Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution:

Concept: Linear Transformation

2. (1 point) Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose $T(\begin{bmatrix}4 & 8 & 12\end{bmatrix}^\top) = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ and $T(\begin{bmatrix}3 & 12 & 27\end{bmatrix}^\top) = \begin{bmatrix}2 & 2 & 2\end{bmatrix}^\top$. Find $T(\begin{bmatrix}-2 & -6 & -12\end{bmatrix}^\top)$

Solution:

3. (1 point) Prove that if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ then $T(b\mathbf{x}+c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$.

Solution:

4. (2 points) In the lecture, we mentioned that a system of linear equations can have 0, 1 or ∞ solutions. Can you formally argue why a system of linear equations cannot have exactly 2 solutions? (Hint: If $\mathbf{x}$ and $\mathbf{y}$ are two solutions then ...)

Solution:

5. (2 points) Suppose $A \in \mathbf{R}^{3\times 3}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3 (\mathbf{x} \neq \mathbf{0}, \mathbf{y} \neq \mathbf{0})$. Further, suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$. If $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ is one solution for $A\mathbf{x} = b$, write down at least one more solution (you are welcome to write down all the infinite solutions if you want :-)).

Solution:

Concept: Matrix multiplication

6. (1 point) True or False: If A, B, C are matrices and if AC = BC then A = B. **Explain your answer**.

Solution:

7.

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & -1 & -2 & -1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$$

For each of the equations below, find **x**

(a) (½ point) $A\mathbf{x} = \begin{bmatrix} 1 & 4 & -1 & 2 \end{bmatrix}^\top$

Solution:

(b) (½ point) $A\mathbf{x} = \begin{bmatrix} 1 & 2 & 0.5 & 0 \end{bmatrix}^\top$

Solution:

8. (1 point) Prove that $(AB)^\top = B^\top A^\top$

Solution:

9. If, A, B,C are matrices (assume appropriate dimensions) prove that

(a) (½ point) $A(B + C) = AB + AC$

Solution:

(b) (½ point) $(AB)C = A(BC)$

Solution:

10. (1 point) Let A be any matrix. In the lecture we saw that $A^\top A$ is a square symmetric matrix. Is $AA^\top$ also a square symmetric matrix? (Hint: The answer is either "Yes, except when ..." or "No, except when ...".)

Concept: Inverse

11. (1 point) Let A and B be square invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

12. What is the inverse of the following two matrices? (Hint: I don't want you to compute the inverse using some method. Instead think of the linear transformation that these matrices do and think how you would reverse that transformation. **You will have to explain your answer in words clearly stating the linear transformations being performed.**)

(a) (½ point)

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Solution:

(b) (½ point)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

(c) (1 point)

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Solution:

Concept: System of linear equations

13. (1 point) Argue why the following system of linear equations will not have any solutions.

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

14. Consider the following 3 planes

$$\begin{aligned} 3x + 2y - z &= 2 \\ x - 4y + 3z &= 1 \\ 4x - 2y + 2z &= 3 \end{aligned}$$

(a) (½ point) Plot these planes in geogebra and paste the resulting figure here (you can download the figure as .png and paste it here)

Solution:

(b) (½ point) How many solutions does the above system of linear equations have? (based on visual inspection in geogebra)

Solution:

(c) (1 point) Notice that the third equation can be obtained by adding the first two equations. Based on this observation, can you explain your answer for the number of solutions in the previous part of the question. (Note that I am looking for an answer in plain English which does not include terms like "linear independence" or "dependence of columns/rows". In other words, your answer should be based only on concepts/ideas which have already been discussed in the class)

Solution:

15. Consider the following system of linear equations:

$$\begin{aligned} x + y - z &= 1 \\ x - y + z &= 2 \end{aligned}$$

Add one more equation to the above system such that the resulting system of 3 linear equations has

(a) ($\frac{1}{2}$ point) 0 solutions

Solution:

(b) (1 point) exactly 1 solution

Solution:

(c) ($\frac{1}{2}$ point) infinite solutions

Solution: