1. Let A be a matrix $\in \mathbb{R}^{m \times n}$. Prove that any matrix of the type $A^T A$ is a positive semidefinite matrix.

   Solution:

2. Prove that the eigen values of a positive semidefinite matrix are non-negative.

   Solution:

3. Prove that the eigen vectors of a symmetric matrix are orthogonal.

   Solution:

4. Prove that the eigen values of a symmetric matrix are real.

   Solution:

5. Consider a matrix $A \in \mathbb{R}^{m \times n}$ where $m \neq n$. Does this matrix have eigen vectors? Explain your answer.

   Solution:

6. Can a non-square matrix be positive semidefinite ? Explain your answer.

   Solution:

7. Prove that n linearly independent vectors span $\mathbb{R}^n$

   Solution:

8. Argue why an orthonormal basis is the most convenient basis one can hope for.

Solution:

9. Let $u_1, u_2, ..., u_k$ be k distinct non-zero vectors $\in \mathbb{R}^k$. Similarly, let $v_1, v_2, ..., v_k$ be k distinct non-zero vectors $\in \mathbb{R}^k$. Further, let $\sigma_1, \sigma_2, ..., \sigma_k$ be k distinct non-zero scalars $\in \mathbb{R}$. Prove that $\sum_k (\sigma_k u_k v_k^T)$ is a rank k matrix if u_i is orthogonal to u_j $(\forall i \neq j)$ and v_i is orthogonal to v_j $(\forall i \neq j)$.

Solution:

10. $w_0, w_1, w_2, ..., w_n$ are a series of vectors $\in \mathbb{R}^n$ related by the following recursive equation:

$$w_t = (I - \eta Q \Lambda Q^T) w_{t-1} + \eta Q \Lambda Q^T w^*$$

where I is the identity matrix $\in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$ is an orthonormal matrix and $\Lambda \in \mathbb{R}^{n \times n}$ is a diagonal matrix. Prove that if we start with $w_0 = 0$ then,

$$w_t = Q(I - (I - \eta \Lambda)^t) Q^T w^*$$

Solution:

11. Prove that the only subspaces of $\mathbb{R}^2$ are $\{\mathbf{0}\}$, $\mathbb{R}^2$ and any set L of the form $L = \{cu : c \in \mathbb{R}, u \neq 0\}$ consisting of all scalar multiples of a nonzero vector u.

Solution: