	79	Extracting	45 min

Patient	Scheduled arrival	Service start	Service duration (in min)	Service end	Waiting (in min)	Idle time
1	8:00	8:00	60	9:00	0	0
2	8:30	9:00	15	9:15	30	0
3	9:00	9:15	45	10:00	15	0
4	9:30	10:00	45	10:45	30	0
5	10:00	10:45	45	11:30	45	0
6	10:30	11:30	15	11:45	60	0
7	11:00	11:45	45	12:30	45	0
8	11:30	12:30	45	1:15	60	0

One-sided limit

From Wikipedia, the free encyclopedia

In calculus, a **one-sided limit** is either of the two limits of a function *f*(*x*) of a real variable *x* as *x* approaches a specified point either from below or from above. One should write either:

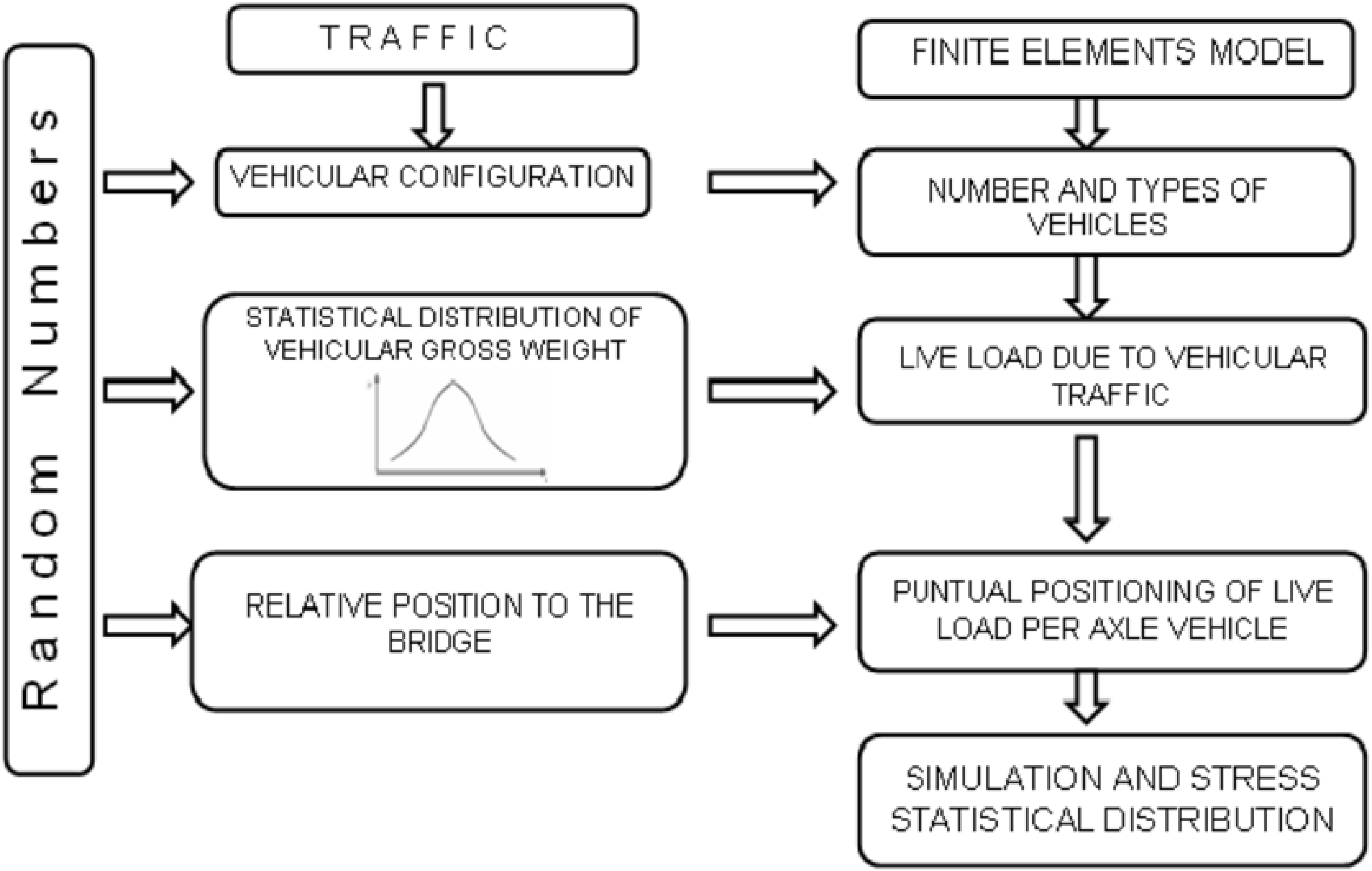
$$\lim_{x\to a^+} f(x) \text{ or } \lim_{x\downarrow a} f(x) \text{ or } \lim_{x\searrow a} f(x) \text{ or } \lim_{x \underset{>}{\longrightarrow} a} f(x)$$

for the limit as *x* decreases in value approaching *a* (*x* approaches *a* "from the right" or "from above"), and similarly

$$\lim_{x\to a^-} f(x) \text{ or } \lim_{x\uparrow a} f(x) \text{ or } \lim_{x\nearrow a} f(x) \text{ or } \lim_{x \underset{<}{\longrightarrow} a} f(x)$$

for the limit as *x* increases in value approaching *a* (*x* approaches *a* "from the left" or "from below"). In probability theory it is common to use the short notation

$f(x-)$ for the left limit and $f(x+)$ for the right limit.


Random Numbers
TRAFFIC
VEHICULAR CONFIGURATION
STATISTICAL DISTRIBUTION OF VEHICULAR GROSS WEIGHT
RELATIVE POSITION TO THE BRIDGE
FINITE ELEMENTS MODEL
NUMBER AND TYPES OF VEHICLES
LIVE LOAD DUE TO VEHICULAR TRAFFIC
PUNTUAL POSITIONING OF LIVE LOAD PER AXLE VEHICLE
SIMULATION AND STRESS STATISTICAL DISTRIBUTION