Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

   Solution:

Eigenstory: Special Properties

2. (1 point) Prove that for any square matrix A the eigenvectors corresponding to distinct eigenvalues are always independent.

   Solution:

3. (2 points) Prove the following.

   (a) The sum of the eigenvalues of a matrix is equal to its trace.

   Solution:

   (b) The product of the eigenvalues of a matrix is equal to its determinant.

   Solution:

4. (2 points) What is the relationship between the rank of a matrix and the number of non-zero eigenvalues? Explain your answer.

   Solution: I think the answer to this question is "The rank of a matrix is equal to the number of non-zero eigenvalues if $\cdots$"

5. (1 point) If A is a square symmetric matrix then prove that the number of positive pivots it has is the same as the number of positive eigenvalues it has.

Solution:

Eigenstory: Special Matrices

6. (2 points) Consider the matrix $R = I - 2\mathbf{u}\mathbf{u}^\top$ where $\mathbf{u}$ is a unit vector $\in \mathbb{R}^n$.

   (a) Show that R is symmetric and orthogonal. (How many independent vectors will R have?)

   Solution:

   (b) Let $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$. Draw the line passing through this vector in geogebra (or any tool of your choice). Now take any vector in $\mathbf{R}^3$ and multiply it with the matrix R (i.e., the matrix R as defined above with $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$). What do you observe or what do you think the matrix R does or what would you call matrix R? (Hint: the name starts with R)

   Solution:

   (c) Compute the eigenvalues and eigenvectors of the matrix R as defined above with $\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$

   Solution:

   (d) I believe that irrespective of what $\mathbf{u}$ is any such matrix R will have the same eigenvalues as you obtained above (with one of the eigenvalues repeating). Can you reason why this is the case? (Hint: think about how we reasoned about the eigenvectors of the projection matrix P even without computing them.)

   Solution:

7. (2 points) Let Q be a $n \times n$ real orthogonal matrix (i.e., all its elements are real and its columns are orthonormal). State with reason whether the following statements are True or False (provide a proof if the statement is True and a counter-example if it is False).

   (a) If λ is an eigenvalue of Q then $|\lambda| = 1$

Solution:

(b) The eigenvectors of Q are orthogonal

Solution:

(c) Q is always diagonalizable.

Solution:

8. ($1\frac{1}{2}$ points) Any rank one matrix can be written as $\mathbf{u}\mathbf{v}^\top$.

(a) Prove that the eigenvalues of any rank one matrix are $\mathbf{v}^\top\mathbf{u}$ and 0.

Solution:

(b) How many times does the value 0 repeat?

Solution:

(c) What are the eigenvectors corresponding to these eigenvalues?

Solution:

9. (2 points) Consider a $n \times n$ Markov matrix.

(a) Prove that the dominant eigenvalue of a Markov matrix is 1

Solution:
Proof (part 1): 1 is an eigenvalue of a Markov matrix
Proof (part 2): all other eigenvalues are less than 1
(If you have a simpler way of proving this instead of proving it in two parts then feel free to do so but your proof should convince me about both these parts.)

(b) Consider any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = c + d$. Show that one of the eigenvalues of such a matrix is 1. (I hope you notice that a Markov matrix is a special case of such a matrix where $a + b = c + d = 1$.)

Solution:

(c) Does the result extend to $n \times n$ matrices where the sum of the elements of a row is the same for all the n rows? (Explain with reason)

Solution:

(d) What is the corresponding eigenvector?

Solution:

Eigenstory: Special Relations

10. (4 points) For each of the statements below state True or False with reason.

(a) The eigenvalues of A^T are **always** the same as that of A.

Solution:

(b) The eigenvectors of A^T are **always** the same as that of A

Solution:

(c) The eigenvalues of A^{-1} are **always** the reciprocal of the eigenvalues of A.

Solution:

(d) The eigenvectors of A^{-1} are **always** the same as the eigenvectors of A.

Solution:

(e) If $\mathbf{x}$ is an eigenvector of A and B then it is also an eigenvector of both AB and BA, even if the eigenvalues of A and B corresponding to $\mathbf{x}$ are different.

Solution:

(f) If $\mathbf{x}$ is and eigenvector of A and B then it is also an eigenvector of $A + B$

Solution:

(g) If λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$.

Solution:

(h) The non-zero eigenvalues of $AA^\top$ and $A^\top A$ are equal.

Solution:

Eigenstory: Change of basis

11. (2 points) Consider the following two basis. Basis 1: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, and Basis 2: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}$,. Consider a vector $\mathbf{x} = \begin{bmatrix}a\\b\end{bmatrix}$ in Basis 1 (i.e., $\mathbf{x} = a\mathbf{u_1} + b\mathbf{u_2}$). How would you represent it in Basis 2?

Solution:

12. (1 point) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the standard basis. Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be the representation of these vectors in a different basis. Prove that $\mathbf{u} \cdot \mathbf{v} = T(\mathbf{u}) \cdot T(\mathbf{v})$ if and only if the basis represented by T is an orthonormal basis (i.e., dot products are preserved only when the new basis is orthonormal).

Solution:

Eigenstory: PCA and SVD

13. (1 point) How are PCA and SVD related? (no vague answers please, think and answer very precisely with mathematical reasoning)

Solution:

14. ($1\frac{1}{2}$ points) Consider the matrix $\begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}$

(a) Find Σ and V, *i.e.*, the eigenvalues and eigenvectors of $A^\top A$

Solution:

(b) Find Σ and U, *i.e.*, the eigenvalues and eigenvectors of $AA^\top$

Solution:

(c) Now compute $U\Sigma V^\top$. Did you get back A? If yes, good! If not, what went wrong?

Solution: Please refer to following lectures of Prof. Gilbert Strang to understand what went wrong and then correct your answer (if it was wrong):

- https://www.youtube.com/watch?v=TX_vooSnhm8&t=1177s (starts at 1177 seconds)
- https://www.youtube.com/watch?v=HgC1l_6ySkc&feature=youtu.be&t=1731) (starts at 1731 seconds)

15. (2 points) Prove that the matrices U and V that you get from the SVD of a matrix A contain the basis vectors for the four fundamental subspaces of A. (this is where the whole course comes together: fundamental subspaces, basis vectors, orthonormal vectors, eigenvectors, and our special symmetric matrices $AA^\top$, $A^\top A$!)

Solution:

16. (2 points) Fun with flags.

    (a) Browse through the flags of all countries and paste 5 rank one flags below.

    Solution:

    (b) What is the rank of the flag of Greece?

    Solution:

17. (2 points) Consider the LFW dataset (Labeled Faces in the Wild).

    (a) Perform PCA using this dataset and plot the first 25 eigenfaces (in a 5×5 grid)

    Solution: Here is something to get you started.

```
import matplotlib.pyplot as plt
from sklearn.datasets import fetch_lfw_people
from sklearn.decomposition import PCA


# Load data
lfw_dataset = fetch_lfw_people(min_faces_per_person=100)

_, h, w = lfw_dataset.images.shape
X = lfw_dataset.data

# Compute a PCA
n_components = 100
pca = PCA(n_components=n_components, whiten=True).fit(X)
```

```
Beyond this you are on your own. Good Luck!
```

(b) Take your close-up photograph (face only) and reconstruct it using the first 25 eigenfaces :-). If due to privacy concerns, you do not want to to use your own photo then feel free to use a publicly available close-up photo (face only) of your favorite celebrity.

Solution:

...And that concludes the story of *How I Met Your Eigenvectors :-)* (I hope you enjoyed it!)