Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

   Solution:

Concept: Projection

2. (2 points) Consider a matrix A and a vector $\mathbf{b}$ which does not lie in the column space of A. Let $\mathbf{p}$ be the projection of $\mathbf{b}$ on to the column space of A. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \\ 11 \\ 8 \end{bmatrix}$, find $\mathbf{b}$.

   Solution:

3. (2 points) Consider the following statement: Two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ cannot have the same projection $\mathbf{p}$ on the column space of A.

   (a) Give one example where the above statement is True.

   Solution:

   (b) Give one example where the above statement is False.

   Solution:

   (c) Based on the above examples, state the generic condition under which the above statement will be True or False.

Solution: Any one of the following statements will do:

The condition is True except when ....

The condition is False except when ....

(and then explain your statement)

4. (2 points) (a) Find the projection matrix P_1 that projects onto the line through $\mathbf{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to $\mathbf{a}$.

Solution:

(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain the result.

Solution:

Concept: Dot product of vectors

5. (1 point) Consider two vectors $\mathbf{u}$ and $\mathbf{v}$. Let θ be the angle between these two vectors. Prove that

$$cos\theta = \frac{\mathbf{u}^\top \mathbf{v}}{||\mathbf{u}||_2||\mathbf{v}||_2}$$

Solution:

Concept: Vector norms

6. (1 point) The L_p-norm of a vector $\mathbf{x} = [x_1, x_2, \dots, x_n] \in \mathbb{R}^n$ is defined as:

$$||\mathbf{x}||_p = (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

(a) Prove that $||\mathbf{x}||_\infty = max_{1 \le i \le n}|x_i|$

Solution:

(b) True or False (explain with reason): $||\mathbf{x}||_0$ is a norm.

Solution:

Concept: Orthogonal/Orthornormal vectors and matrices

7. (1 point) Consider the following questions:

   (a) Construct a 2×2 matrix, such that all its entries are +1 and -1 and its columns are orthogonal.

   Solution:

   (b) Now, construct a 4×4 matrix, such that all its entries are +1 and -1, its columns are orthogonal and it contains the above matrix within it.

   Solution:

8. (1 point) Consider the vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

   (a) What multiple of $\mathbf{a}$ is closest to $\mathbf{b}$?

   Solution:

   (b) Find orthonormal vectors $\mathbf{q_1}$ and $\mathbf{q_2}$ that lie in the plane formed by $\mathbf{a}$ and $\mathbf{b}$?

   Solution:

9. (1 point) Suppose $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are orthogonal vectors. Prove that they are also independent.

   Solution:

10. (1 point) If Q_1 and Q_2 are orthogonal matrices,show that their product Q_1Q_2 is also an orthogonal matrix.

    Solution:

Concept: Determinants

11. (2 points) A tri-diagonal matrix is a matrix which has 1's on the main diagonal as well as on the diagonals to the left and right of the main diagonal. For example,

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let A_n be an $n \times n$ tri-diagonal matrix. Prove that $|A_n| = |A_{n-1}| + |A_{n-2}|$

Solution:

12. (1 point) State True or False and explain your answer: $det(A + B) = det(A) + det(B)$

Solution:

13. (1 point) This question is about properties 9 and 10 of determinants.
    (a) Prove that $det(AB) = det(A)det(B)$

    Solution:

    (b) (2 points) Prove that $det(A^\top) = det(A)$

    Solution:

14. (1 point) Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.
    (a) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$

    Solution:

    (b) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{v}$

    Solution:

15. (2 points) The determinant of the following matrix can be computed as a sum of 120 (5!) terms.

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

State true or false with an appropriate explanation: All the 120 terms in the determinant will be 0.

Solution: