Module 21.2: Variational Autoencoders: The Neural Network Perspective

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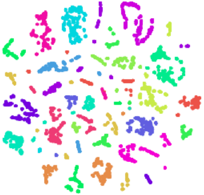

Abstraction

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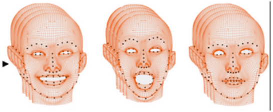

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- We are also interested in generation (*i.e.*, given a hidden representation generate an X)

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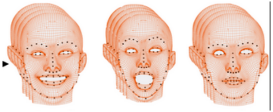

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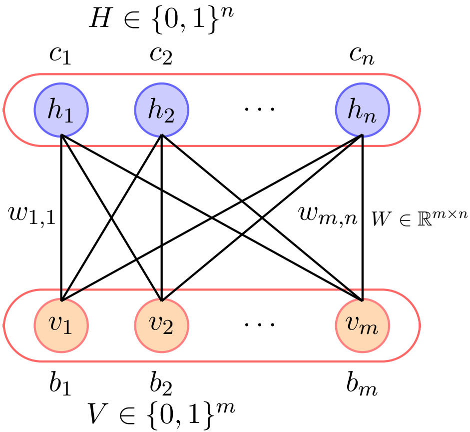
- We are interested in learning an abstraction (i.e., given an X find the hidden representation z)
- We are also interested in generation (*i.e.*, given a hidden representation generate an X)
- In probabilistic terms we are interested in $P(z|X)$ and $P(X|z)$ (to be consistent with the literation on VAEs we will use z instead of H and X instead of V)



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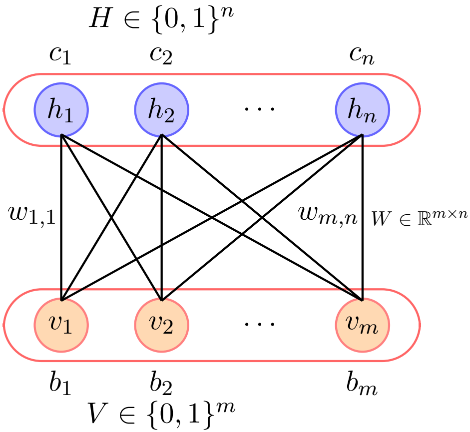

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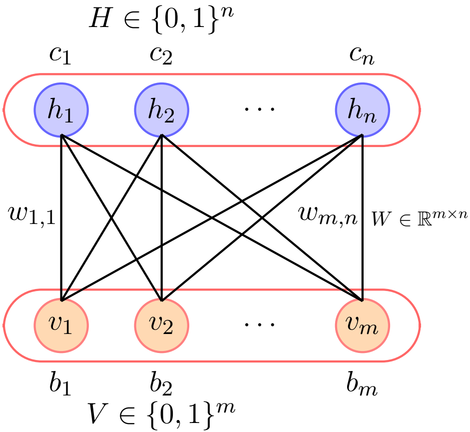

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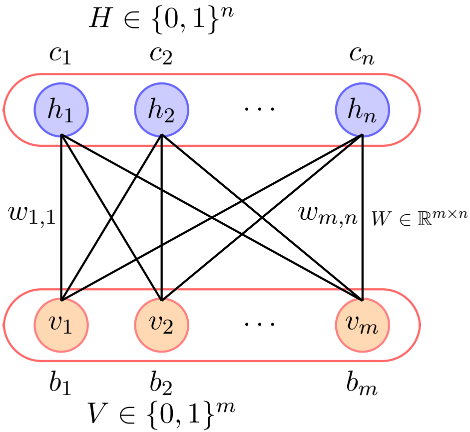

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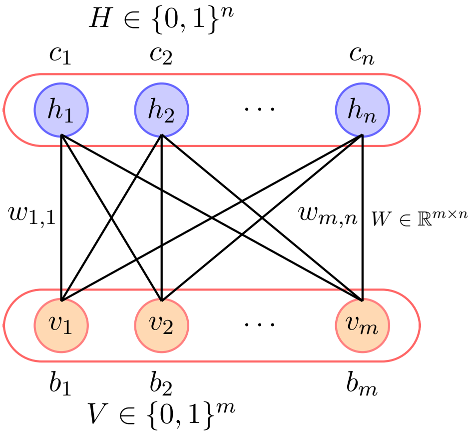

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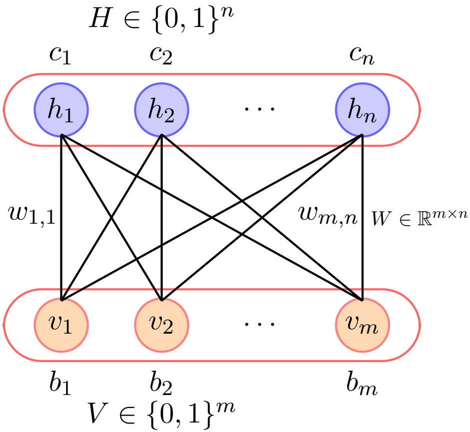

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- (Nothing wrong with the above but we just mention them to make the reader aware of these characteristics)

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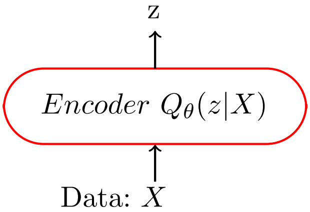

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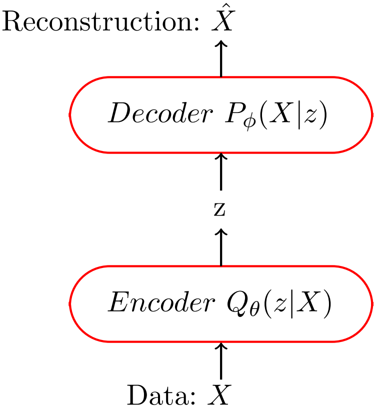

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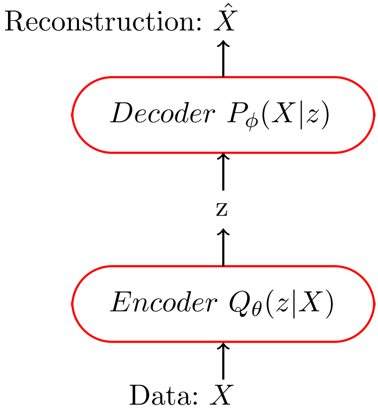

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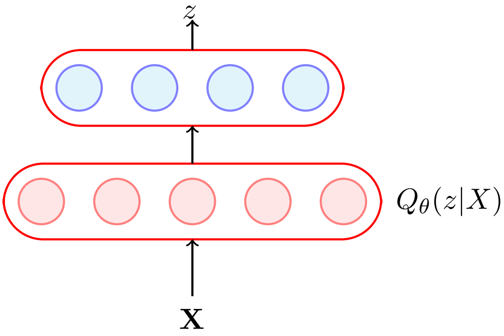
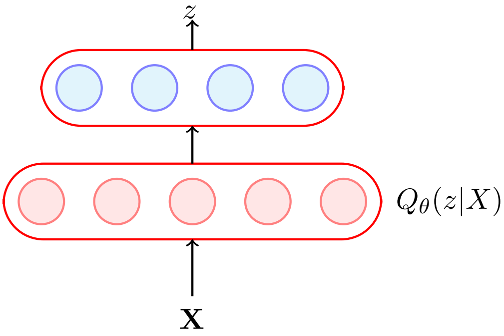
- We will look at the encoder first



- **Encoder:** What do we mean when we say we want to learn a distribution?



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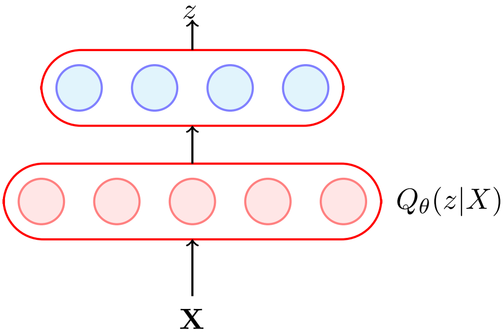

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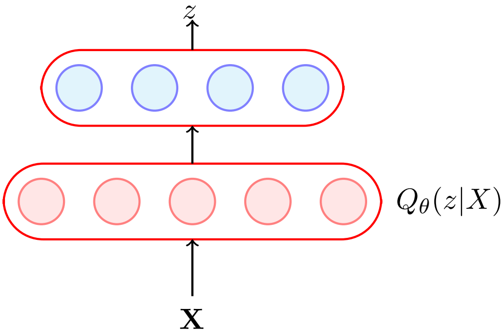

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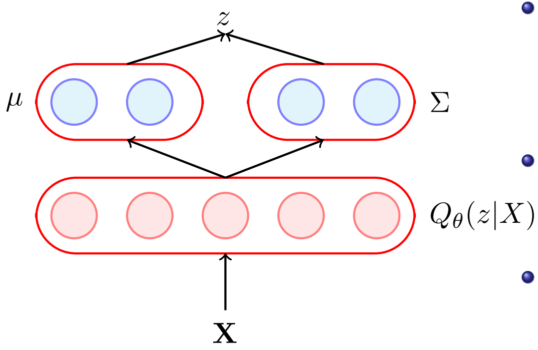
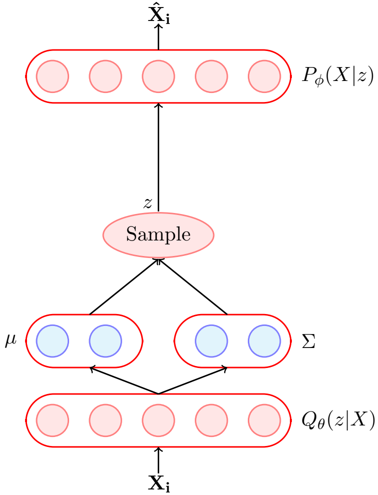


$X \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ and $\Sigma \in \mathbb{R}^{m \times m}$

- **Encoder:** What do we mean when we say we want to learn a distribution? We mean that we want to learn the parameters of the distribution
- But what are the parameters of $Q(z|X)$? Well it depends on our modeling assumption!
- In VAEs we assume that the latent variables come from a standard normal distribution $\mathcal{N}(0, I)$ and the job of the encoder is to then predict the parameters of this distribution



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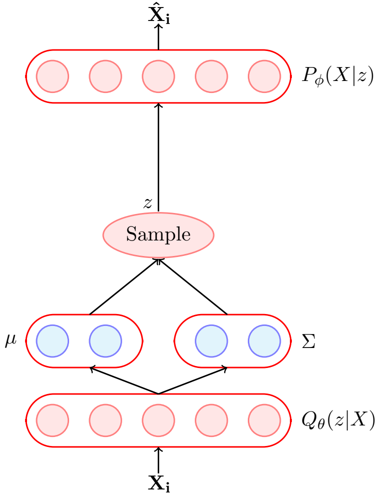

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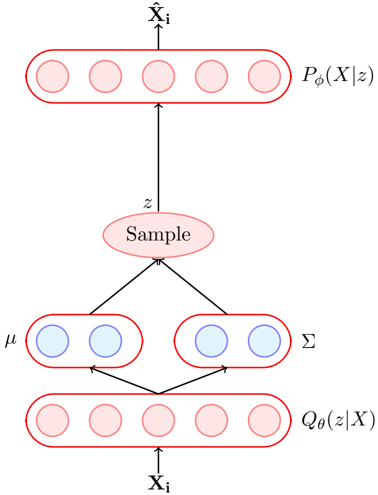

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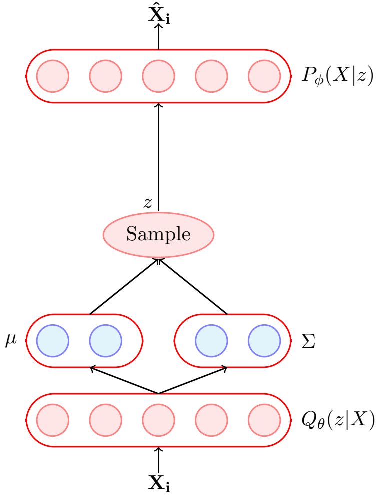

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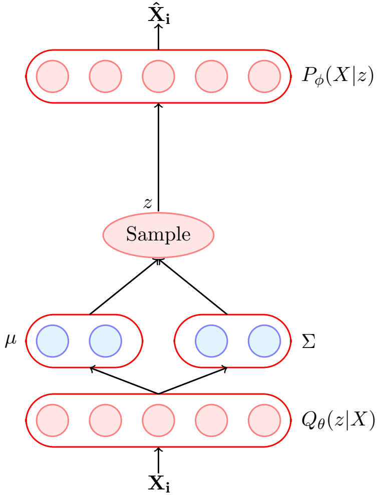

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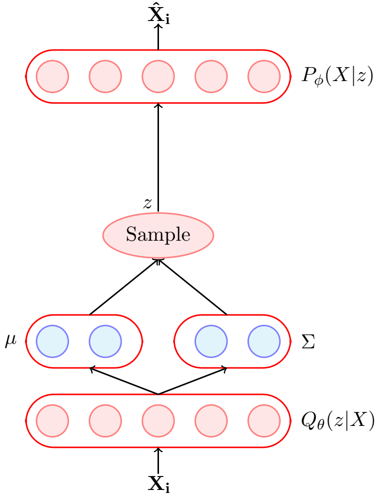

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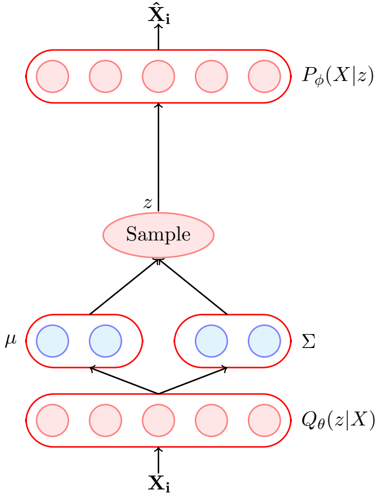
- The job of the decoder f would then be to predict the mean of this distribution as $f_\phi(z)$



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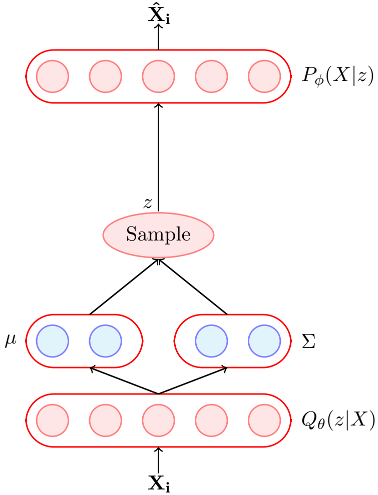

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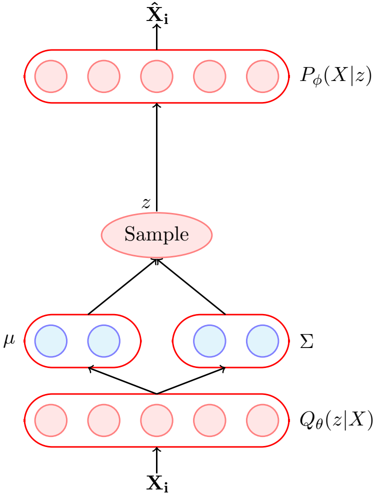

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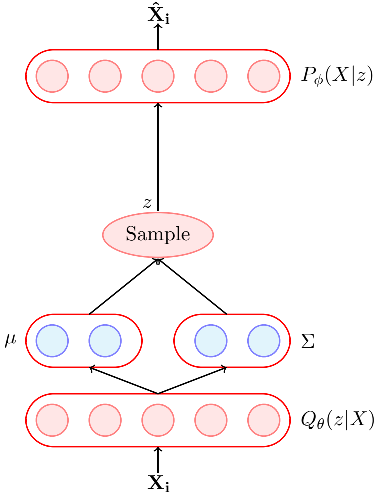

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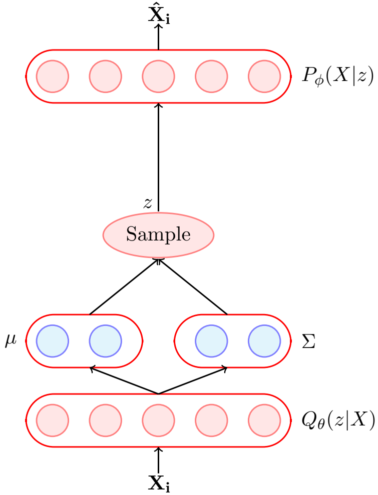
- (As usual we take log for numerical stability)



- This is the loss function for one data point ($l_i(\theta)$) and we will just sum over all the data points to get the total loss $\mathscr{L}(\theta)$

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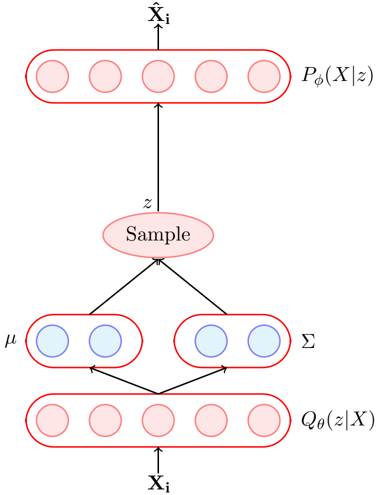

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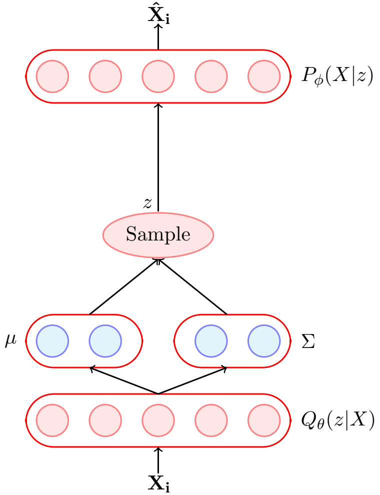

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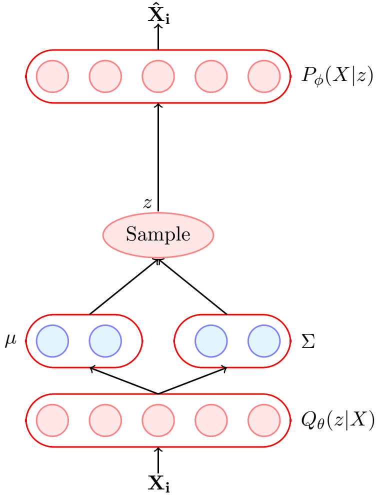

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- Thus, we will modify the loss function such that

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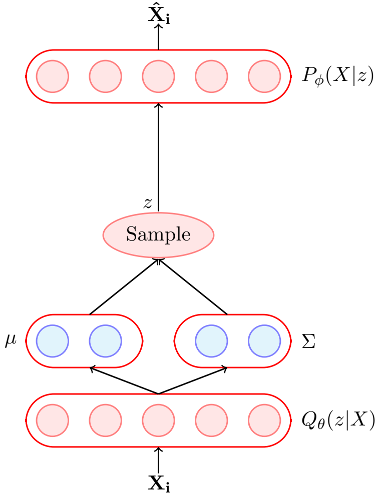

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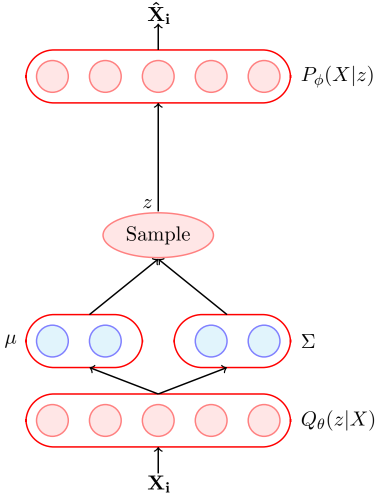

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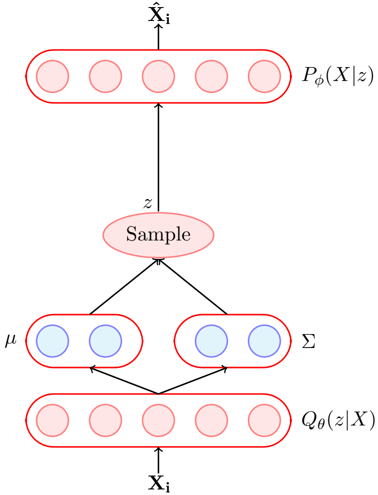

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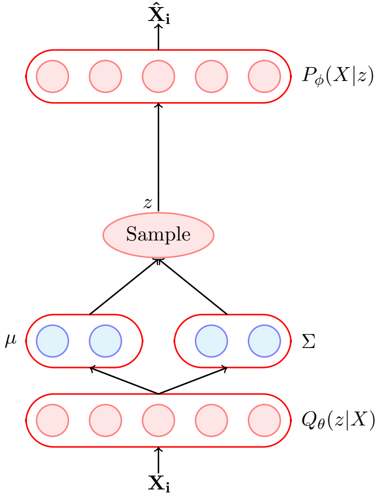

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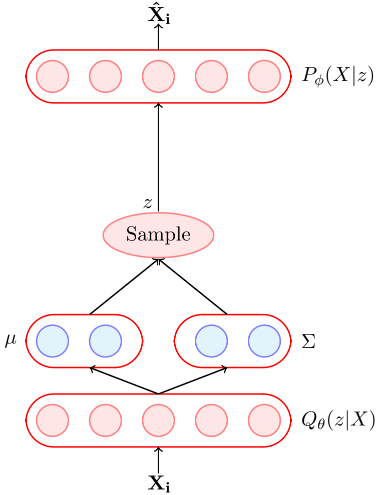

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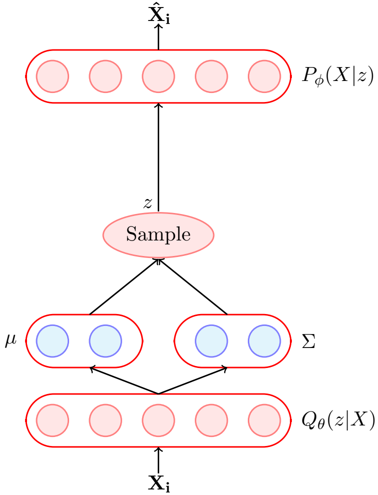

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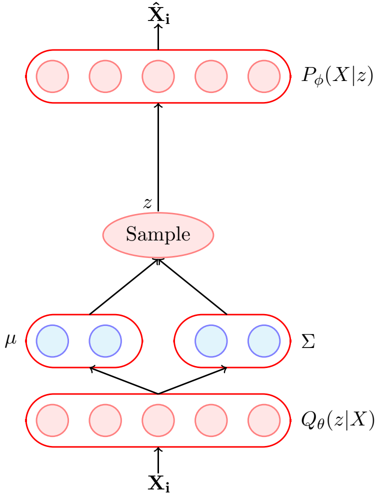

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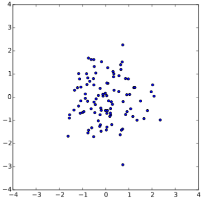
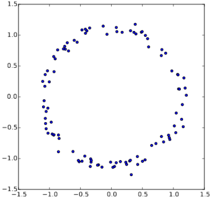
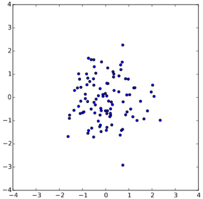
- But why do we choose a normal distribution? Isn't it too simplistic to assume that z follows a normal distribution

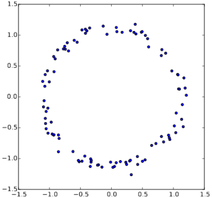




$$l_i(\theta, \phi) = -\mathbb{E}_{z\sim Q_\theta(z|x_i)}[\log P_\phi(x_i|z)] + KL(Q_\theta(z|x_i)||P(z))$$

- Isn't it a very strong assumption that $P(z) \sim \mathcal{N}(0, I)$?





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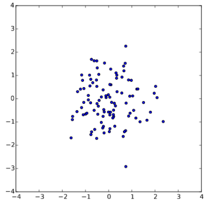
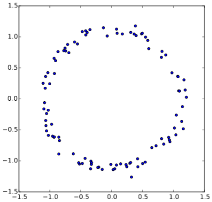
- For example, in the 2-dimensional case how can we be sure that $P(z)$ is a normal distribution and not any other distribution





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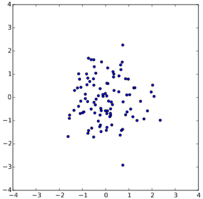

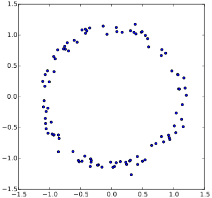

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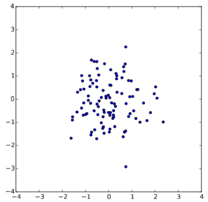

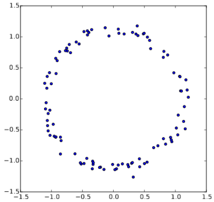

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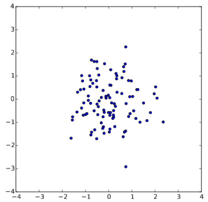
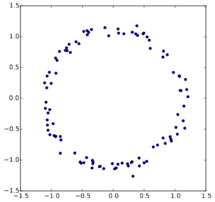
- Step 1: Start with a set of d variables that are normally distributed (that's exactly what we are assuming for $P(z)$)
- Step 2: Mapping these variables through a sufficiently complex function (that's exactly what the first few layers of the decoder can do)





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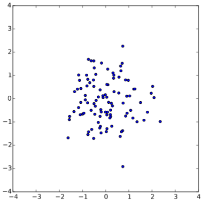
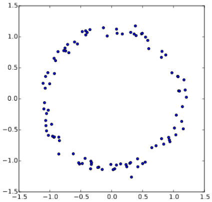
- In particular, note that in the adjoining example if z is 2-D and normally distributed then $f(z)$ is roughly ring shaped (giving us the distribution in the bottom figure)

$$f(z) = \frac{z}{10} + \frac{z}{||z||}$$





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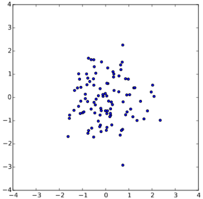

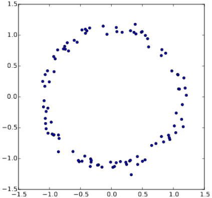

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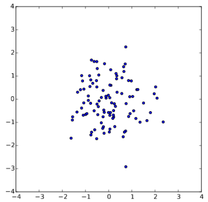

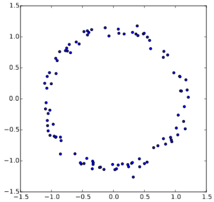

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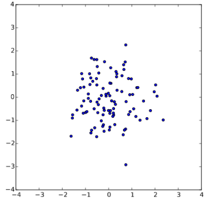

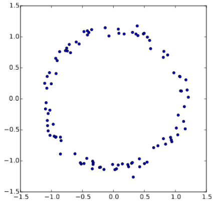

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- The above argument suggests that even if we start with normally distributed variables the initial layers of the decoder could learn a complex transformation of these variables say $f_\phi(z)$ if required
- The objective function of the decoder will ensure that an appropriate transformation of z is learnt to reconstruct X