Advanced Counting Techniques

CS1200, CSE IIT Madras

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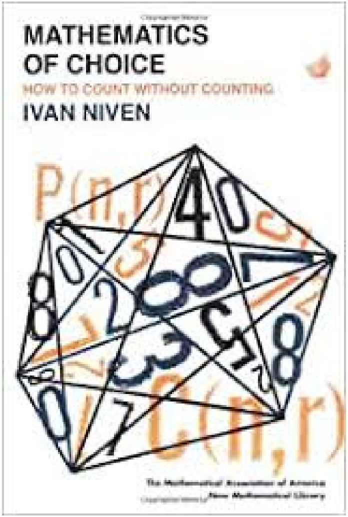

- Principle of Inclusion-Exclusion
- Recurrences and its applications
- Solving Recurrences

Example: Number of solutions to an equation

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All the above can be solved using combinations with repetitions.

- How many integral solutions if each $x_i \geq 0$ and $x_1 \leq 3$, $x_2 \leq 4$ and $x_3 \leq 6$?

Verify that this cannot be solved using the same. We will use the principle of inclusion exclusion.

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Thus, $k = \binom{r}{0} = 1$, that is x gets counted exactly once on the RHS.

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We see examples of this form next.

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N: Total number of arrangements $= 6!$

Goal: Compute $N - |A_1 \cup A_2 \cup A_3|$.

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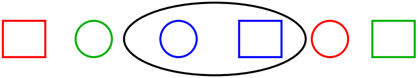
We think of "gluing" the couple together.



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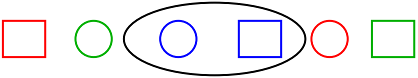

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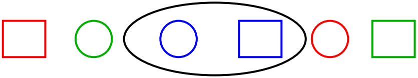
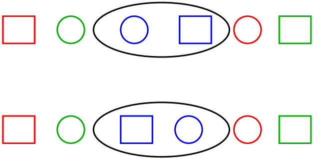


Now we have only 5 "objects" to permute. This can be done in 5! ways. However, for each such permutation, we can have two ways of arranging the couple.



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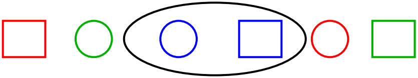

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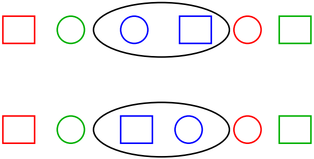

$$|A_1| = 2 \cdot 5!$$

This holds for each $i = 1, 2, 3$.

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- Let $A_1 \cap A_2$ denote the arrangements in which first and second couple stand next to each other.

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Ex: Calculate the same when there are 5 couples instead of 3.

Example: Counting number of solutions to an equation

$$x_1 + x_2 + x_3 = 11$$

How many integral solutions if each $x_i \geq 0$ and $x_1 \leq 3$, $x_2 \leq 4$ and $x_3 \leq 6$?

Can we cast this as properties that we wish to avoid?
What is our universe (that is N)?

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The our solution $k = N - |A_1 \cup A_2 \cup A_3|$.

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  Clearly $N(P_1P_2P_3) = 0$.
- Compute $|A_1 \cup A_2 \cup A_3| = \sum_{i=1}^{3} N(P_i) - \sum_{1 \leq i < j \leq 3} N(P_iP_j) + N(P_1P_2P_3)$

Summary

- Principle of inclusion exclusion with applications.
- Allows us to count elements avoiding certain properties.
- Need to come up with appropriate properties (specific to the example)
- Once properties are identified (correctly) use known techniques to count sets satisfying properties.
    - In arranging couples, we used product rule.
    - In number of solutions to equation, we used combinations with repetition.
- References Section 8.6[KR]