On Saving Energy in Boolean Circuits via Negations

Jayalal Sarma[1] Kei Uchizawa[2]

[1]Indian Institute of Technology Madras, Chennai, India

[2]Graduate School of Science and Engineering, Yamagata University, Japan

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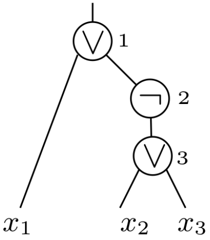

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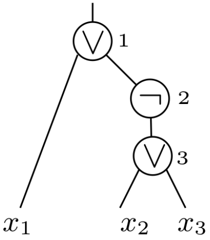

Energy Complexity of Boolean Functions

The energy complexity of the Boolean function f,

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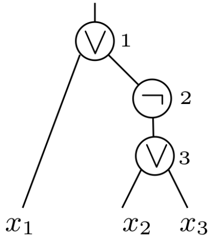

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For $n \in \mathbb{N}$, $\mathrm{EC}_{\mathcal{B}}(n) = \max_f \mathrm{EC}_{\mathcal{B}}(f)$ where the max is over all n-bit functions.

History of Energy Complexity

Vaintsvaig (1961)

For any finite basis $\mathcal{B}$, for every $n \in \mathbb{N}$, $\Omega(n) \leq \mathrm{EC}_{\mathcal{B}}(n) \leq O(2^n/n)$

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- $\mathsf{EC}(f) \leq \mathsf{DT}(f)^3$ (Dinesh *et al* 2020).
- $\sqrt{\mathsf{DT}(f)} \leq \mathsf{EC}(f) \leq \mathsf{DT}(f)^2$ (Sun *et al* 2022)

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A "close variant" - Dichotomy Theorem - Kasim-zade (1992)

Fix a finite basis $\mathcal{B}$ (with d-ary gates for constant d), one of the following must hold:

- For every n, $\Omega(n) \leq \widetilde{\mathrm{EC}}_{\mathcal{B}}(n) \leq O(n^2)$.
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$\widetilde{\mathrm{EC}}_{\mathcal{B}}(n)$ counts the gates where the output is 1 or at least one input is 1.

It was also shown that such a dichotomy does not hold for energy complexity $\mathrm{EC}_{\mathcal{B}}(n)$.

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Threshold Basis - $\mathcal{B}_{th}$ with linear threshold functions - (Uchizawa *et al*)

There exists explicit Boolean functions $f : \{0,1\}^n \to \{0,1\}$ such that

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Bounds parameterized by Fanin - Suzuki *et al* (2013)

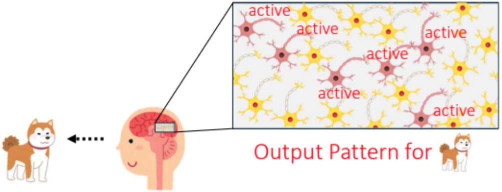
$\mathcal{B}_\ell$ consisting of arbitrary Boolean functions of fanin ℓ showed that

$$\Omega\left(\frac{n - m_f}{\ell}\right) \leq \mathsf{EC}_{\mathcal{B}_\ell}(f) \leq O\left(\frac{n}{\ell}\right)$$

for any symmetric function f, where m_f is the maximum number of consecutive 0s or 1s in the value vector of f.

Another Motivation for Threshold Basis

- Modeling neural networks as threshold circuits.
- A neuran firing uses energy.
- The brain needs to construct energy efficient output patterns.



Question: *Can we evaluate computational power of a neural network with few number of active neurons?* This leads to energy complexity.

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- More recently, we had shown (2024) that for the standard basis $\mathcal{B}_* = \{\wedge, \vee, \neg\}$ where $\wedge$ and $\vee$ are of unbounded-fanin.

$$\left\{ \begin{array}{c} f \text{ can be computed by a circuit } C \\ \text{of size } s \text{ and energy } e \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{Number of output patterns} \\ \text{is at most } 2^{e \log e} \end{array} \right\}$$

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This was further used in to establish lower bounds for the energy over the basis $\mathcal{B}_*$.

Role of Negations in Saving Energy : The $\mathcal{N}_*$-basis

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Our Basis - $\mathcal{N}_*$

Conjunction and disjunction with unbounded fan-in, where any input variable can be negated.

Our Results

Our Results 1: Power of $\mathcal{N}_*$-Circuits

Universality

For any Boolean function $f : \{0,1\}^n \to \{0,1\}$, there exists and $\mathcal{N}_*$-circuit of size $|f^{-1}(0)|$ and energy one. $\mathsf{EC}_{\mathcal{N}_*}(n) = 1$.

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Contrast: $\mathcal{B}_*$-circuits require:

- Energy $\Omega(\sqrt{n})$ to compute PAR_n (defined as $\bigoplus_{i\in[n]}$).
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Contrast : Any $\mathcal{B}_*$-circuit of energy e has at most $2^{O(e \log e)}$ output patterns.

Our Results 2: From Decision Trees to $\mathcal{N}_*$-Circuits

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If a Boolean function f is computable by a decision tree of size s and rank r, then f is also computable by a $\mathcal{N}_*$-circuit of size $O(s)$ and energy $O(r)$.

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Contrast: There is a Boolean function that can be computed by a rank 1 Decision tree but every $\mathcal{B}_*$ circuit computing it requires $\Omega(\log n)$ energy.

Our Results 3: Lower Bounds in terms of Size and Depth

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Lower Bounds for DeMorgan Circuits

Let C be a DeMorgan $\mathcal{N}_*$-circuit computing $\mathrm{PAR}_{[n]}$. Then

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where s and d are the size and depth of C, respectively.

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Lower Bounds for Layered Circuits

Let C be a $\mathcal{N}_*$-circuit of size s and depth d that computes $\mathrm{PAR}_{[n]}$. If every input variable is connected to a gate in the bottom layer, then for sufficiently large n:

$$\mathsf{EC}_{\mathcal{N}_*}(C) \geq \frac{n}{d \log s}$$

Our Results 4: Lower Bounds in terms of Matrix Rank

Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a Boolean function. The communication matrix M_f is a $2^n \times 2^n$ matrix with entry $M_f(x,y) = f(x,y)$.

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Lower Bounds from Rank of the Communication Matrix

If a circuit C on $2n$ vars computes a Boolean function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, and has size s, depth d, then

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- Any $\mathcal{N}_*$-circuit C is a polynomial-size and constant-depth circuit computing DISJ_n, then C must have energy at least $\Omega(n/\log n)$.
- This is tight! - DISJ_n can indeed be computed by a $\mathcal{N}_*$-circuit of size $O(n^2/\log n)$, depth 3 and energy $O(n/\log n)$.

A Couple of Proof Ideas

A Distinction between $\mathcal{N}_*$-circuit and $\mathcal{B}_*$-circuit

Simplying $\mathcal{B}_*$-circuit of small energy

If a Boolean function f is computable by a $\mathcal{B}_*$-circuit of energy e, then we can force f to be constant by fixing at most $e^2 + e$ variables.

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Corollaries

- Energy $\Omega(\sqrt{n})$ required to compute PAR_n (defined as $\bigoplus_{i \in [n]}$).
- Energy $\Omega(\sqrt{\log n})$ required for computing MUX_n.

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- Write $\neg f$ in the sum of products form and attach a negation to get back f.
- Replace the root $\vee$ gate with negated $\wedge$ gate from $\mathcal{N}_*$.
- For $a \in \{0,1\}^n$, if $f(a) = 0$, exactly one of the $\wedge$ gates output 1 if $f(a) = 0$.
- For $a \in \{0,1\}^n$, if $f(a) = 1$, only the root gate outputs 1.
- The size is $|f^{-1}(0)|$ and energy is 1.

From Rank-bounded Decision Trees to $\mathcal{N}_*$-circuits

Decision tree T of rank r.

S_T : the "addresses" of the nodes of the tree.
L_T : the addresses of the leaves of the tree.

Uchizawa *et al* (2008)

$$\left\{ \begin{array}{c} \text{Decision tree } T \text{ of} \\ \text{size } s \text{ and rank } r \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Equivalent DT } T' \text{ of size } s \\ \text{such that } \forall a \in L_T, wt(a) \leq r \end{array} \right\}$$

- For each $s \in S_{T'}$, we make a $\wedge$-gate g_s with fan-in $|s| + 1$, receives the outputs of g_t for each $t \prec s$ (negated if $t0$ is a prefix of s) and the variable x_s.
- Output gate g receives the output of every gate g_t for t satisfying that either

$$\text{“}t0 \in L_{T'} \text{ and } \ell_{t0} = 1\text{” or “}t1 \in L_{T'} \text{ and } \ell_{t1} = 1\text{”}$$

  and the output is negated for the former case.
- This circuit has size $O(s)$ and energy $r + 1$.

Lower bounds for DeMorgan Circuits over $\mathcal{N}_*$

Theorem

Let C be a DeMorgan $\mathcal{N}_*$-circuit computing $\mathrm{PAR}_{[n]}$. Then $\mathsf{EC}_{\mathcal{N}_*}(C) \geq n/(d \log s)$ where s and d are the size and depth of C, respectively.

Aim : To show an input assignment $a \in \{0,1\}^n$ such that at least $n/(d \log s)$ gates output 1.

- Construction is iterative : While $d \log s \leq |I|$, we repeatedly apply a procedure to find a partial assignment of the desired a, and obtain a circuit C' computing $\mathrm{PAR}_{I'}$ for some $I' \subseteq I$ towards the next step.
- In a stage, we will set at most $d \log s$ variables.
- If an $\vee$ gate directly receives an input literal, set the bit and proceed to next stage.

Proof Idea (Contd.)

- No $\vee$ gate receives a direct input literal. Bottom layer is only $\wedge$ gates.
- We will show that there exists an $\wedge$ gate in the bottom later with fanin at most $d \log s$. We will set the variables and proceed to the next stage.
- We can assume that the circuit has alternating $\vee$ and $\wedge$ gates.
- Suppose for contradiction that every $\wedge$ gate in the bottom layer has fanin larger than $d \log s$.
- Since the circuit is computing parity on $|I|$ variables, we have for each g in the bottom layer:

$$S_1^C(g) \leq 2^{|I| - d \log s} = \frac{2^{|I|}}{s^d}$$

where S_1 is the number of input settings to the circuit which makes g output 1.

Proof Idea (Contd.)

- By an induction on the layers, for each $g \in G_\ell$,

$$S_1^C(g) \leq \frac{2^{|I|}}{s^{d-\ell+1}}$$

- This is a contradiction since the number of assignments that makes the root output 1 will be at most $\frac{2^{|I|}}{s} < 2^{|I|-1}$.
- Hence there must exist an $\wedge$ gate in the bottom layer with fanin at most $d \log s$.
- We set those $d \log s$ bits such that $\wedge$ gate outputs 1 and continue to the next stage.
- Repeat the stages until $d \log s > |I|$.
- Since we are setting the gates to value $1.$ we should be able to do this only for e steps before hitting the contraint $d \log s > |I|$.
- Hence $n \leq ed \log s$.

A Connection to Rank of the Communication Matrix

- Let C be a $\mathcal{N}_*$-circuit. For every gate g in C:
  - $I(g)$: input variables that are connected to g
  - $I'(g)$: set of the gates in G_C that feed into g.
- G_C : functions obtained from g by fixing the outputs of the gates in $I'(g)$.
- Define

$$r_C = \max_{g_1,\ldots,g_e \in G_C} rk(M)_{\wedge[g_1,\ldots,g_e]})$$

Lemma - (Implicit in Uchizawa and Abe (2023))

Let C be a circuit of size s, depth d and energy e. Then it holds that

$$rk(M_C) \leq (s^e \cdot r_C)^{O(d)}.$$

A Connection to Rank of the Communication Matrix

Observe that our functions in G_C can be only $\wedge$, $\vee$ and constant functions.
We show $r_C \leq 2^e$ and this derives our result.

Theorem

If a $\mathcal{N}_*$-circuit C computes a Boolean function of $2n$ variables, has size s, depth d and energy e, then it holds that

$$\log(rk(M_C)) = O(ed \log s)$$

Other Applications

- Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. Let $g : \{0,1\}^2 \to \{0,1\}$ by a 2-bit gadget function. Define $F_g : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ defined as $F_g(x,y) = f(z_1, z_2, \ldots z_n)$ where $z_i = g(x_i, y_i)$.
- Let M_f^g for the matrix M_{F_g}.
- Let M_f^* be a matrix that satisfies

$$rk(M_f^*) = \max(rk(M_f^\wedge), rk(M_f^\vee))$$

- Shrestov (2010): for any Boolean function f, the $\deg(f)$ is upper bounded by $rk(M_f^*)$.

Corollary

If a circuit C of size s, depth d, computes a Boolean function F such that $M_F = M_f^*$, then it holds that $\text{EC}_{\mathcal{N}_*}(C) \geq \Omega\left(\frac{\deg(f)}{d \log s}\right)$

Conclusion and Open Problems

Conclusion and Open Problems

- We showed upper and lower bounds for Energy complexity over the $\mathcal{N}_*$.
- Motivation is to understand the power and limiations of negations in saving energy.
- Shortcoming: ideally we would like to have design techniques where negations can be used to switch off "irrelavant parts" of the circuit.
- Can we relate the energy over $\mathcal{N}_*$ to other Boolean complexity measures?
- Lower bound techniques for energy complexity? Limitations?

Thank you for your attention.

Questions?