1. (1 point) Honor code

Concept: Linear Transformation

2. (1 point) Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose $T(\begin{bmatrix}4 & 8 & 12\end{bmatrix}^\top) = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ and $T(\begin{bmatrix}3 & 12 & 27\end{bmatrix}^\top) = \begin{bmatrix}2 & 2 & 2\end{bmatrix}^\top$. Find $T(\begin{bmatrix}-2 & -6 & -12\end{bmatrix}^\top)$

Solution: Let, $\vec{u} = \begin{bmatrix}4 & 8 & 12\end{bmatrix}^\top$, $\vec{v} = \begin{bmatrix}3 & 12 & 27\end{bmatrix}^\top$, $\vec{w} = \begin{bmatrix}-2 & -6 & -12\end{bmatrix}^\top$

If we can write $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$ as, $\vec{w} = a\vec{u} + b\vec{v}$, then

$$\begin{aligned}T(\vec{w}) &= T(a\vec{u} + b\vec{v}) \\ &= T(a\vec{u}) + T(b\vec{v}) \\ &= aT(\vec{u}) + bT(\vec{v})\end{aligned}$$

Therefore, once we find values of a and b (if possible) we will get the value of $T(\vec{w})$ We get the following equation,

$$a\begin{bmatrix}4 & 8 & 12\end{bmatrix}^\top + b\begin{bmatrix}3 & 12 & 27\end{bmatrix}^\top = \begin{bmatrix}-2 & -6 & -12\end{bmatrix}^\top$$

Solving the above we get, $a = \frac{-1}{4}$ and $b = \frac{-1}{3}$. Using a and b, we get the value of $T(\vec{w})$ as,

$$\begin{aligned}T(\begin{bmatrix}-2 & -6 & -12\end{bmatrix}^\top) &= \frac{-1}{4}T(\vec{u}) + \frac{-1}{3}T(\vec{v}) \\ &= \begin{bmatrix}\frac{-11}{12} & \frac{-11}{12} & \frac{-11}{12}\end{bmatrix}^\top\end{aligned}$$

3. (1 point) Prove that if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ then $T(b\mathbf{x} + c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$.

Solution: To prove: $T(bx + cy) = bT(x) + cT(y)$
Let $bx = u$, and $cy = v$.

LHS

$$
\begin{aligned}
&T(bx + cy) \\
&= T(u + v) \\
&= T(u) + T(v) \quad \text{[Given T(x+y) = T(x) + T(y)]} \\
&= T(bx) + T(cy) \\
&= bT(x) + cT(y) \quad \text{[Given T(ax) = aT(x)]} \\
&= RHS
\end{aligned}
$$

4. (2 points) In the lecture, we mentioned that a system of linear equations can have 0, 1 or ∞ solutions. Can you formally argue why a system of linear equations cannot have exactly 2 solutions? (Hint: If $\mathbf{x}$ and $\mathbf{y}$ are two solutions then ...)

Solution: Let the following matrix eq^n represent any generic system of k linear equations in n variables.

$$Ax = b$$

Let x = u be a solution,
$\therefore Au = b$ [Equation 1]
Let x = v also be a solution,
$\therefore Av = b$ [Equation 2]
Performing 2*Eq1 - Eq2, we get

$$
\begin{aligned}
2Au - Av &= 2b - b \\
\Rightarrow A(2u - v) &= b
\end{aligned}
$$

The above equation is of the form $Ax = b$, therefore $2u - v$ is also a solution. If we repeat the process, we can obtain infinitely more solutions which will be some linear combination of u and v. Therefore if we have two solutions, then infinite solutions exist to the system of linear equations.

5. (2 points) Suppose $A \in \mathbf{R}^{3\times 3}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3 (\mathbf{x} \neq \mathbf{0}, \mathbf{y} \neq \mathbf{0})$. Further, suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$. If $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ is one solution for $A\mathbf{x} = b$, write down at least one more solution (you are welcome to write down all the infinite solutions if you want :-)).

Solution: Let $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top = u$.
Given:
$x = u$ is a solution to Ax = b.
$Ay = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$ [Equation 1]
$Au = b$ [Equation 2]

Performing c * Eq1 + Eq2 (c is any non zero real number), we get

$$\begin{aligned} cAy + Au &= \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top + b \\ \Rightarrow A(cy + u) &= b \end{aligned}$$

The above equation is of the form $Ax = b$, where $x = cy + u$. Therefore $cy + u$ is a solution. Since c can be any real number, there are infinite solutions to the equation. Taking c = 1, we get one such solution as $x = y + u = y + \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$.

Concept: Matrix multiplication

6. (1 point) True or False: If A, B, C are matrices and if AC = BC then A = B. **Explain your answer**.

Solution: False.
Let us view the matrix multiplication equation $AC = BC$ in the row picture.
The elements in the i^{th} of A give us the scalar coefficients by which to combine the rows of C to give us the i^{th} row of the product matrix AC.
Now let us take a generic mxn matrix C as such,

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & . & . & . & c_{1,n} \\ c_{2,1} & c_{2,2} & . & . & . & c_{2,n} \\ . & . & & & & . \\ . & . & & & & . \\ c_{m,1} & c_{m,2} & . & . & . & c_{m,n} \end{bmatrix} = \begin{bmatrix} c_{r1} \\ c_{r2} \\ . \\ . \\ c_{rm} \end{bmatrix} \qquad [c_{ri} \text{ is the } i^{th} \text{ row of Matrix C}]$$

If we can write any row i of Matrix C as a linear combination of the other rows of C,

$$C_{ri} = \sum_{j=1, j\neq i}^{m} p_j * C_{rj} \qquad \text{[Equation 1]}$$

where p_j are scalar coefficients for the row C_{rj}.
Then any linear combination of the rows of C can be rewritten as another linear

combination of rows of C. This is because using Equation 1, we can replace C_{ri} with the linear combination of the rest of the rows.
This gives us,

$$\sum_{j=1}^{m} q_j * C_{rj} = \sum_{j=1, j \neq i}^{m} s_j * C_{rj}$$

where q_j and s_j are scalar coefficients which need not be equal for all j. Since we already established before that elements of a row of Matrix A gives us the scalar coefficients of linear combination of rows of C, conversely, the scalar coefficients also give us the elements of a row of A. Since q $\neq s$, the row they produce are also unequal. Therefore there can exist two such rows which when multiplied to C give us the same result. Therefore there can be two different matrices, which when multiplied to C give us the same result.

7.

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & -1 & -2 & -1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$$

For each of the equations below, find $\mathbf{x}$

(a) (½ point) $A\mathbf{x} = \begin{bmatrix} 1 & 4 & -1 & 2 \end{bmatrix}^\top$

Solution: $x = \begin{bmatrix} \frac{9}{23} & \frac{25}{23} & \frac{13}{23} & \frac{-1}{23} \end{bmatrix}^\top$

(b) (½ point) $A\mathbf{x} = \begin{bmatrix} 1 & 2 & 0.5 & 0 \end{bmatrix}^\top$

Solution: $x = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \end{bmatrix}^\top$

8. (1 point) Prove that $(AB)^\top = B^\top A^\top$

Solution: Let A be a mxn matrix and B be a nxp matrix
For any Matrix M, Let $M_{i,j}$ represent the element of Matrix M at i^{th} row and j^{th} column

LHS
The $(i,j)^{th}$ element of $(AB)^\top$ is $(j,i)^{th}$ element of (AB) is

$$\sum_{k=1}^{n} A_{j,k} * B_{k,i}$$

RHS
The $(i,j)^{th}$ element of $(B^\top A^\top)$ is

$$\begin{aligned} &\sum_{k=1}^{n} B^\top_{i,k} * A^\top_{k,j} \\ =&\sum_{k=1}^{n} B_{k,i} * A_{j,k} \\ =&\sum_{k=1}^{n} A_{j,k} * B_{k,i} \end{aligned}$$

LHS = RHS

9. If, A, B,C are matrices (assume appropriate dimensions) prove that
   (a) (½ point) $A(B+C) = AB + AC$

   Solution: Let A be a mxn matrix, B and C be nxp size matrices
   For any Matrix M, Let $M_{i,j}$ represent the element of Matrix M at i^{th} row and j^{th} column.

   The $(i,j)^{th}$ element of matrix $A(B+C)$ is:

$$\begin{aligned} A(B+C)_{i,j} =& \sum_{k=1}^{n} A_{i,k} * (B+C)_{k,j} \\ =& \sum_{k=1}^{n} A_{i,k} * B_{k,j} + \sum_{k=1}^{n} A_{i,k} * C_{k,j} \\ =& (AB)_{i,j} + (AC)_{i,j} \end{aligned}$$

   Since the $(i,j)^{th}$ element of A(B+C) is equal to $(i,j)^{th}$ element of (AB + AC), we proved
   $A(B+C) = AB + AC$

   (b) (½ point) $(AB)C = A(BC)$

Solution: Let A be a mxn matrix, B be a nxp matrix and C be a pxr matrix
For any Matrix M, Let $M_{i,j}$ represent the element of Matrix M at i^{th} row and j^{th} column.

The $(i,j)^{th}$ element of matrix $[(AB)C]$ is:

$$[(AB)C]_{i,j} = \sum_{k=1}^{p} (AB)_{i,k} * (C)_{k,j}$$
$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} A_{i,l} * B_{l,k} \right) * C_{k,j}$$

Rearranging the Sum terms we get

$$= \sum_{l=1}^{n} A_{i,l} * \left(\sum_{k=1}^{p} B_{l,k} * C_{k,j} \right)$$
$$= \sum_{l=1}^{n} (A)_{i,l} * (BC)_{l,j}$$
$$= [A(BC)]_{i,j}$$

Since the $(i,j)^{th}$ element of [(AB)C] is equal to the $(i,j)^{th}$ element of [A(BC)], we proved
$(AB)C = A(BC)$

10. (1 point) Let A be any matrix. In the lecture we saw that $A^\top A$ is a square symmetric matrix. Is $AA^\top$ also a square symmetric matrix? (Hint: The answer is either "Yes, except when ..." or "No, except when ...".)

Solution: Yes.

Concept: Inverse

11. (1 point) Let A and B be square invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: Let C = AB,

Post Multiplying both sides by $B^{-1}A^{-1}$, we get,

$$\begin{aligned}
C * (B^{-1}A^{-1}) &= AB * B^{-1}A^{-1} \\
\Rightarrow C * (B^{-1}A^{-1}) &= A * (B * B^{-1}) * A^{-1} \\
\Rightarrow C * (B^{-1}A^{-1}) &= A * I * A^{-1} \\
\Rightarrow C * (B^{-1}A^{-1}) &= A * A^{-1} \\
\Rightarrow C * (B^{-1}A^{-1}) &= I
\end{aligned}$$

Since $C * (B^{-1}A^{-1}) = I$, $(B^{-1}A^{-1})$ is the inverse of C. Therefore,
$C^{-1} = (B^{-1}A^{-1}) \Rightarrow (AB)^{-1} = (B^{-1}A^{-1})$

12. What is the inverse of the following two matrices? (Hint: I don't want you to compute the inverse using some method. Instead think of the linear transformation that these matrices do and think how you would reverse that transformation. **You will have to explain your answer in words clearly stating the linear transformations being performed.**)

(a) (½ point)

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Since Matrix A scales the diagonal elements by $\frac{1}{2}$, its inverse will reverse this operation by scaling the diagonal elements by 2 given by the matrix A^{-1}.

(b) (½ point)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since Matrix A is performing the operation $C1 \rightarrow C1 + 2C2$, its inverse should perform the operation $C1 \rightarrow C1 - 2C2$, given by the matrix A^{-1}

(c) (1 point)

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

Since Matrix A is rotating a vector counter-clockwise by θ, its inverse should rotate the vector in the same direction by $-\theta$, given by Matrix A^{-1}

Concept: System of linear equations

13. (1 point) Argue why the following system of linear equations will not have any solutions.

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Solution: The given system of linear equations has no solutions because one of the equations is:

$$\begin{aligned} 0x1 + 0x2 + 0x3 + 0x4 = 4 \\ \Rightarrow 0 = 4 \end{aligned}$$

The above equation is incorrect. Therefore the system of linear equations is inconsistent and has 0 solutions.

14. Consider the following 3 planes

$$\begin{aligned} 3x + 2y - z &= 2 \\ x - 4y + 3z &= 1 \\ 4x - 2y + 2z &= 3 \end{aligned}$$

(a) ($\frac{1}{2}$ point) Plot these planes in geogebra and paste the resulting figure here (you can download the figure as .png and paste it here)

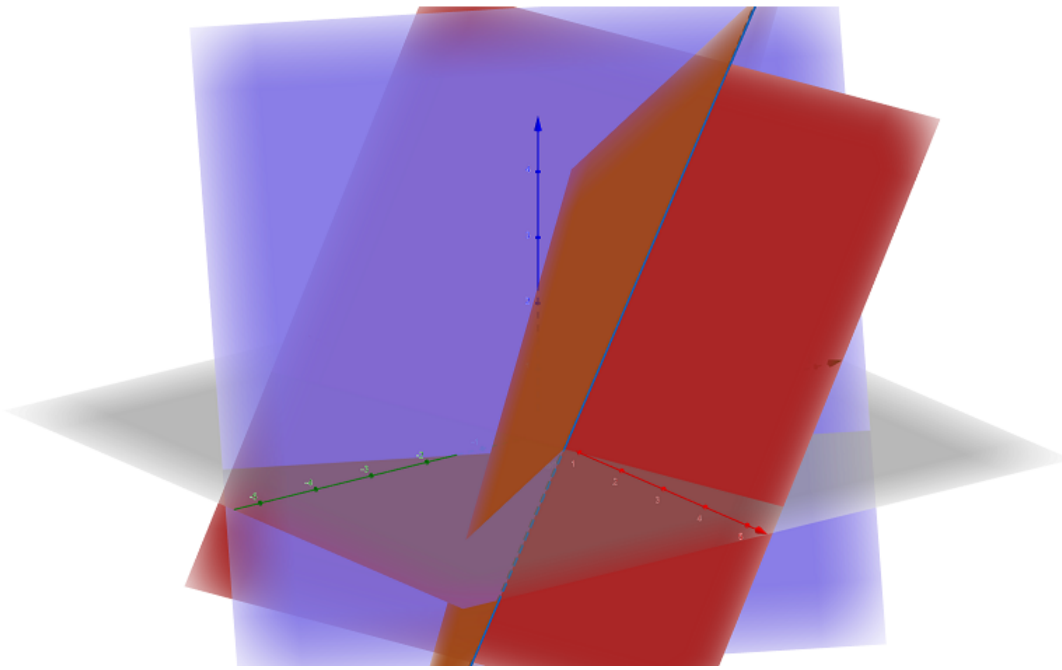

Figure 1: 3D plot of the three equations

(b) (½ point) How many solutions does the above system of linear equations have? (based on visual inspection in geogebra)

> **Solution:** Infinite Solutions

(c) (1 point) Notice that the third equation can be obtained by adding the first two equations. Based on this observation, can you explain your answer for the number of solutions in the previous part of the question. (Note that I am looking for an answer in plain English which does not include terms like "linear independence" or "dependence of columns/rows". In other words, your answer should be based only on concepts/ideas which have already been discussed in the class)

> **Solution:** The third equation doesn't give us any new information, as it is simply a combination of the first two equations. Therefore the set of points that satisfied the first two equations (point on the line of intersection) will also satisfy the new third equation. Therefore the plane given by the third equation will pass through the line of intersection of the previous two planes. This gives us that the intersection between these three planes is the same line that was the intersection between the first two planes. Since there is no single point of intersection between these three planes, there are infinite solutions to the system of equations.

15. Consider the following system of linear equations:

$$
\begin{aligned}
x + y - z &= 1 \\
x - y + z &= 2
\end{aligned}
$$

Add one more equation to the above system such that the resulting system of 3 linear equations has

(a) (½ point) 0 solutions

> **Solution:** $3x + y - z = 1$

(b) (1 point) exactly 1 solution

> **Solution:** $z + y = \frac{1}{2}$

(c) (½ point) infinite solutions

Solution: $3x + y - z = 4$