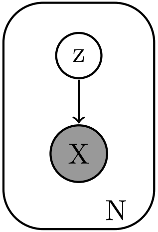
Module 21.3: Variational autoencoders: (The graphical model perspective)



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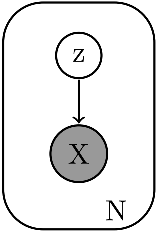

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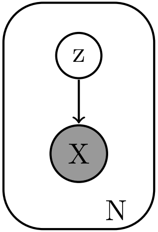

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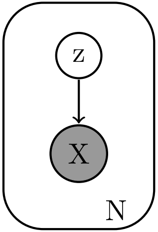

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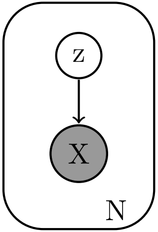

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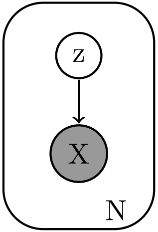
- And of course, unlike RBMs, this is a directed graphical model



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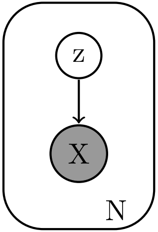

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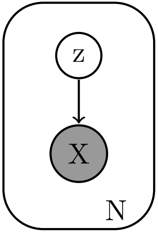

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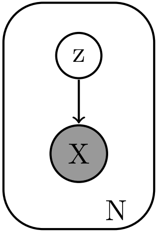

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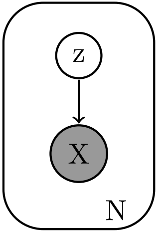

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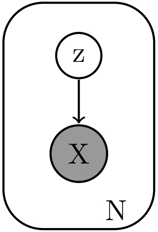
- VAEs, on the other hand, cast this into an optimization problem and learn the parameters of the optimization problem



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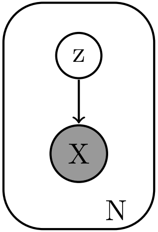

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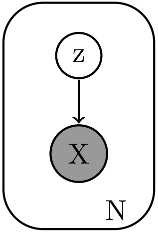

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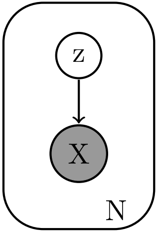

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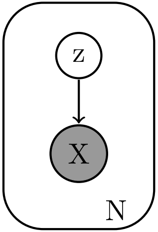
- Our job then is to learn the parameters of this neural network



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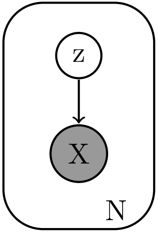

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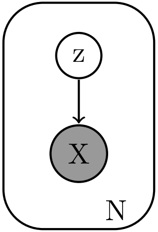

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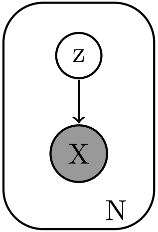

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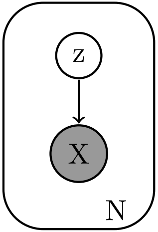

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- What are the parameters of the objective function ? (they are the parameters of the neural network - we will return back to this again)

- Let us expand the KL divergence term

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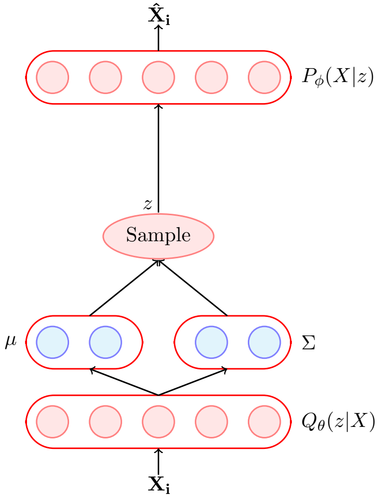
- Why is this any easier? It is easy because of certain assumptions that we make as discussed on the next slide



- First we will just reintroduce the parameters in the equation to make things explicit

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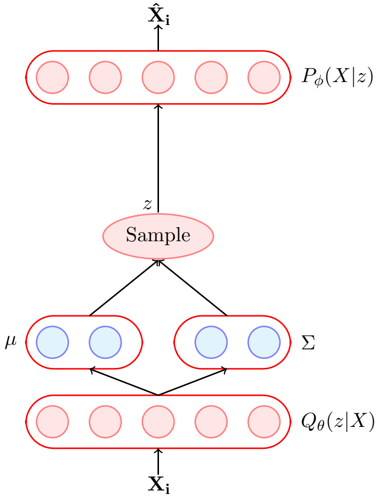

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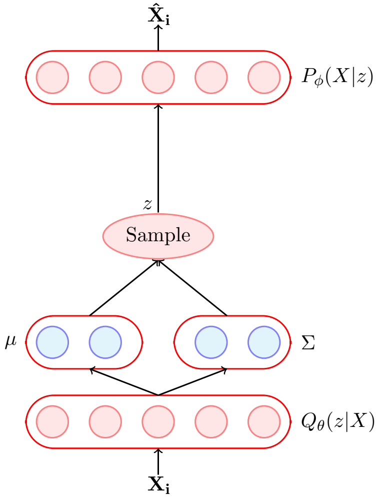

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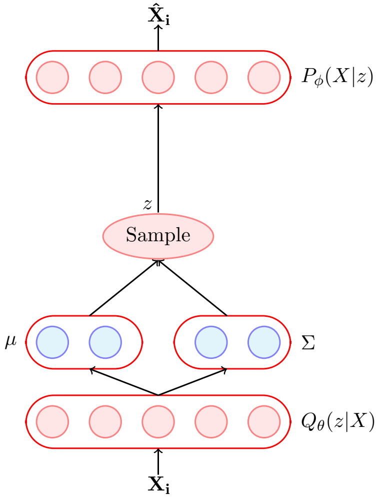

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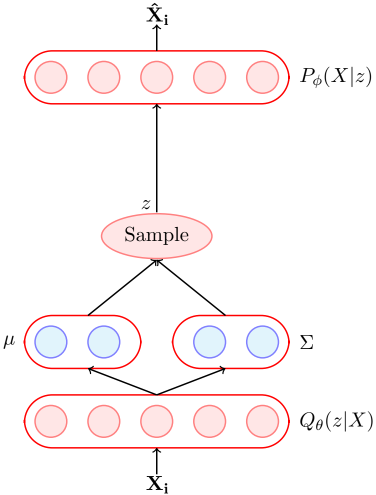
- We will shorthand $P(X = x_i)$ as $P(x_i)$



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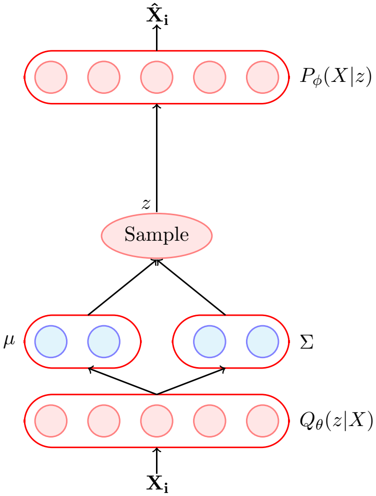
- We will shorthand $P(X = x_i)$ as $P(x_i)$
- However, we will assume that we are using stochastic gradient descent so we need to deal with only one of the terms in the summation corresponding to the current training example



- So our objective function w.r.t. one example is

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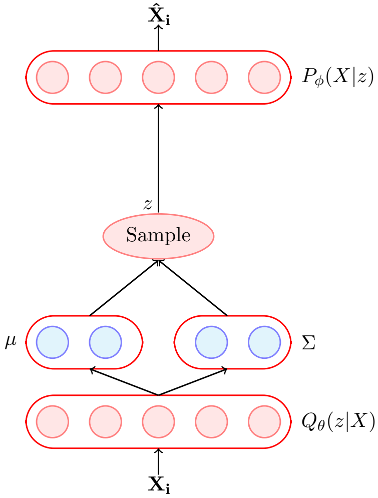

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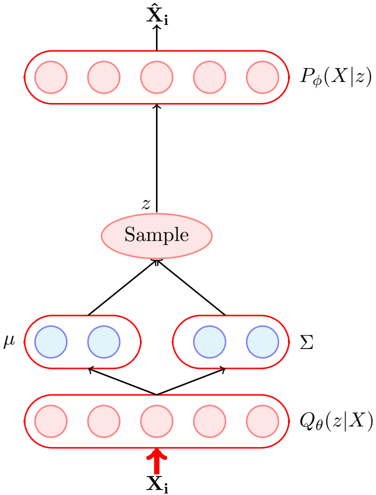

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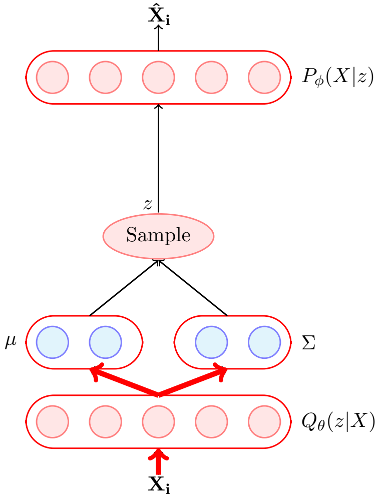

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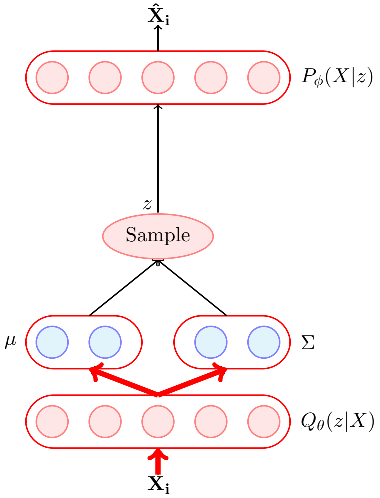

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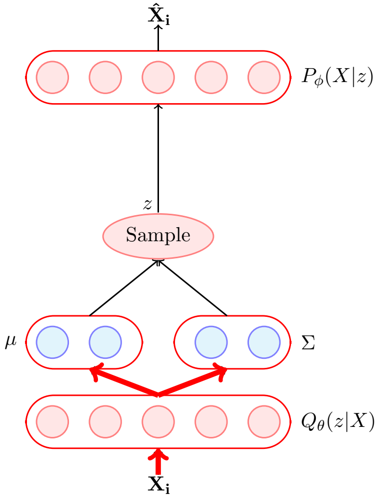
- Now, first we will do a forward prop through the encoder using X_i and compute $\mu(X)$ and $\Sigma(X)$
- The second term in the above objective function is the difference between two normal distribution $\mathcal{N}(\mu(X), \Sigma(X))$ and $\mathcal{N}(0, I)$



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$$D[\mathcal{N}(\mu(X), \Sigma(X))||\mathcal{N}(0, I)]$$
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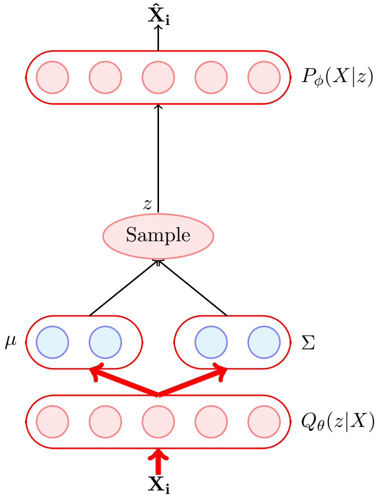
where k is the dimensionality of the latent variables



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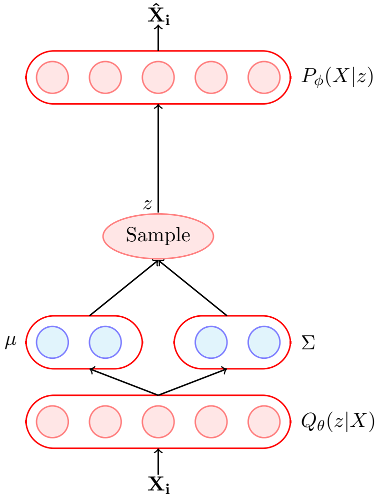
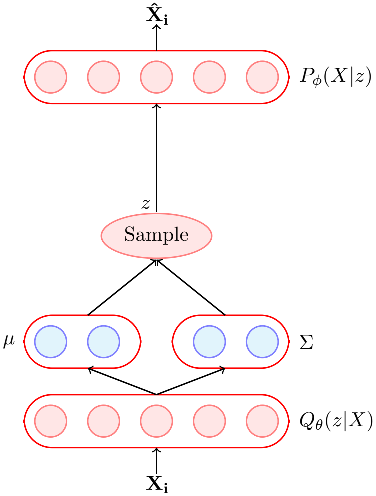
where k is the dimensionality of the latent variables

- This term can be computed easily because we have already computed $\mu(X)$ and $\Sigma(X)$ in the forward pass



- Now let us look at the other term in the objective function

$$\sum_{i=1}^{n} \mathbb{E}_Q[\log P_\phi(X|z)]$$



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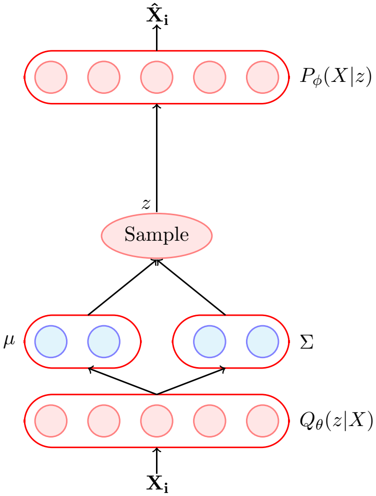

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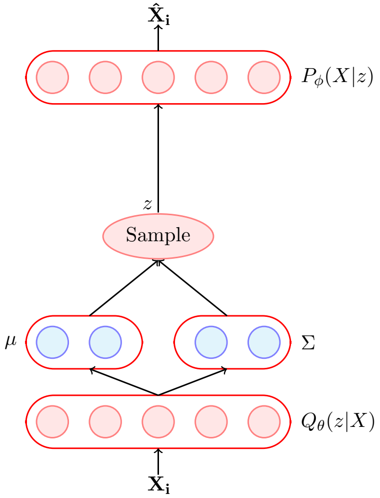

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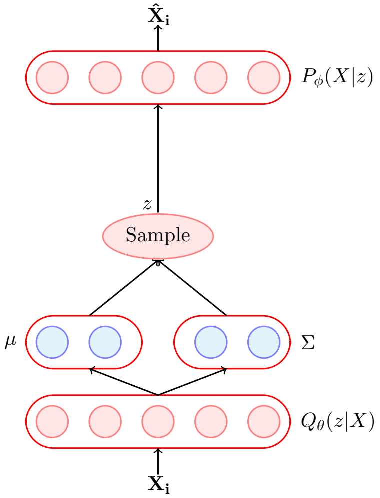
- Hence this term is also easy to compute (of course it is a nasty approximation but we will live with it!)



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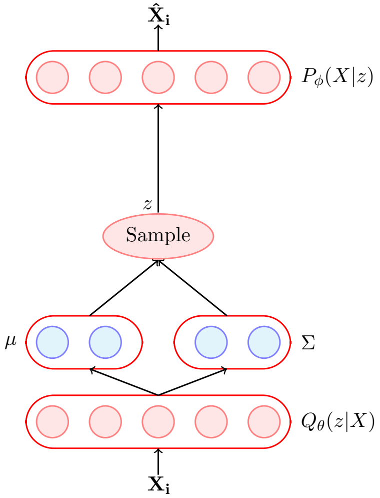

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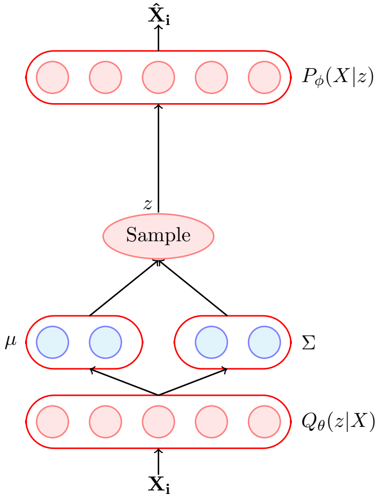
- For example, if we assume that $P(X|z)$ is a Gaussian with mean $\mu(z)$ and variance I then

$$\log P(X = X_i|z) = C - \frac{1}{2}||X_i - \mu(z)||^2$$



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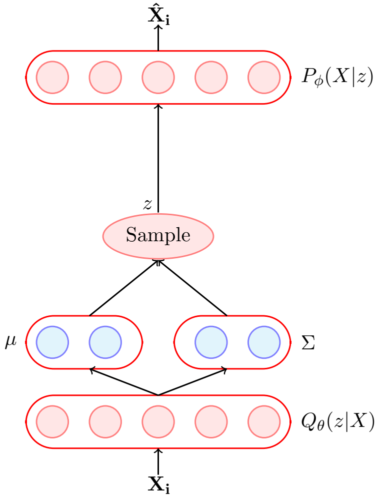

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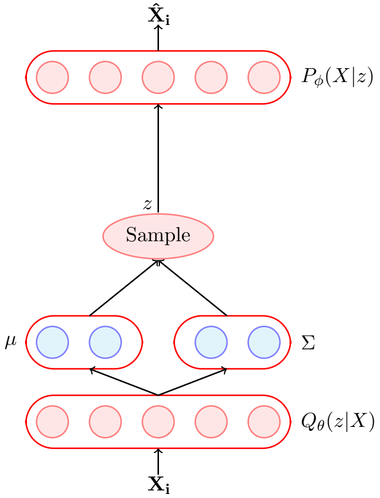

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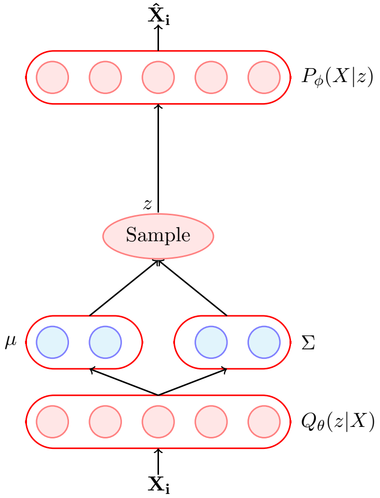
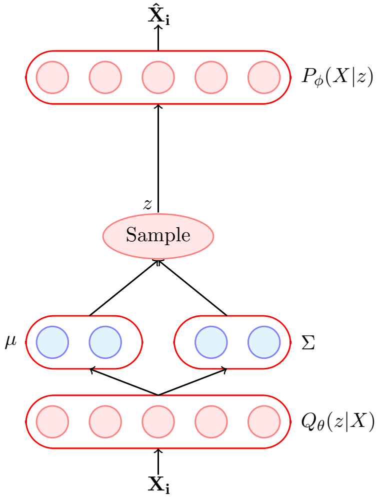
- Our effective objective function thus becomes

$$\underset{\theta,\phi}{minimize} \sum_{n=1}^{N} \Big[\frac{1}{2}(tr(\Sigma(X_i)) + (\mu(X_i))^T[\mu(X_i)) - k - \log det(\Sigma(X_i))] + ||X_i - f_\phi(z)||^2\Big]$$



- The above loss can be easily computed and we can update the parameters θ of the encoder and ϕ of decoder using backpropagation



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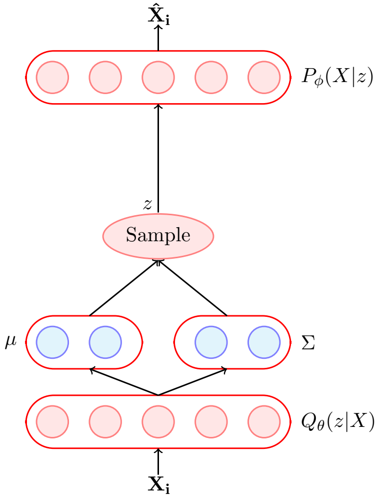

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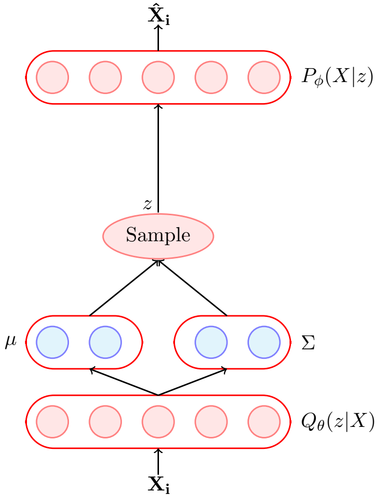

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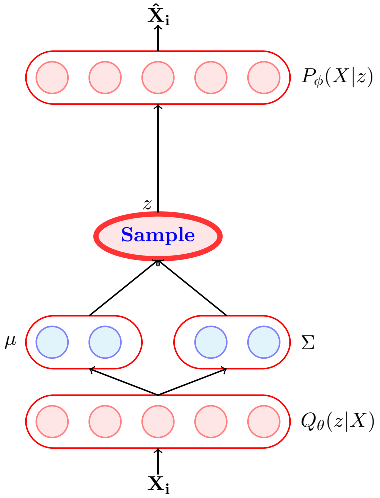
- Why?



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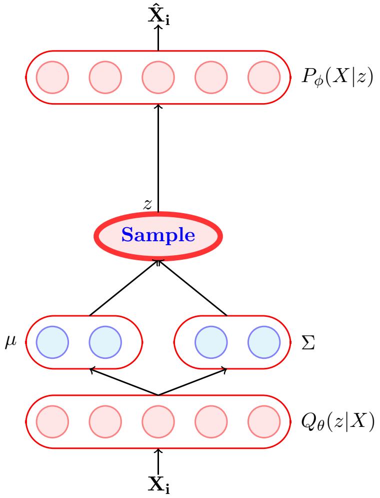

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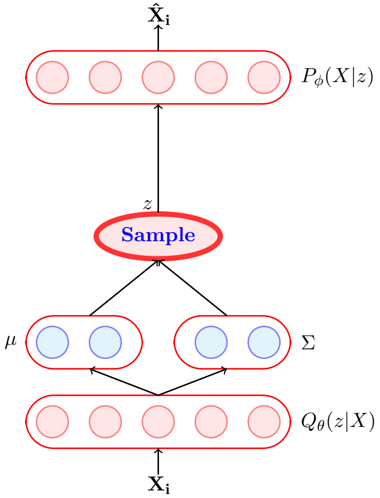
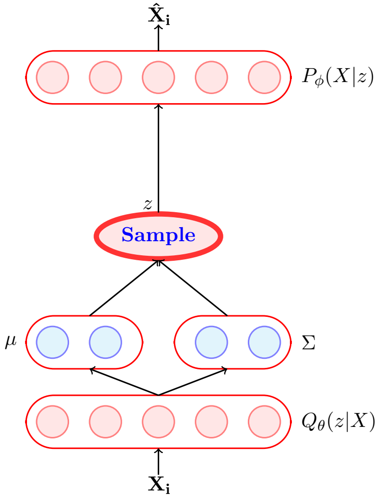
- Why? because after passing X through the network we simply compute $\mu(X)$ and $\Sigma(X)$ and then sample a z to be fed to the decoder
- This makes the entire process non-deterministic and hence $f_\phi(z)$ is not a continuous function of the input X

- VAEs use a neat trick to get around this problem





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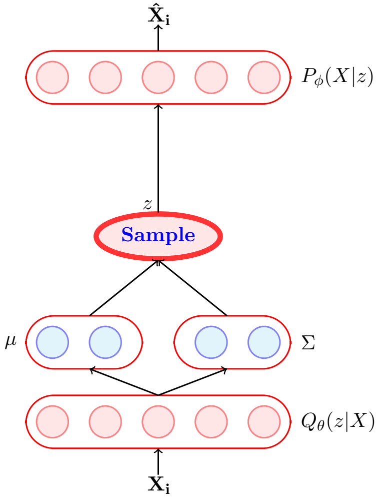

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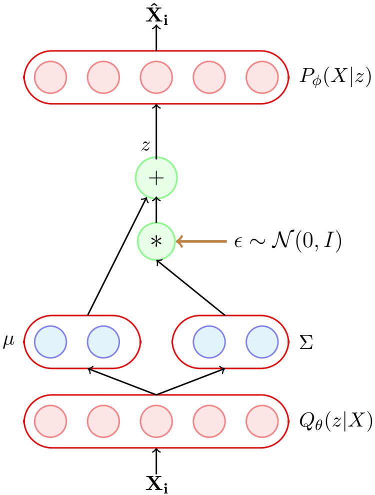
- For 1 dimensional case, given μ and σ we can sample from $\mathcal{N}(\mu, \sigma)$ by first sampling $\epsilon \sim \mathcal{N}(0, 1)$, and then computing

$$z = \mu + \sigma * \epsilon$$



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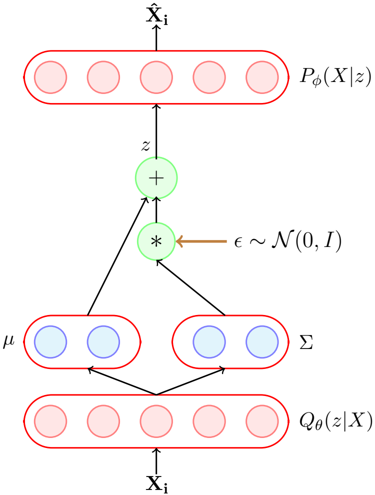

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- The adjacent figure shows the difference between the original network and the reparamterized network
- The randomness in $f_\phi(z)$ is now associated with ϵ and not X or the parameters of the model

- With that we are done with the process of training VAEs

- **Data:** $\{X_i\}_{i=1}^N$
- **Model:** $\hat{X} = f_\phi(\mu(X) + \Sigma(X) * \epsilon)$
- **Parameters:** θ, ϕ
- **Algorithm:** Gradient descent
- **Objective:**

$$\sum_{n=1}^{N} \left[\frac{1}{2}(tr(\Sigma(X_i)) + (\mu(X_i))^T [\mu(X_i)) - k - \log det(\Sigma(X_i))] + ||X_i - f_\phi(z)||^2\right]$$

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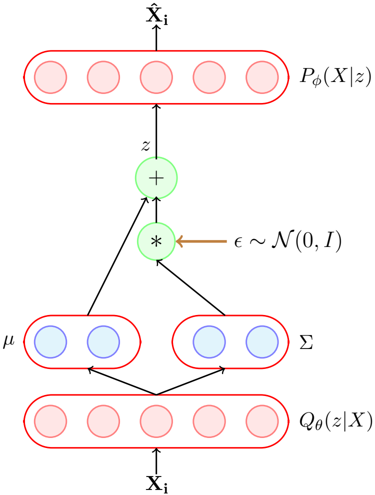
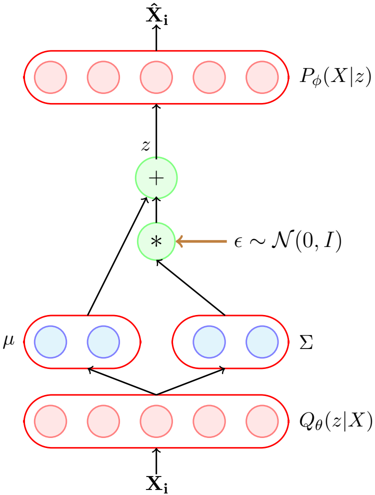
- Let us look at each of these goals

Abstraction



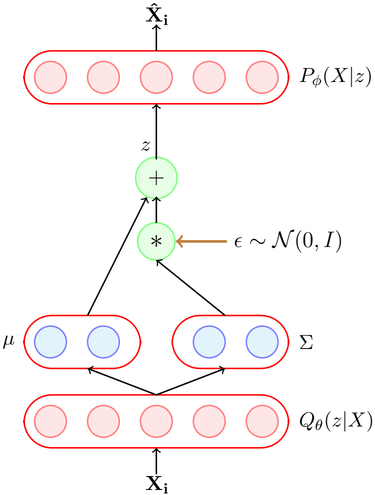
- After the model parameters are learned we feed a X to the encoder

Abstraction

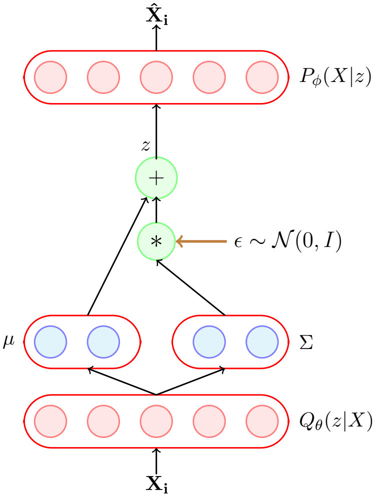


- After the model parameters are learned we feed a X to the encoder
- By doing a forward pass using the learned parameters of the model we compute $\mu(X)$ and $\Sigma(X)$

Abstraction



- After the model parameters are learned we feed a X to the encoder
- By doing a forward pass using the learned parameters of the model we compute $\mu(X)$ and $\Sigma(X)$
- We then sample a z from the distribution $\mu(X)$ and $\Sigma(X)$ or using the same reparameterization trick



Abstraction

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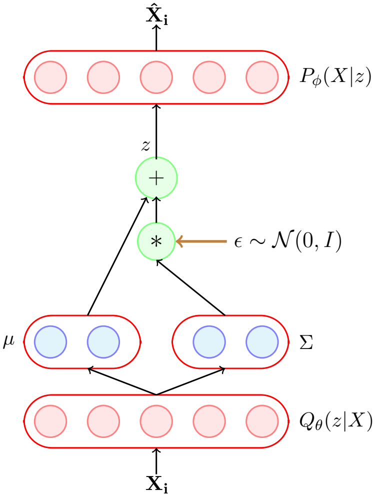
- By doing a forward pass using the learned parameters of the model we compute $\mu(X)$ and $\Sigma(X)$
- We then sample a z from the distribution $\mu(X)$ and $\Sigma(X)$ or using the same reparameterization trick
- In other words, once we have obtained $\mu(X)$ and $\Sigma(X)$, we first sample $\epsilon \sim \mathcal{N}(\mu(X), \Sigma(X))$ and then compute z

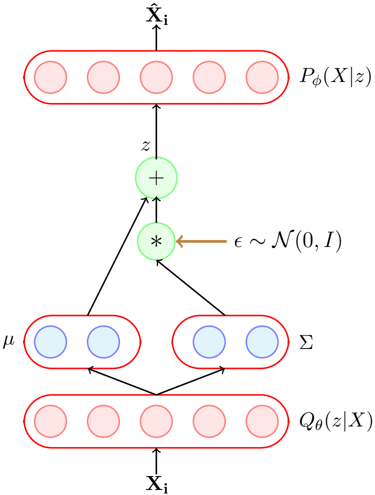
$$z = \mu + \sigma * \epsilon$$



Generation

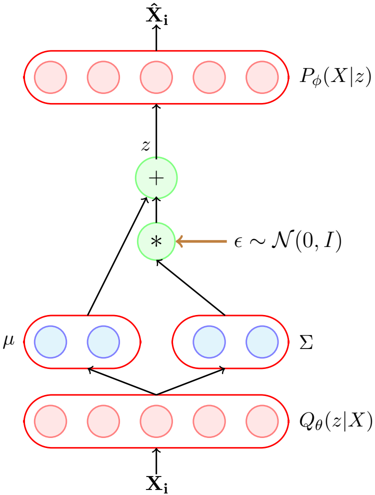
- After the model parameters are learned we remove the encoder and feed a $z \sim \mathcal{N}(0, I)$ to the decoder

Generation



- After the model parameters are learned we remove the encoder and feed a $z \sim \mathcal{N}(0, I)$ to the decoder
- The decoder will then predict $f_\phi(z)$ and we can draw an $X \sim \mathcal{N}(f_\phi(z), I)$

Generation



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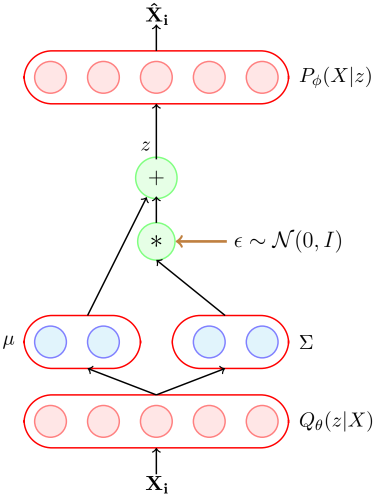

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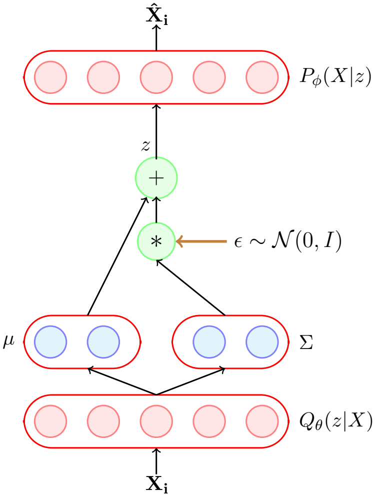
- The decoder will then predict $f_\phi(z)$ and we can draw an $X \sim \mathcal{N}(f_\phi(z), I)$
- Why would this work ?
- Well, we had trained the model to minimize $D(Q_\theta(z|X)||p(z))$ where $p(z)$ was $\mathcal{N}(0, I)$

Generation



- After the model parameters are learned we remove the encoder and feed a $z \sim \mathcal{N}(0, I)$ to the decoder
- The decoder will then predict $f_\phi(z)$ and we can draw an $X \sim \mathcal{N}(f_\phi(z), I)$
- Why would this work ?
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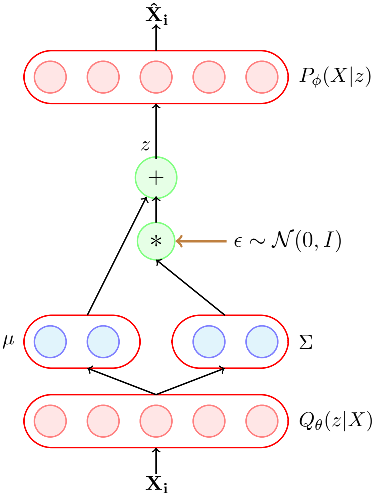

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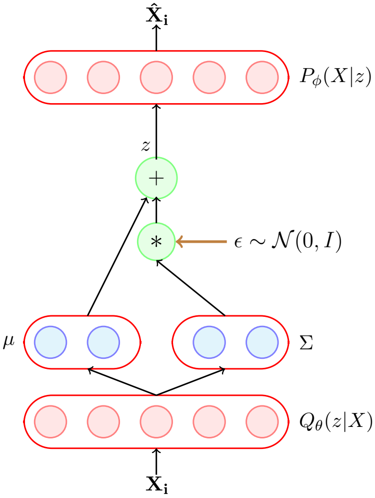
Generation



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- Hence this will work !