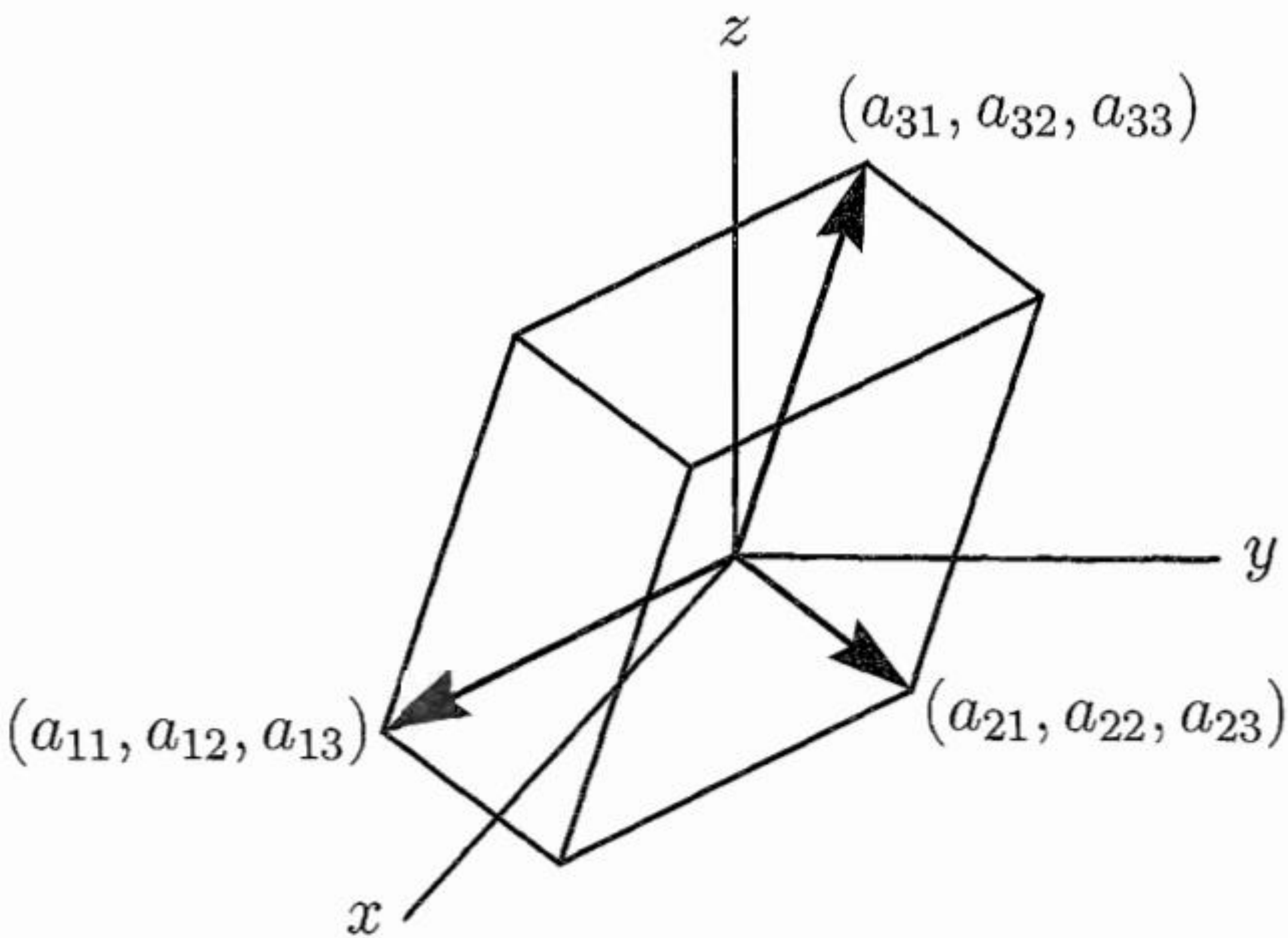
Determinants

Introduction- four uses of determinants

1. They test for invertibility. **If the determinant of $\boldsymbol{A}$ is zero, then $\boldsymbol{A}$ is singular. If $\det \boldsymbol{A} \neq \boldsymbol{0}$, then $\boldsymbol{A}$ is invertible**.
2. **The determinant of A equals the volume of a box in n-dimensional space**. The edges of the box come from the rows of A. The columns of A would give an entirely different box with the same volume.



The box formed from the rows of A:
volume = |determinant|.

3. The determinant gives a formula for each pivot. From the formula **determinant = ± (product of the pivots),** it follows that regardless of the order of elimination, the product of the pivots remains the same apart from sign.
4. The determinant measures the dependence of $A^{-1}b$ on each element of b. If one parameter is changed in an experiment, or one observation is corrected, the "influence coefficient" in A^{-1} is a ratio of determinants.

- The determinant can be (and will be) defined by its three most basic properties:
  - $\det I = 1$
  - The sign is reversed by a row exchange
  - The determinant is linear in each row separately

Properties of the Determinant

A 2 by 2 determinant is defined as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

1. *The determinant of the identity matrix is* 1.

$$\mathbf{det}\ \boldsymbol{I} = \mathbf{1} \qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

2. *The determinant changes sign when two rows are exchanged.*

$$\textbf{Row exchange} \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The determinant of every permutation matrix is $\det P = \pm 1$. By row exchanges, we can turn P into the identity matrix. Each row exchange switches the sign of the determinant, until we reach $\det I = 1$.

3. *The determinant depends linearly on the first row.* Suppose A, B, C are the same from the second row down—and row 1 of A is a linear combination of the first rows of B and C. Then the rule says: det A *is the same combination of* det B *and* det C.

Linear combinations involve two operations—adding vectors and multiplying by scalars. Therefore this rule can be split into two parts:

Add vectors in row 1 $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Multiply by t in row 1 $$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

4. *If two rows of A are equal, then* det $A = 0$.

Equal rows $$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0.$$

This follows from rule 2, since if the equal rows are exchanged, the determinant is supposed to change sign. But it also has to stay the same, because the matrix stays the same. The only number which can do that is zero, so $\det A = 0$.

5. *Subtracting a multiple of one row from another row leaves the same determinant.*

Row operation $\begin{vmatrix} a-\ell c & b-\ell d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$

Rule 3 says that there is a further term $-l\begin{vmatrix} c & d \\ c & d \end{vmatrix}$, but that term is zero by rule 4.

6. *If A has a row of zeros, then* det $A = 0$.

Zero row $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$

Add some other row to the zero row. The determinant is unchanged, by rule 5. Because the matrix will now have two identical rows, $\det A = 0$ by rule 4.

7. *If A is triangular, then* det A *is the product* $a_{11}a_{22}\cdots a_{nn}$ *of the diagonal entries. If the triangular A has* 1s *along the diagonal, then* det $A = 1$.

Triangular matrix $\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \qquad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$

Proof:

- Suppose the diagonal entries are nonzero. Then elimination can remove all the off-diagonal entries, without changing the determinant (by rule 5).
- If A is lower triangular, the steps are downward as usual.
- If A is upper triangular, the last column is cleared out first- using multiples of a_{nn}.
- Either way we reach the diagonal matrix D:

$$D = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix} \quad \text{has} \quad \det D = a_{11}a_{22}\cdots a_{nn} \det I = a_{11}a_{22}\cdots a_{nn}$$

- To find $\det D$ apply rule 3. Factoring out a_{11} and then a_{22} and finally a_{nn} leaves the identity matrix.
- At last use rule 1: $\mathbf{det}\,\boldsymbol{I} = \mathbf{1}$

- If a diagonal entry is zero then elimination will produce a zero row.
- By rule 5 these elimination steps do not change the determinant.
- By rule 6 the zero row means a zero determinant.
- This means: When a triangular matrix is singular (because of a zero on the main diagonal) its determinant is zero.
- **All singular matrices have a zero determinant.**

8. *If A is singular, then* $\det A = 0$. *If A is invertible, then* $\det A \neq 0$.

Singular matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if and only if $ad - bc = 0$.

- If A is singular, elimination leads to a zero row in U. Then $\det A = \det U = 0$.
- If A is nonsingular, elimination puts the pivots $d_1, \dots, d_n$ on the main diagonal.
- We have a "product of pivots" formula for $\det A$.

- The sign depends on whether the number of row exchanges is even or odd:

Product of Pivots: $\mathbf{det}\,\boldsymbol{A} = \pm\,\mathbf{det}\,\boldsymbol{U} = \pm\boldsymbol{d_1 d_2 \ldots d_n}$

9. *The determinant of AB is the product of* det A *times* det B.

$$\textbf{Product rule } |A||B| = |AB| \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{vmatrix}.$$

- A particular case of this rule gives the determinant of A^{-1}. It must be 1 / $\det A$:
- $\det A^{-1} = \dfrac{1}{\det A}$ because $(\det A)(\det A^{-1}) = \det AA^{-1} = \det I = 1$.
- In the 2 by 2 case, the product rule is:

$$(ad - bc)(eh - fg) = (ae + bg)(cf + dh) - (af + bh)(ce + dg)$$

- In the n by n case there are two possible proofs. Both proofs assume that A and B are nonsingular.
- Otherwise if AB is singular, the equation $\det AB = (\det A)(\det B)$ is easily verified. By rule 8, it becomes $0 = 0$.

Proof 1:

- We prove that the ratio $d(A) \;=\; \det AB / \det B$ has properties 1-3.
- Then $d(A)$ must equal $\det A$. For example, $d(I) = \det B / \det B = 1$; rule 1 is satisfied.
- If two rows of A are exchanged, so are the same two rows of AB, and the sign of d changes as required by rule 2.
- A linear combination in the first row of A gives the same linear combination in the first row of AB. Then rule 3 for the determinant of AB, divided by the fixed quantity $\det B$, leads to rule 3 for the ratio $d(A)$.
- Thus $d(A) = \det AB / \det B$ coincides with $\det A$, which is our product formula.

Proof 2:

- For a diagonal matrix, $\det DB = (\det D)(\det B)$ follows by factoring each d_i from its row.
- Reduce a general matrix A to D by elimination-from A to U as usual, and from U to D by upward elimination.
- The determinant does not change, except for a sign reversal when rows are exchanged.
- The same steps reduce AB to DB, with precisely the same effect on the determinant.
- But for DB it is already confirmed that rule 9 is correct.

10. *The transpose of A has the same determinant as A itself:* $\det A^{\mathrm{T}} = \det A$.

$$\textbf{Transpose rule} \qquad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = |A^{\mathrm{T}}|.$$

- Again the singular case is separate; A is singular if and only if A^T is singular, and we have $0 = 0$.
- If A is nonsingular, then it allows the factorization $PA = LDU$, and we apply rule 9 for the determinant of a product:

$$\det P \det A \;=\; \det L \det D \det U$$

- Transposing $PA = LDU$ gives $A^T P^T = U^T D^T L^T$, and again by rule 9,

$$\det A^T \det P^T = \det U^T \det D^T \det L^T$$

- L, U, L^T, and U^T are triangular with unit diagonal.
- By rule 7, their determinants all equal to 1.

- Also, any diagonal matrix is the same as its transpose: $D = D^T$.
- We only have to show that $\det P = \det P^T$.
- Certainly $\det P$ is 1 or -1, because P comes from I by row exchanges.
- Also $PP^T = I$. (The 1 in the first row of P matches the 1 in the first column of P^T, and misses the 1's in the other columns.)
- Therefore $\det P \det P^T = \det I = 1$, and P and P^T must have the same determinant: both 1 or both -1.
- Thus $\mathbf{det}\, \boldsymbol{A} = \mathbf{det}\, \boldsymbol{A}^{\boldsymbol{T}}$.
- This shows that **every rule that are applied to the rows can now be applied to the columns**:
  - The determinant changes sign when two columns are exchanged,
  - Two equal columns (or a column of zeros) produce a zero determinant,
  - The determinant depends linearly on each individual column.

  The proof is just to transpose the matrix and work with the rows.

Formulas for the determinant

4A If A is invertible, then $PA = LDU$ and $\det P = \pm 1$. The product rule gives

$$\det A = \pm \det L \det D \det U = \pm(\textbf{product of the pivots}). \qquad (1)$$

- The sign ± 1 depends on whether the number of row exchanges is even or odd. The triangular factors have $\det L = \det U = 1$ and $\det D = d_1 \dots d_n$.
- In the 2 by 2 case, the standard LDU factorization is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & 0 \\ 0 & \boldsymbol{(ad-bc)/a} \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}.$$

- The product of the pivots is $(ad - bc) = \det \mathrm{A}$. That is the determinant of the diagonal matrix D. If the first step is a row exchange, the pivots are c and $(-\det A)/c$.

- **Example 1**. The $-1, 2, -1$ second difference matrix has pivots 2/1, 3/2,….in D:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} = LDU = L \begin{bmatrix} 2 & & & & \\ & 3/2 & & & \\ & & 4/3 & & \\ & & & \cdot & \\ & & & & (n+1)/n \end{bmatrix} U.$$

- Its determinant is the product of its pivots. The numbers $2, \ldots, n$ all cancel:

$$\det A = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\ldots\left(\frac{n+1}{n}\right) = n+1$$

- But concentrating all information into the pivots makes it impossible to figure out how a change in one entry would affect the determinant.
- An explicit expression for the determinant is required in terms of the n^2 entries.

- For n = 2, $ad - bc$ is the determinant.
- For n = 3, the determinant formula is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{array}$$

- **Goal**: to derive these formulas directly from the defining properties 1–3 of $\det A$.
- If we can handle $\mathrm{n} = 2$ and $\mathrm{n} = 3$ in an organized way, we will see the pattern.
- To start, each row can be broken down into vectors in the coordinate directions:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \end{bmatrix}.$$

- Then we apply the property of linearity, first in row 1 and then in row 2:

Separate into $n^n = 2^2$ easy determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}.$$

- Every row splits into n coordinate directions, so this expansion has n^n terms.
- Most of those terms (all but $n!$) will be automatically zero.
- When two rows are in the same coordinate direction, one will be a multiple of the other, and

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0.$$

- We pay attention only when the rows point in different directions.
- The nonzero terms have to come in different columns.
- Suppose the first row has a nonzero term in column α, the second row is nonzero in column β, and finally the nth row in column v. The column numbers $\alpha, \beta, \dots, v$ are all different.
- They are a reordering, or permutation, of the numbers $1, 2, \dots, n$.
- The 3 by 3 case produces $3! = 6$ determinants:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}.$$

- All but these $n!$ determinants are zero, because a column is repeated. (There are n choices for the first column α, n − 1 remaining choices for β, and finally only one choice for the last column v).
- In other words, there are $n!$ ways to permute the numbers $1,2, \ldots, n$. The column numbers give the permutations:
- **Column numbers**

  $(\alpha, \beta, v) = (1,2,3), (2,3,1), (3,1,2), (1,3,2), (2,1,3), (3,2,1).$

- Those are the $3! = 6$ permutations of $(1,2,3)$; the first one is the identity.
- The determinant of A is now reduced to six separate and much simpler determinants.
- Factoring out the a_{ij}, there is a term for every one of the six permutations:

$$\det A = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + \boldsymbol{a_{12}a_{23}a_{31}}\begin{vmatrix} & \mathbf{1} & \\ & & \mathbf{1} \\ \mathbf{1} & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$
$$+ a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}.$$

- Every term is a product of $\mathrm{n} = 3$ entries a_{ij}, with each row and column represented once.
- If the columns come in the order $(\alpha, \ldots, v)$, that term is the product $a_{1\alpha} \cdots a_{nv}$ times the determinant of a permutation matrix P.

- The determinant of the whole matrix is the sum of these $n!$ terms, and that sum is the formula:

$$\textbf{Big Formula} \qquad \det A = \sum_{\text{all } P\text{'s}} (a_{1\alpha} a_{2\beta} \cdots a_{nv}) \det P.$$

- For an n by n matrix, this sum is taken over all $n!$ permutations $(\alpha, \ldots, v)$ of the numbers $(1, \ldots, n)$.
- The permutation gives the column numbers as we go down the matrix. The 1s appear in P at the same places where the a's appeared in A.
- It remains to find the determinant of P. Row exchanges transform it to the identity matrix, and each exchange reverses the sign of the determinant: $\mathbf{det}\boldsymbol{P} = \mathbf{+1 \ or\ -1}$ for an even or odd number of row exchanges.

$$(1,3,2) \quad \text{is odd so} \quad \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} = -1 \qquad (3,1,2) \quad \text{is even so} \quad \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} = 1$$

- (1,3,2) requires one exchange and (3,1,2) requires two exchanges to recover (1,2,3). These are two of the six ± signs.
- For $\mathrm{n} = 2$, we only have (1,2) and (2,1):

$$\det A = a_{11}a_{22} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_{12}a_{21} \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (\text{or } ad - bc).$$

- For $A = I$, every product of the a_{ij} will be zero, except for the column sequence $(1, 2, \dots, n)$.
- This term gives $\det I = 1$. The determinant should depend linearly on the first row $a_{11}, a_{12}, \dots, a_{1n}$.
- Look at all the terms $a_{1\alpha}a_{2\beta} \dots\ a_{nv}$ involving a_{11}. The first column is $\alpha = 1$. This leaves some permutation $(\beta, \dots, v)$ of the remaining columns $(2, \dots, n)$. All these terms are collected together as $a_{11}C_{11}$, where the coefficient of a_{11} is a smaller determinant— with row 1 and column 1 removed:

Cofactor of a_{11} $$C_{11} = \sum (a_{2\beta} \cdots a_{n\nu}) \det P = \det(\text{submatrix of } A).$$

- Similarly, the entry a_{12} is multiplied by C_{12}. Grouping all the terms that start with the same a_{1j} it becomes

Cofactors along row 1 $$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

- This shows that $\det A$ depends linearly on the entries $a_{11}, \dots, a_{1n}$ of the first row.

Example 2. For a 3 by 3 matrix,

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

The cofactors C_{11}, C_{12}, C_{13} are the 2 by 2 determinants in parentheses.

Expansion of det A in Cofactors

- C_{1j} depends on rows $2, \ldots, n$. Row 1 is already accounted for by a_{1j} .
- Furthermore, a_{1j} also accounts for the jth column, so its cofactor C_{1j} must depend entirely on the other columns.
- No row or column can be used twice in the same term.
- Split the determinant into the following sum:

Cofactor splitting

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

- For a determinant of order n, this splitting gives n smaller determinants (minors) of order $n - 1$.
- The submatrix M_{1j} is formed by throwing away row 1 and column j.
- Its determinant is multiplied by a_{1j}—and by a plus or minus sign. These signs alternate as in $\det M_{11}, -\det M_{12}, \det M_{13}$:

Cofactors of row 1 $\qquad C_{1j} = (-1)^{1+j} \det M_{1j}.$

- The second cofactor C_{12} is $a_{23}a_{31} - a_{21}a_{33}$, which is $\det M_{12}$ times -1.
- This same technique works on every n by n matrix.
- The splitting above confirms that C_{11} is the determinant of the lower right corner M_{11}.
- There is a similar expansion on any other row, say row i. It could be proved by exchanging row i with row 1. Remember to delete row i and column j of A for M_{ij}:

4B The determinant of A is a combination of any row i times its cofactors:

$$\textbf{det}\,A\ \textbf{by cofactors} \qquad \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor C_{1j} is the determinant of M_{ij} with the correct sign:

$$\textbf{delete row } i \textbf{ and column } j \qquad C_{ij} = (-1)^{i+j} \det M_{ij}.$$

- These formulas express $\det A$ as a combination of determinants of order $\mathrm{n} - 1$.
- There is one more consequence of $\det A = \det A^T$. We can expand in cofactors of a column of A, which is a row of A^T.

$$\mathbf{det}\,\boldsymbol{A} = \boldsymbol{a_{1j}C_{1j}} + \boldsymbol{a_{2j}C_{2j}} + \cdots + \boldsymbol{a_{nj}C_{nj}}$$

- **Example 3**. The 4 by 4 second difference matrix A_4 has only two non zeros in row 1:

Use cofactors $$A4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

- C_{11} comes from erasing row 1 and column 1, which leaves the $-1, 2, -1$ pattern:

$$C_{11} = \det A_3 = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- For $a_{12} = -1$ it is column 2 that gets removed, and we need its cofactor C_{12}:

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = +\det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \det A_2.$$

- Altogether row 1 has produced $2C_{11} - C_{12}$:
  $\det A_4 = 2(\det A_3) - \det A_2 = 2(4) - 3 = 5$
- The same idea applies to A_5 and A_6, and every A_n:

Recursion by cofactors $\qquad \det A_n = 2(\det A_{n-1}) - \det A_{n-2}.$

- This gives the determinant of increasingly bigger matrices. At every step the determinant of A_n is $\mathrm{n} + 1$, from the previous determinants n and $\mathrm{n} - 1$:

−1, 2, −1 matrix $\qquad \det A_n = 2(n) - (n-1) = n+1.$

- The answer $\mathrm{n} + 1$ agrees with the product of pivots (discussed at the start of this section).

Applications of Determinants

1. Computation of A^{-1}.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

- The number $C_{11} = d$ is the cofactor of a.
- The number $C_{12} = -c$ is the cofactor of b. C_{12} goes in row 2, column 1.
- The row a, b times the column C_{11}, C_{12} produces $ad - bc$. This is the cofactor expansion of $\det A$.
- **A^{-1} divides the cofactors by $\det A$.**

Cofactor matrix
C is transposed

$$A^{-1} = \frac{C^{\mathrm{T}}}{\det A} \quad \text{means} \quad (A^{-1})_{ij} = \frac{C_{ji}}{\det A}.$$

- $AC^T = (\det A)I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \det A \end{bmatrix}$$

2. **The Solution of Ax = b**. The multiplication $x = A^{-1}b$ is just $C^T b$ divided by $\det A$.

4C ***Cramer's rule***: The jth component of $x = A^{-1}b$ is the ratio

$$x_j = \frac{\det B_j}{\det A}, \quad \text{where} \quad B_j = \begin{bmatrix} a_{11} & a_{12} & b_1 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{bmatrix} \text{ has } b \text{ in column } j.$$

Example

The solution of

$$x_1 + 3x_2 = 0$$
$$2x_1 + 4x_2 = 6$$

has 0 and 6 in the first column for x_1 and in the second column for x_2:

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9, \qquad x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{6}{-2} = -3.$$