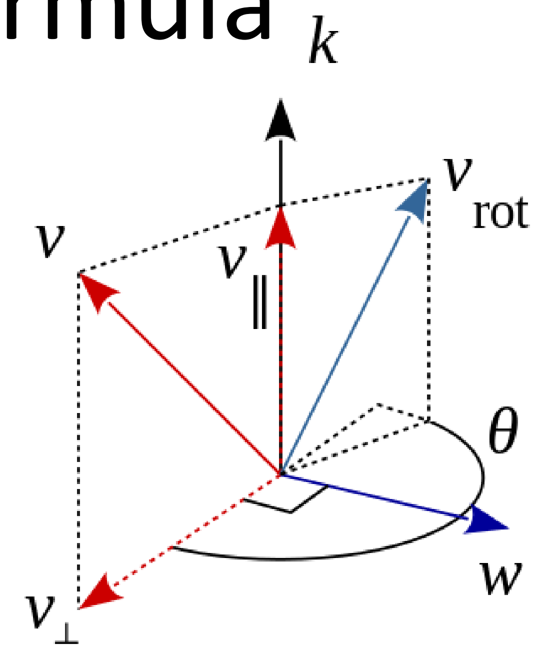
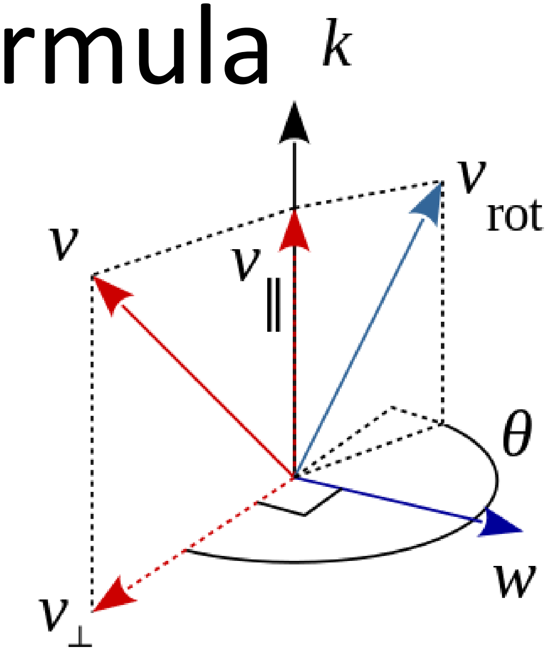
Rodrigues' Rotation Formula



- k – unit vector (rotation axis)
- v – vector to rotate about k by θ
- $v = v_{par} + v_{per}$
- $v_{par} = (v.k)k$ $(\mathbf{1})$ [projection of v on k]
- $v_{per} = (v - v_{par})$
  $= v - (k.v)k = (k.k)v - k.v(k) = -k \times (k \times v)$ **(2)**
- $v_{par}^{rot} = v_{par}$(parallel comp. will not change direction or magnitude Under rotation)
- $\left|v_{per}^{rot}\right| = \left|v_{per}\right|$ (perpendicular comp. changes its direction but retains its magnitude under rotation)

Rodrigues' Rotation Formula



- $\left|v_{per}^{rot}\right| = \left|v_{per}\right|$

  $\Longrightarrow v_{per}^{rot} = cos\theta v_{per} + sin\theta k \times v_{per}$ **(3)**
- $k \times v_{per} = k \times \left(v - v_{par}\right)$

  $= k \times v - k \times v_{par}$

  $= k \times v$ **(4)** [as, $k \times v_{par} = 0$]
- Putting in Eq.3, $v_{per}^{rot} = cos\theta v_{per} + sin\theta k \times v$
- Finally, the full rotated vector is

$$\begin{aligned} v_{rot} &= v_{par}^{rot} + v_{per}^{rot} \\ &= v_{par} + cos\theta v_{per} + sin\theta k \times v \\ &= v_{par} + cos\theta\left(v - v_{par}\right) + sin\theta k \times v \\ &= cos\theta v + (1 - cos\theta) v_{par} + sin\theta k \times v \\ &= cos\theta v + (1 - cos\theta)(k.v)k + sin\theta k \times v \quad (\mathbf{5}) \end{aligned}$$

Rodrigues' Rotation Formula

- $k \times v$ can be expressed as a matrix product

$$\begin{bmatrix} (k \times v)_x \\ (k \times v)_y \\ (k \times v)_z \end{bmatrix} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = [k]_X v$$

- So, $[k]_X v = k \times v$
- $(k.v)k = v + k \times (k \times v)$

$$= v + [k]_X([k]_X v) = v + [k]_X^2 v$$

- Putting in Eq.5

$$\begin{aligned} v_{rot} &= cos\theta v + (1 - cos\theta)(v + [k]_X^2 v) + sin\theta k \times v \\ &= v + (sin\theta)[k]_X v + (1 - cos\theta)[k]_X^2 v \\ &= Rv \end{aligned}$$

where, $\boldsymbol{R = I + (sin\theta)[k]_X + (1 - cos\theta)[k]_X^2}$

Monasse 3-step Rectification

- INPUT : Fundamental Matrix, *F* by DLT.

$e = (e_x, e_y, 1)^T$ Applying, $Fe = 0$, find e.

$e' = (e'_x, e'_y, 1)^T$ Applying, $e'^T F = 0$, find e'.

- Orientation of a camera can be adjusted by,

$$H = KRK^{-1}$$

- Since the image is not calibrated,

$$K = \begin{bmatrix} f & 0 & w/2 \\ 0 & f & h/2 \\ 0 & 0 & 1 \end{bmatrix}$$ where, w = width of the image,

h = height of the image.

Monasse 3-step Rectification

- **Step 1:**

$$H_1 e = (e_x, e_y, 0)^T = e_1 \qquad \text{where,} \quad H_1 = K R_1 K^{-1}$$

$$H'_1 e' = (e'_x, e'_y, 0) = e'_1 \qquad \text{where,} \quad H'_1 = K R'_1 K^{-1}$$

$$H_1 e = (e_x, e_y, 0)^T$$

$$K R_1 K^{-1} e = (e_x, e_y, 0)^T$$

$$R_1 K^{-1} e = K^{-1} (e_x, e_y, 0)^T$$

$$R_1 \mathbf{a} = \mathbf{b}$$

According to Rodrigues' formulae,

$$R_1(\theta, t) = I + \sin\theta [t]_\times + (1 - \cos\theta)[t]_\times^2$$

where,

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \quad \text{and}$$

$$\text{rotation axis, } t = \frac{\mathbf{a} \text{ x } \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Monasse 3-step Rectification

- **Step 2:**

$$H_2 e_1 = (1,0,0)^T = e_2 \qquad \text{where,} \quad H_2 = K R_2 K^{-1}$$

$$H'_2 e'_1 = (1,0,0)^T = e'_2 \qquad \text{where,} \quad H'_2 = K R'_2 K^{-1}$$

$\therefore H_1, H'_1, H_2, H'_2$ are all parameterized by f.

- **Step 3:**

The remaining relationship between the two cameras of the rectified image is characterized by a rotation, $\hat{R}$ around the baseline.

Finding the Essential Matrix

- According to Zisserman and Hartley,

$\hat{F}$ of a rectified image is given by

$$\hat{F} = K^{-T}[i]_{\times}\hat{R}K^{-1} = K^{-T}\hat{E}K^{-1}$$

$$\therefore \hat{E} = [i]_{\times}\hat{R}$$

$\hat{E}$ is also parameterized by f.

Now, $\hat{E}$ is decomposed into $\hat{E} = UDV^T$

Following the definition of Essential Matrix,

$$\hat{\tilde{E}} = U\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}V^T$$

$$\therefore \hat{\tilde{F}} = K^{-T}\hat{\tilde{E}}K^{-1}$$

The optimization step

$$e'_2 \hat{\tilde{F}} e_2 = 0$$

$$(H'_2 H'_1 e')^T \hat{\tilde{F}} (H_2 H_1 e) = 0$$

$$e'^T H'^T_1 H'^T_2 \hat{\tilde{F}} H_2 H_1 e = 0 \qquad \text{and, } e'^T \tilde{F} e = 0$$

$$\therefore \tilde{F} = H'^T_1 H'^T_2 \hat{\tilde{F}} H_2 H_1$$

Now an optimization function, S is defined as :

$$S(f) = \sum_{i=1}^{N} d(\mathrm{x}'_i, \tilde{F}\mathrm{x}_i) + d(\mathrm{x}_i, \tilde{F}^T \mathrm{x}'_i)$$

where, N is the no. of pixels in the image.

$d(p, q)$ is the Euclidean distance between p and q.

A minimization of $S(f)$ is done to estimate K in terms of f.

From K, P and P' is estimated.

$$\therefore \mathrm{X} = P^+ \mathrm{x} \quad \text{or} \quad \mathrm{X} = P'^+ \mathrm{x}'$$

BMVC 2010 doi:10.5244/C.24.89

The idea is to transform both images so that the fundamental matrix gets the form $[\mathbf{i}]_\times$. Unlike the other methods which directly parameterize the homographies from the constraints $\mathbf{He} = \mathbf{i}$, $\mathbf{H}'\mathbf{e}' = \mathbf{i}$ and $\mathbf{H}'^T[\mathbf{i}]_\times\mathbf{H} = \mathbf{F}$ and find an optimal pair by minimizing a measure of distortion, we shall compute the homography by explicitly rotating each camera around its optical center. The algorithm is decomposed into three steps (Fig. 1):

1. Com
   left
   $(e'_x, $
2. Rota
   $(1,0$
3. Rota
   tion
   denc

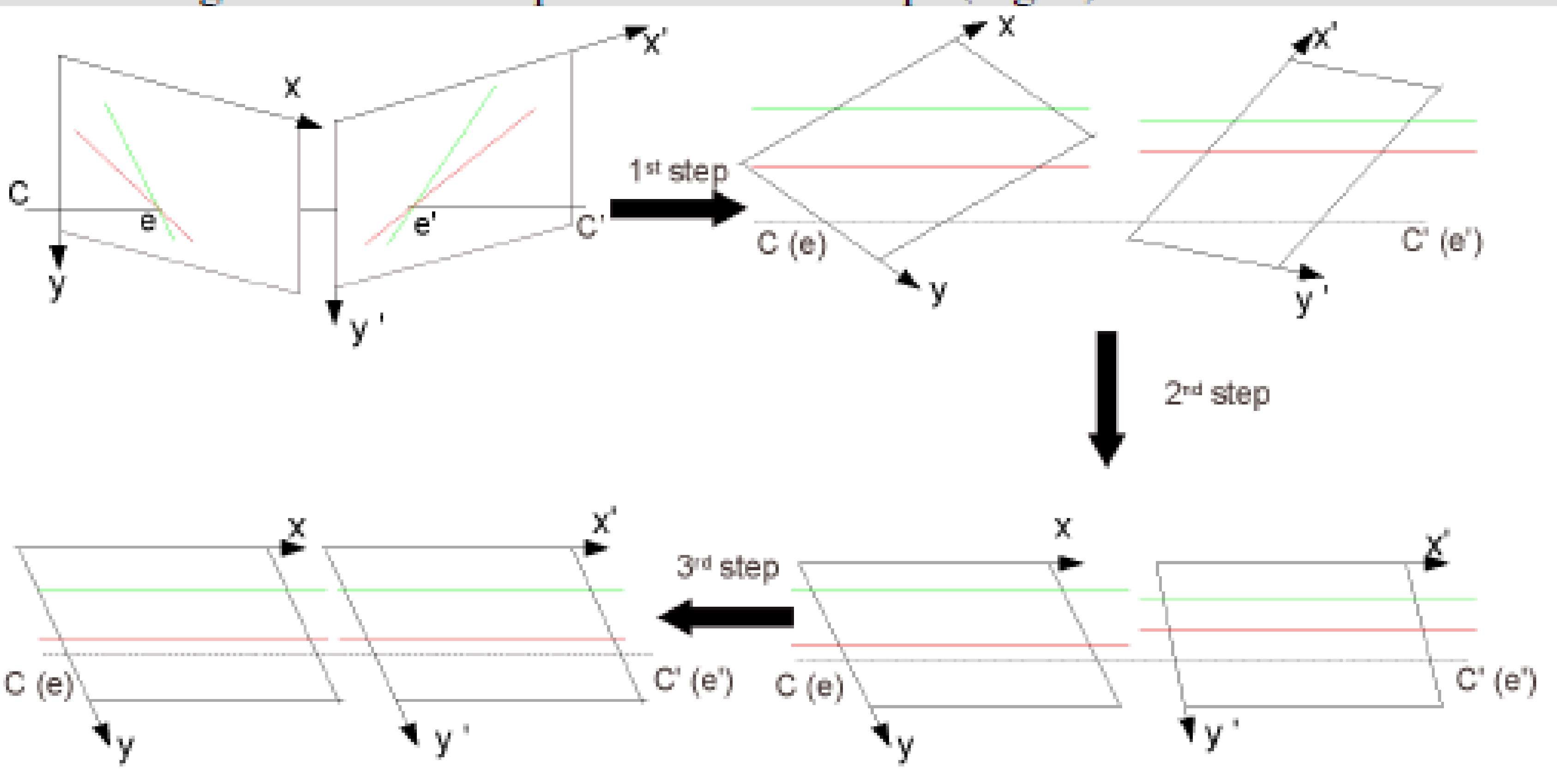


Figure 1: Three-step rectification. First step: the image planes become parallel to CC'. Second step: the images rotate in their own plane to have their epipolar lines also parallel to CC'. Third step: a rotation of one of the image planes around CC' aligns corresponding epipolar lines in both images. Note how the pairs of epipolar lines become aligned.

 BMVC 2010 doi:10.5244/C.24.89

Input: F, computed using correspondences;
which gives epipoles e and e';

Let,

$$\mathbf{x}_1 = K[\mathbf{I} \mid 0]\mathbf{X}; \Rightarrow K^{-1}\mathbf{x}_1 = [\mathbf{I} \mid 0]\mathbf{X}$$

$$\mathbf{x}_2 = K.\mathbf{R}[\mathbf{I} \mid 0]\mathbf{X};$$

$$\Rightarrow \mathbf{x}_2 = K.\mathbf{R}K^{-1}\mathbf{x}_1 = H\mathbf{x}_1;$$

where, **Homography is:**

$$H = K.\mathbf{R}K^{-1}$$

Steps: 1 & 2:

$$\mathbf{H}_1\mathbf{e} = (e_x, e_y, 0)^T \text{ and } \mathbf{H}'_1\mathbf{e}' = (e'_x, e'_y, 0)^T$$

$$\mathbf{H}_1 = \mathbf{K}\mathbf{R}\mathbf{K}^{-1} \text{ and } \mathbf{H}'_1 = \mathbf{K}\mathbf{R}'\mathbf{K}^{-1}$$

$$\mathbf{R}\mathbf{K}^{-1}\mathbf{e} = \mathbf{K}^{-1}(e_x, e_y, 0)^T$$

rotates the vector $\mathbf{a} = \mathbf{K}^{-1}\mathbf{e}$ to $\mathbf{b} = \mathbf{K}^{-1}(e_x, e_y, 0)^T$

$$\mathbf{K} = \begin{bmatrix} f & 0 & \frac{w}{2} \\ 0 & f & \frac{h}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}(\theta, \mathbf{t}) = \mathbf{I} + \sin\theta[\mathbf{t}]_\times + (1 - \cos\theta)[\mathbf{t}]_\times^2$$

minimal angle θ is $\operatorname{acos}(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|})$ and the rotation axis $\mathbf{t}$ is $\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$\mathbf{H}_1, \mathbf{H}'_1, \mathbf{H}_2$ and $\mathbf{H}'_2$ are all parametrized by f

Step 3: Rotation R^, of one camera about baseline: $\hat{\mathbf{F}} = \mathbf{K}^{-T}[\mathbf{i}]_\times \hat{\mathbf{R}}\mathbf{K}^{-1}$

H_3 is obtained after obtaining optimal K (or f)

Examples

Image-1 Image-2 Rectified Image