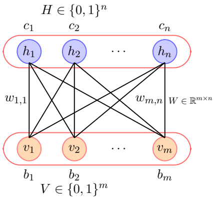
CS7015 (Deep Learning) : Lecture 23

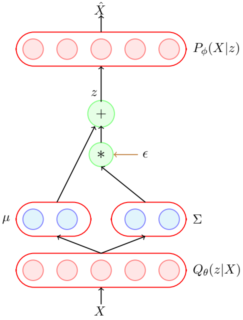
Generative Adversarial Networks (GANs)

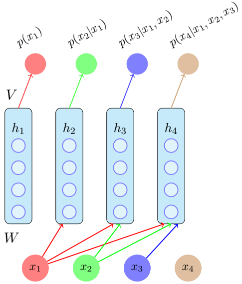
Mitesh M. Khapra

Department of Computer Science and Engineering
Indian Institute of Technology Madras

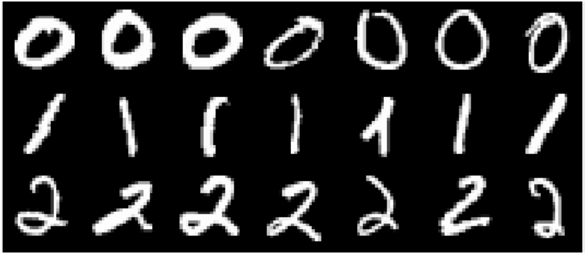
Module 23.1: Generative Adversarial Networks - The intuition



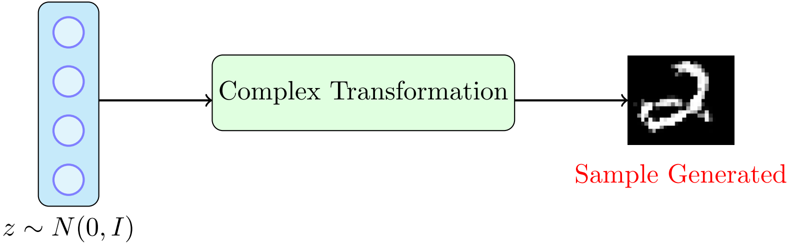




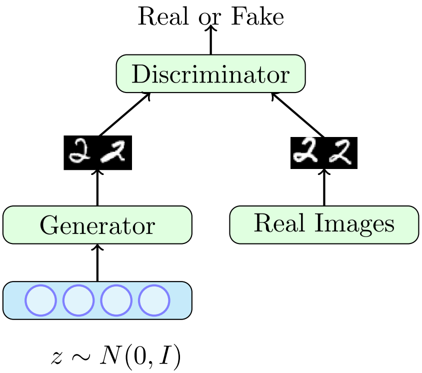
- So far we have looked at generative models which explicitly model the joint probability distribution or conditional probability distribution
- For example, in RBMs we learn $P(X, H)$, in VAEs we learn $P(z|X)$ and $P(X|z)$ whereas in AR models we learn $P(X)$
- What if we are only interested in sampling from the distribution and don't really care about the explicit density function $P(X)$?
- What does this mean? Let us see



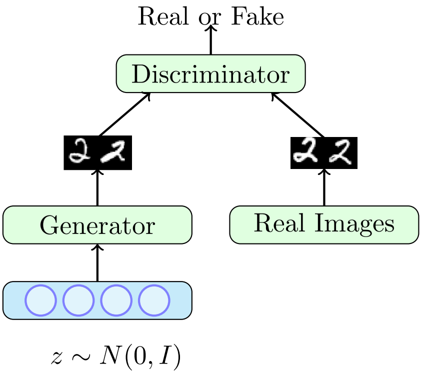
- As usual we are given some training data (say, MNIST images) which obviously comes from some underlying distribution
- Our goal is to generate more images from this distribution (*i.e.*, create images which look similar to the images from the training data)
- In other words, we want to sample from a complex high dimensional distribution which is intractable (recall RBMs, VAEs and AR models deal with this intractability in their own way)



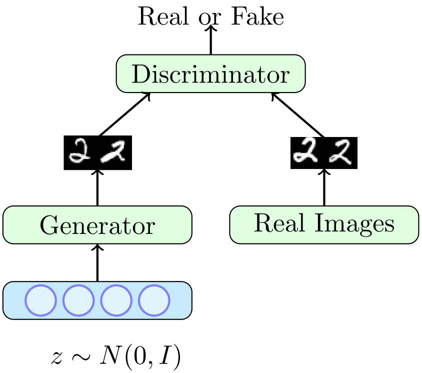
- GANs take a different approach to this problem where the idea is to sample from a simple tractable distribution (say, $z \sim N(0, I)$) and then learn a complex transformation from this to the training distribution
- In other words, we will take a $z \sim N(0, I)$, learn to make a series of complex transformations on it so that the output looks as if it came from our training distribution



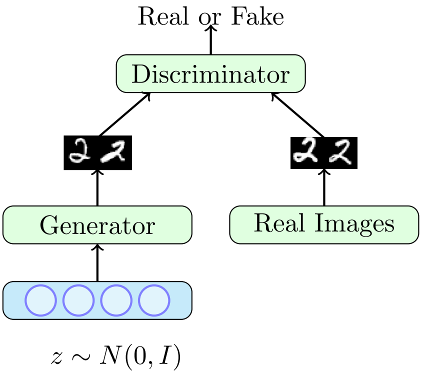
- What can we use for such a complex transformation? A Neural Network
- How do you train such a neural network? Using a two player game
- There are two players in the game: a generator and a discriminator
- The job of the generator is to produce images which look so natural that the discriminator thinks that the images came from the real data distribution
- The job of the discriminator is to get better and better at distinguishing between true images and generated (fake) images



- So let's look at the full picture
- Let G_ϕ be the generator and D_θ be the discriminator (ϕ and θ are the parameters of G and D, respectively)
- We have a neural network based generator which takes as input a noise vector $z \sim N(0, I)$ and produces $G_\phi(z) = X$
- We have a neural network based discriminator which could take as input a real X or a generated $X = G_\phi(z)$ and classify the input as real/fake



- What should be the objective function of the overall network?
- Let's look at the objective function of the generator first
- Given an image generated by the generator as $G_\phi(z)$ the discriminator assigns a score $D_\theta(G_\phi(z))$ to it
- This score will be between 0 and 1 and will tell us the probability of the image being real or fake
- For a given z, the generator would want to maximize $\log D_\theta(G_\phi(z))$ (log likelihood) or minimize $\log(1 - D_\theta(G_\phi(z)))$



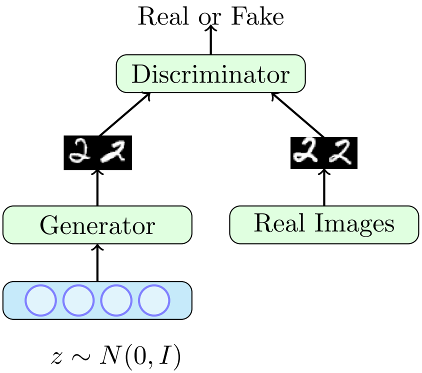
- This is just for a single z and the generator would like to do this for all possible values of z,
- For example, if z was discrete and drawn from a uniform distribution (*i.e.*, $p(z) = \frac{1}{N} \ \forall z$) then the generator's objective function would be

$$\min_{\phi} \sum_{i=1}^{N} \frac{1}{N} \log(1 - D_\theta(G_\phi(z)))$$

- However, in our case, z is continuous and not uniform ($z \sim N(0, I)$) so the equivalent objective function would be

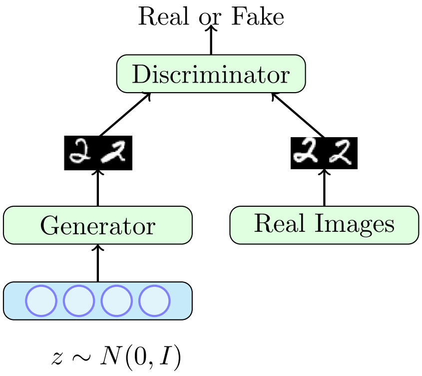
$$\min_{\phi} \int p(z) \log(1 - D_\theta(G_\phi(z)))$$

$$\min_{\phi} E_{z \sim p(z)}[\log(1 - D_\theta(G_\phi(z)))]$$



- Now let's look at the discriminator
- The task of the discriminator is to assign a high score to real images and a low score to fake images
- And it should do this for all possible real images and all possible fake images
- In other words, it should try to maximize the following objective function

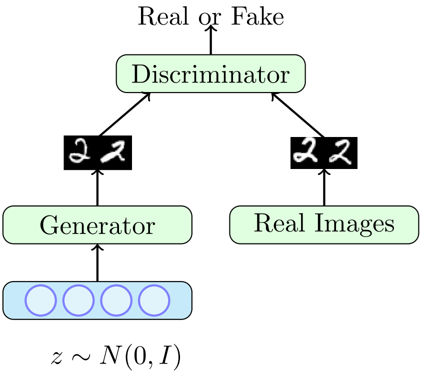
$$\max_{\theta} E_{x \sim p_{data}}[\log D_\theta(x)] + E_{z \sim p(z)}[\log(1 - D_\theta(G_\phi(z)))]$$



- If we put the objectives of the generator and discriminator together we get a minimax game

$$\min_{\phi} \max_{\theta} \left[\mathbb{E}_{x \sim p_{data}} \log D_\theta(x) + \mathbb{E}_{z \sim p(z)} \log(1 - D_\theta(G_\phi(z)))\right]$$

- The first term in the objective is only w.r.t. the parameters of the discriminator (θ)
- The second term in the objective is w.r.t. the parameters of the generator (ϕ) as well as the discriminator (θ)
- The discriminator wants to maximize the second term whereas the generator wants to minimize it (hence it is a two-player game)



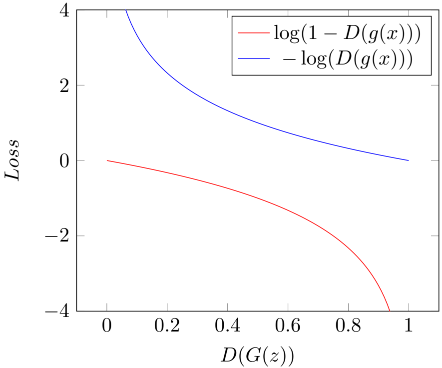
- So the overall training proceeds by alternating between these two step
- **Step 1:** Gradient Ascent on Discriminator

$$\max_{\theta} \left[\mathbb{E}_{x \sim p_{data}} \log D_\theta(x) + \mathbb{E}_{z \sim p(z)} \log(1 - D_\theta(G_\phi(z)))\right]$$

- **Step 2:** Gradient Descent on Generator

$$\min_{\phi} \ \mathbb{E}_{z \sim p(z)} \log(1 - D_\theta(G_\phi(z)))$$

- In practice, the above generator objective does not work well and we use a slightly modified objective
- Let us see why



- When the sample is likely fake, we want to give a feedback to the generator (using gradients)
- However, in this region where $D(G(z))$ is close to 0, the curve of the loss function is very flat and the gradient would be close to 0
- Trick: Instead of minimizing the likelihood of the discriminator being correct, maximize the likelihood of the discriminator being wrong
- In effect, the objective remains the same but the gradient signal becomes better

With that we are now ready to see the full algorithm for training GANs

procedure GAN TRAINING
  for number of training iterations **do**
    for k steps **do**
      - Sample minibatch of m noise samples $\{\mathbf{z}^{(1)}, .., \mathbf{z}^{(m)}\}$ from noise prior $p_g(\mathbf{z})$
      - Sample minibatch of m examples $\{\mathbf{x}^{(1)}, .., \mathbf{x}^{(m)}\}$ from data generating distribution $p_{data}(\mathbf{x})$
      - Update the discriminator by ascending its stochastic gradient:

$$\nabla_\theta \frac{1}{m} \sum_{i=1}^{m} \left[\log D_\theta\left(x^{(i)}\right) + \log\left(1 - D_\theta\left(G_\phi\left(z^{(i)}\right)\right)\right)\right]$$

    end for
    - Sample minibatch of m noise samples $\{\mathbf{z}^{(1)}, .., \mathbf{z}^{(m)}\}$ from noise prior $p_g(\mathbf{z})$
    - Update the generator by ascending its stochastic gradient

$$\nabla_\phi \frac{1}{m} \sum_{i=1}^{m} \left[\log\left(D_\theta\left(G_\phi\left(z^{(i)}\right)\right)\right)\right]$$

  end for
end procedure

Module 23.2: Generative Adversarial Networks - Architecture

- We will now look at one of the popular neural networks used for the generator and discriminator (Deep Convolutional GANs)
- For discriminator, any CNN based classifier with 1 class (real) at the output can be used (e.g. VGG, ResNet, etc.)

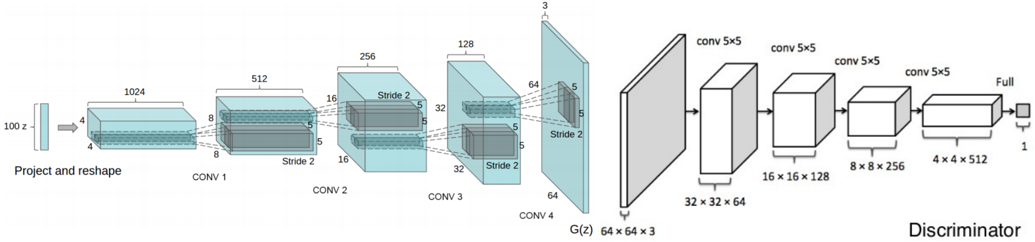


Figure: Generator (Redford et al 2015) (left) and discriminator (Yeh et al 2016) (right) used in DCGAN

Architecture guidelines for stable Deep Convolutional GANs

- Replace any pooling layers with strided convolutions (discriminator) and fractional-strided convolutions (generator).
- Use batchnorm in both the generator and the discriminator.
- Remove fully connected hidden layers for deeper architectures.
- Use ReLU activation in generator for all layers except for the output, which uses tanh.
- Use LeakyReLU activation in the discriminator for all layers

Module 23.3: Generative Adversarial Networks - The Math Behind it

- We will now delve a bit deeper into the objective function used by GANs and see what it implies
- Suppose we denote the true data distribution by $p_{data}(x)$ and the distribution of the data generated by the model as $p_G(x)$
- What do we wish should happen at the end of training?

$$p_G(x) = p_{data}(x)$$

- Can we prove this formally even though the model is not explicitly computing this density?
- We will try to prove this over the next few slides

Theorem

The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if and only if** $p_G = p_{data}$

is equivalent to

Theorem

1. **If** $p_G = p_{data}$ then the global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **and**
2. The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **only if** $p_G = p_{data}$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

(b) Find the value of $V(D, G)$ for other values of the generator *i.e.*, for any p_G such that $p_G \neq p_{data}$

(c) Show that $a < b \; \forall \; p_G \neq p_{data}$(and hence the minimum $V(D, G)$ is achieved when $p_G = p_{data}$)

The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **only if** $p_G = p_{data}$

- Show that when $V(D, G)$ is minimum then $p_G = p_{data}$

- First let us look at the objective function again

$$\min_{\phi} \max_{\theta} \quad [\mathbb{E}_{x\sim p_{data}} \log D_\theta(x) + \mathbb{E}_{z\sim p(z)} \log(1 - D_\theta(G_\phi(z)))]$$

- We will expand it to its integral form

$$\min_{\phi} \max_{\theta} \int_x p_{data}(x) \log D_\theta(x) + \int_z p(z) \log(1 - D_\theta(G_\phi(z)))$$

- Let $p_G(X)$ denote the distribution of the X's generated by the generator and since X is a function of z we can replace the second integral as shown below

$$\min_{\phi} \max_{\theta} \int_x p_{data}(x) \log D_\theta(x) + \int_x p_G(x) \log(1 - D_\theta(x))$$

- The above replacement follows from the *law of the unconscious statistician* (click to link of wikipedia page)

- Okay, so our revised objective is given by

$$\min_{\phi} \max_{\theta} \int_x \left(p_{data}(x) \log D_\theta(x) + p_G(x) \log(1 - D_\theta(x))\right) dx$$

- Given a generator G, we are interested in finding the optimum discriminator D which will maximize the above objective function
- The above objective will be maximized when the quantity inside the integral is maximized $\forall x$
- To find the optima we will take the derivative of the term inside the integral w.r.t. D and set it to zero

$$\begin{aligned}
\frac{d}{d(D_\theta(x))}\left(p_{data}(x) \log D_\theta(x) + p_G(x) \log(1 - D_\theta(x))\right) &= 0 \\
p_{data}(x)\frac{1}{D_\theta(x)} + p_G(x)\frac{1}{1 - D_\theta(x)}(-1) &= 0 \\
\frac{p_{data}(x)}{D_\theta(x)} &= \frac{p_G(x)}{1 - D_\theta(x)} \\
(p_{data}(x))(1 - D_\theta(x)) &= (p_G(x))(D_\theta(x)) \\
D_\theta(x) &= \frac{p_{data}(x)}{p_G(x) + p_{data}(x)}
\end{aligned}$$

- This means for any given generator

$$D_G^*(G(x)) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$$

- Now the if part of the theorem says "if $p_G = p_{data}$ ...."
- So let us substitute $p_G = p_{data}$ into $D_G^*(G(x))$ and see what happens to the loss functions

$$D_G^* = \frac{p_{data}}{p_{data} + p_G} = \frac{1}{2}$$

$$\begin{aligned}
V(G, D_G^*) &= \int_x p_{data}(x) \log D(x) + p_G(x) \log\left(1 - D(x)\right) dx \\
&= \int_x p_{data}(x) \log \frac{1}{2} + p_G(x) \log\left(1 - \frac{1}{2}\right) dx \\
&= \log 2 \int_x p_G(x) dx - \log 2 \int_x p_{data}(x) dx \\
&= -2 \log 2 \qquad = -\log 4
\end{aligned}$$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

(b) Find the value of $V(D, G)$ for other values of the generator *i.e.*, for any p_G such that $p_G \neq p_{data}$

(c) Show that $a < b \ \forall \ p_G \neq p_{data}$(and hence the minimum $V(D, G)$ is achieved when $p_G = p_{data}$)

The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **only if** $p_G = p_{data}$

- Show that when $V(D, G)$ is minimum then $p_G = p_{data}$

- So what we have proved so far is that if the generator is optimal ($p_G = p_{data}$) the discriminator's loss value is $-\log 4$
- We still haven't proved that this is the minima
- For example, it is possible that for some $p_G \neq p_{data}$, the discriminator's loss value is lower than $-\log 4$
- To show that the discriminator achieves its lowest value "if $p_G = p_{data}$", we need to show that for all other values of p_G the discriminator's loss value is greater than $-\log 4$

- To show this we will get rid of the assumption that $p_G = p_{data}$

$$
\begin{aligned}
C(G) &= \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log \left(1 - \frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right] dx \\
&= \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log \left(\frac{p_G(x)}{p_G(x) + p_{data}(x)} \right) + (\log 2 - \log 2)(p_{data} + p_G) \right] dx \\
&= -\log 2 \int_x (p_G(x) + p_{data}(x))\, dx \\
&\quad + \int_x \left[p_{data}(x) \left(\log 2 + \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right) + p_G(x) \left(\log 2 + \log \left(\frac{p_G(x)}{P p_G(x) + p_{data}(x)} \right) \right) \right] dx \\
&= -\log 2(1+1) \\
&\quad + \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{\frac{p_G(x) + p_{data}(x)}{2}} \right) + p_G(x) \log \left(\frac{p_G(x)}{\frac{p_G(x) + p_{data}(x)}{2}} \right) \right] dx \\
&= -\log 4 + KL \left(p_{data} \| \frac{p_G(x) + p_{data}(x)}{2} \right) + KL \left(p_G \| \frac{p_G(x) + p_{data}(x)}{2} \right)
\end{aligned}
$$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

(b) Find the value of $V(D, G)$ for other values of the generator *i.e.*, for any p_G such that $p_G \neq p_{data}$

(c) Show that $a < b \ \forall \ p_G \neq p_{data}$(and hence the minimum $V(D, G)$ is achieved when $p_G = p_{data}$)

The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **only if** $p_G = p_{data}$

- Show that when $V(D, G)$ is minimum then $p_G = p_{data}$

- Okay, so we have

$$C(G) = -\log 4 + KL\left(p_{data}||\frac{p_{data}+p_g}{2}\right) + KL\left(p_G||\frac{p_{data}+p_G}{2}\right)$$

- We know that KL divergence is always ≥ 0

$$\therefore C(G) \geq -\log 4$$

- Hence the minimum possible value of $C(G)$ is $-\log 4$
- But this is the value that $C(G)$ achieves when $p_G = p_{data}$ (and this is exactly what we wanted to prove)
- We have, thus, proved the **if part** of the theorem

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

(b) Find the value of $V(D, G)$ for other values of the generator *i.e.*, for any p_G such that $p_G \neq p_{data}$

(c) Show that $a < b \; \forall \; p_G \neq p_{data}$(and hence the minimum $V(D, G)$ is achieved when $p_G = p_{data}$)

The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **only if** $p_G = p_{data}$

- Show that when $V(D, G)$ is minimum then $p_G = p_{data}$

- Now let's look at the other part of the theorem
  If the global minimum of the virtual training criterion $C(G) = \underset{D}{max}\ V(G, D)$ is achieved then $p_G = p_{data}$
- We know that

$$C(G) = -\log 4 + KL\left(p_{data} \| \frac{p_{data} + p_g}{2}\right) + KL\left(p_G \| \frac{p_{data} + p_G}{2}\right)$$

- If the global minima is achieved then $C(G) = -\log 4$ which implies that

$$KL\left(p_{data} \| \frac{p_{data} + p_g}{2}\right) + KL\left(p_G \| \frac{p_{data} + p_G}{2}\right) = 0$$

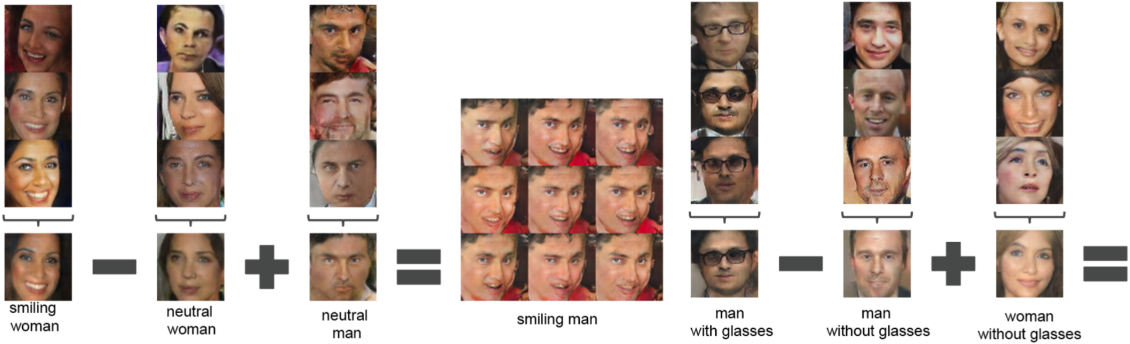
- This will happen only when $p_G = p_{data}$ (you can prove this easily)
- In fact $KL\left(p_{data} \| \frac{p_{data}+p_g}{2}\right) + KL\left(p_G \| \frac{p_{data}+p_G}{2}\right)$ is the Jenson-Shannon divergence between p_G and p_{data}

$$KL\left(p_{data} \| \frac{p_{data} + p_g}{2}\right) + KL\left(p_G \| \frac{p_{data} + p_G}{2}\right) = JSD(p_{data} \| p_G)$$

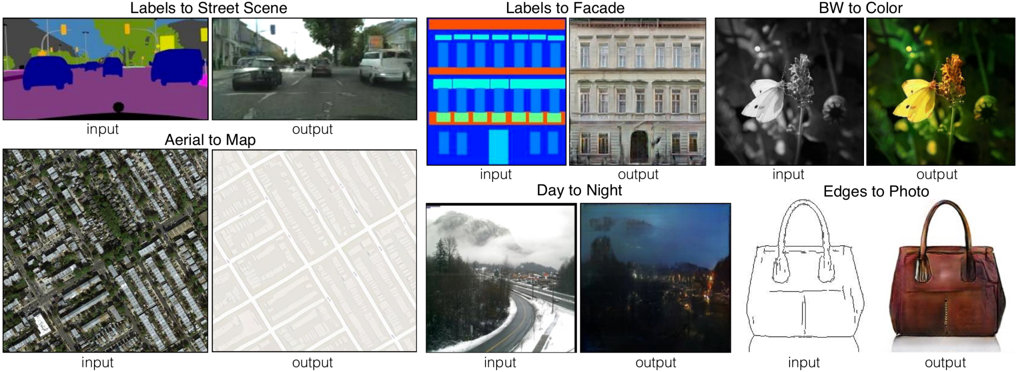
  which is minimum only when $p_G = p_{data}$

Module 23.4: Generative Adversarial Networks - Some Cool Stuff and Applications

- In each row the first image was generated by the network by taking a vector z_1 as the input and the last images was generated by a vector z_2 as the input
- All intermediate images were generated by feeding z's which were obtained by interpolating z_1 and z_2 ($z = \lambda z_1 + (1 - \lambda) z_2$)
- As we transition from z_1 to z_2 in the input space there is a corresponding smooth transition in the image space also



- The first 3 images in the first column were generated by feeding some z_{11}, z_{12}, z_{13} respectively as the input to the generator
- The fourth image was generated by taking an average of $z_1 = z_{11}, z_{12}, z_{13}$ and feeding it to the generator
- Similarly we obtain the average vectors z_2 and z_3 for the 2nd and 3rd columns
- If we do a simple vector arithmetic on these averaged vectors then we see the



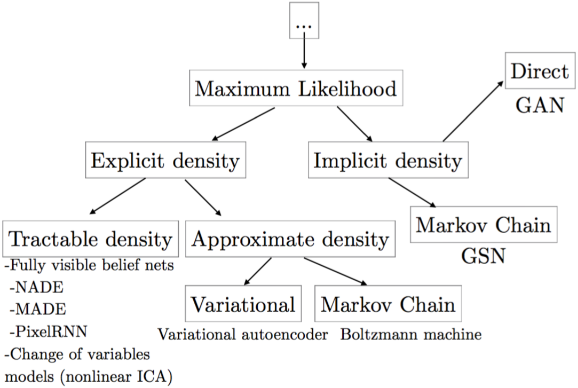
Phillip Isola, Jun-Yan Zhu, Tinghui Zhou, Alexei A. Efros, Image-to-Image Translation with Conditional Adversarial Networks, CVPR, 2017.

Module 23.5: Bringing it all together (the deep generative summary)

	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes
Compute P(X)	Intractable	Intractable	Tractable	No
Sampling	MCMC	Fast	Slow	Fast
Type of GM	Undirected	Directed	Directed	Directed
Loss	KL-divergence	KL-divergence	KL-divergence	JSD
Assumptions	X independent given z	X independent given z	None	N.A.
Samples	Bad	Ok	Good	Good (best)

Table: Comparison of Generative Models

Recent works look at combining these methods: e.g. Adversarial Autoencoders (Makhzani 2015), PixelVAE (Gulrajani 2016) and PixelGAN Autoencoders (Makhzani 2017)



Source: Ian Goodfellow, NIPS 2016 Tutorial: Generative Adversarial Networks