Binary Decision Diagrams

An Introduction and Some Applications

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Binary decision tree for a truth table

x_1	x_2	x_3	x_4	f
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

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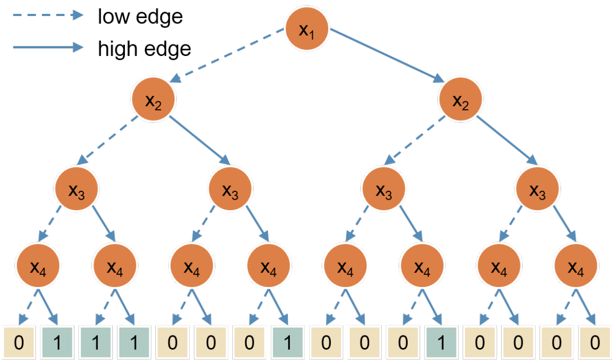

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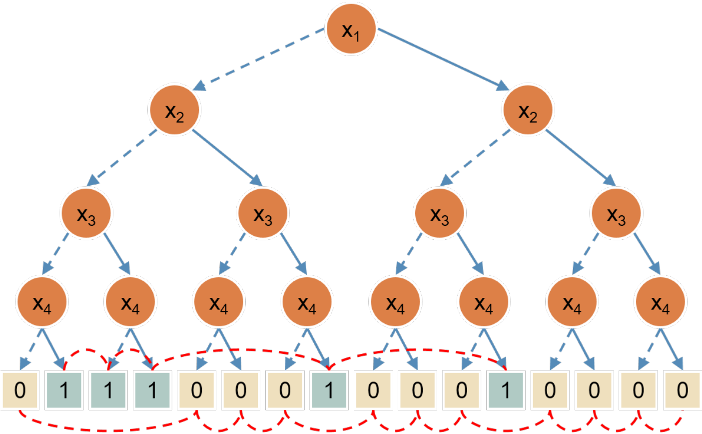
Collapse redundant nodes



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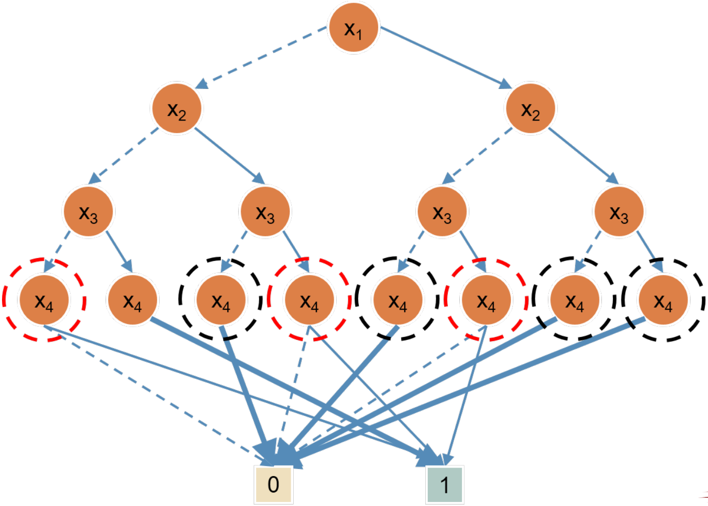

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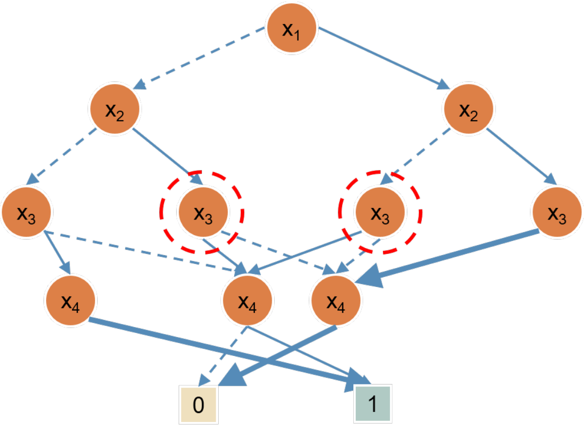

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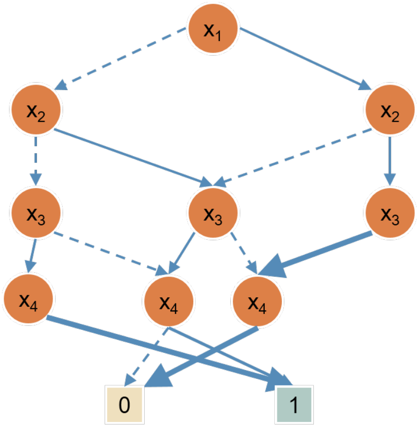

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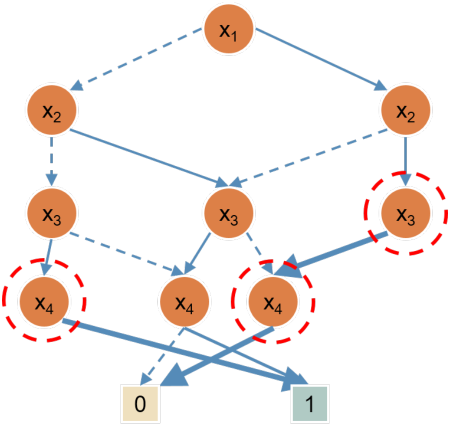
Eliminate unnecessary nodes



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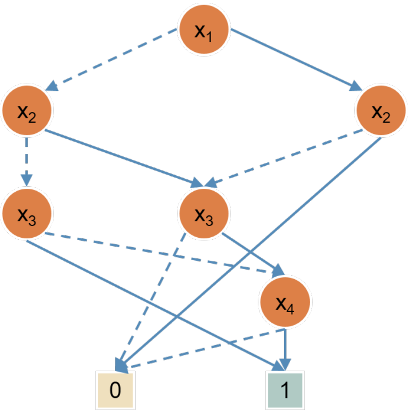
We got an ROBDD!!



Overview

1 Motivating Example

2 Introduction

3 Constructing ROBDDs

4 Applications

5 Conclusion

Binary Decision Diagrams

Definition

A Binary Decision Diagram is a rooted DAG with

- One or two terminal nodes of out-degree zero labeled 0 or 1
- A set of variable nodes of out-degree two

Ordered Binary Decision Diagrams (OBDDs)

- A BDD is ordered if on all paths through the graph, the variables respect a given linear order.

$$b_1 < b_2 < \dots < b_n$$

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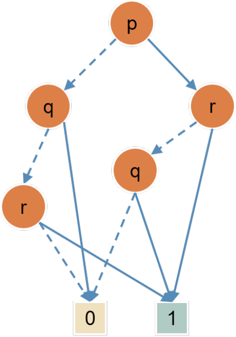
- An unordered BDD



Ordered Binary Decision Diagrams (OBDDs)

- The size of a BDD depends on the variable ordering

b_1	b_2	b_3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
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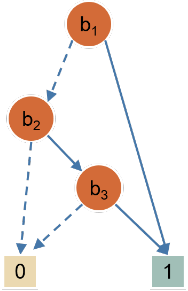

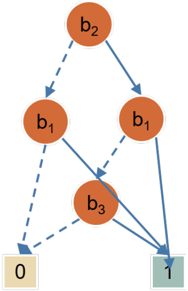

- The problem of finding the best variable ordering in OBDDs is NP-Complete

Reduced Ordered Binary Decision Diagrams (ROBDDs)

Definition

An OBDD is reduced if it satisfies the following properties:

- Uniqueness

$$low(u) = low(v) \text{ and } high(u) = high(v) \text{ implies } u = v$$

- Non-redundant tests

$$low(u) \neq high(u)$$

We already saw an example of ROBDDs!! See it again

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Properties of ROBDDs

- Size is correlated to amount of redundancy, NOT size of relation
  - Insight: As the relation gets larger, the number of dont-care bits increases, leading to fewer necessary nodes (usually)
- **Canonicity:** For every Boolean function, there is exactly one ROBDD representing it
  - Hence, satisfiability, tautology-check, and equivalence can be tested in deterministic time
  - For Boolean expressions, this problem is NP-Complete

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Normal forms for Boolean expressions

- Disjunctive Normal Form (DNF)
  - $(a_1 \wedge a_2 \wedge \ldots \wedge a_n) \vee \ldots \vee (a_1 \wedge a_2 \wedge \ldots \wedge a_n)$
  - Satisfiability: easy; Tautology check: hard

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  - Satisfiability: easy; Tautology check: hard
- Conjunctive Normal Form (CNF)
  - $(a_1 \vee a_2 \vee ... \vee a_n) \wedge ... \wedge (a_1 \vee a_2 \vee ... \vee a_n)$
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- Reduction?
  - No hopes since conversion between CNF and DNF is exponential

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If-then-else Normal Form (INF)

- An If-then-else Normal Form (INF) is a Boolean expression built from the if-then-else operator and the constants 0 and 1, such that all tests are performed only on variables.

$$x \to y_0, y_1 = (x \wedge y_0) \vee (\neg x \wedge y_1)$$

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- More examples

$$x = x \to (1, 0)$$

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$$x = x \rightarrow (1, 0)$$
$$\neg x = (x \rightarrow 0, 1)$$
$$x \vee y = (x \rightarrow 1, y)$$
$$x \wedge y = (x \rightarrow y, 1)$$
$$x \Leftrightarrow y = x \rightarrow (y \rightarrow 1, 0), (y \rightarrow 0, 1)$$

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Example: $t = (x_1 \Leftrightarrow y_1) \wedge (x_2 \Leftrightarrow y_2)$

$$t = x_1 \rightarrow t_1, t_0$$

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$$t_1 = y_1 \rightarrow t_{11}, 0$$

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\begin{aligned}
t &= x_1 \to t_1, t_0 \\
t_0 &= y_1 \to 0, t_{00} \\
t_1 &= y_1 \to t_{11}, 0 \\
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\end{aligned}
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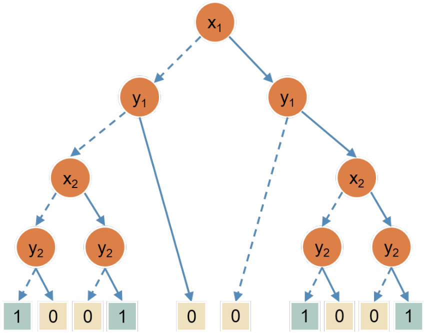

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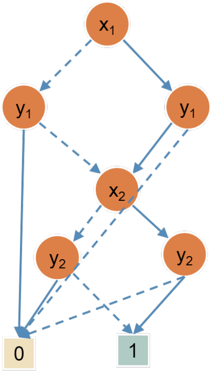

Let's move ahead

1 Motivating Example

2 Introduction

3 Constructing ROBDDs

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5 Conclusion

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Applications of BDDs

- Correctness of Combinational Circuits
- Equivalence of Combinational Circuits
- Model Checking

- And yes, Program Analysis!

BDDs for representing Points-to relation

- Points-to analysis using BDDs. Berndl et al. PLDI'03.
- Let a,b,c be reference variables and A,B,C be object references.
- The points-to relation (a,A),(a,B),(b,A),(b,B),(c,A),(c,B),(c,C) is represented as:

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On the board.

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bddbddb

- Cloning-Based Context-Sensitive Pointer Alias Analysis Using Binary Decision Diagrams. John Whaley and Monica S. Lam. PLDI'04.

bddbddb

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On the Board

1. Represent program information using relations
   Express relations as BDDs

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2. Write program analyses as Datalog queries
   Express queries as BDD operations

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bddbddb

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On the Board

1. Represent program information using relations
   Express relations as BDDs
2. Write program analyses as Datalog queries
   Express queries as BDD operations
3. Get solutions!
   Perform operations on BDDs

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Pointers for the enthusiast

- An Introduction to Binary Decision Diagrams. Tutorial by Henrik Reif Andersen.
- Program Analysis using Binary Decision Diagrams. Ondrej Lhotak's PhD Thesis (2006).
- Context-Sensitive Pointer Analysis using Binary Decision Diagrams. John Whaley's PhD Thesis (2007).
- Fun with Binary Decision Diagrams. Video lecture by Donald Knuth.

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Conclusion

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Conclusion

BDDs are very interesting and useful!