

Module 4.4: Backpropagation (Intuition)

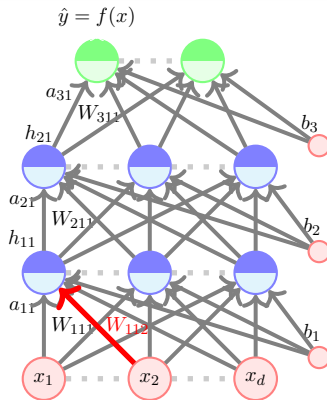
We need to answer two questions

- How to choose the loss function $\mathcal{L}(\theta)$?
- How to compute $\nabla\theta$ which is composed of:
 $\nabla W_1, \nabla W_2, \dots, \nabla W_{L-1} \in \mathbb{R}^{n \times n}, \nabla W_L \in \mathbb{R}^{n \times k}$
 $\nabla b_1, \nabla b_2, \dots, \nabla b_{L-1} \in \mathbb{R}^n$ and $\nabla b_L \in \mathbb{R}^k$?

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- Let us focus on this one weight (W_{112}).



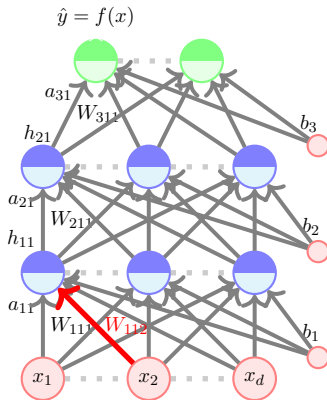
Algorithm: gradient descent()

```

 $t \leftarrow 0;$ 
 $max\_iterations \leftarrow$ 
  1000;
Initialize  $\theta_0;$ 
while
   $t++ < max\_iterations$ 
  do
     $\theta_{t+1} \leftarrow \theta_t - \eta \nabla \theta_t;$ 
  end

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- To learn this weight using SGD we need a formula for $\frac{\partial \mathcal{L}(\theta)}{\partial W_{112}}$.



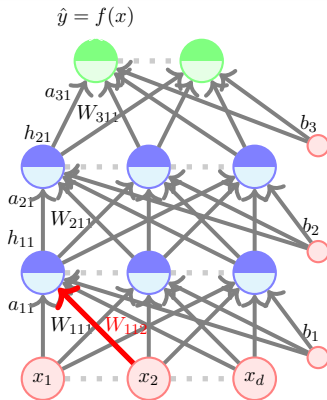
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- To learn this weight using SGD we need a formula for $\frac{\partial \mathcal{L}(\theta)}{\partial W_{112}}$.
- We will see how to calculate this.



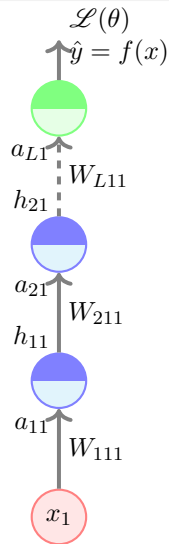
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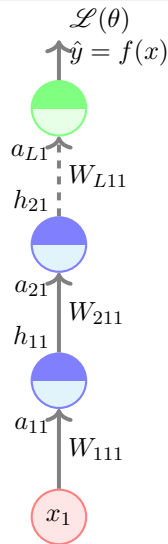
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- First let us take the simple case when we have a deep but thin network.

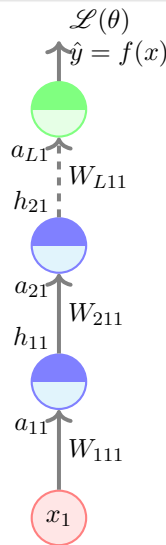


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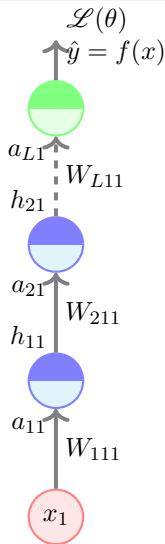
$$\frac{\partial \mathcal{L}(\theta)}{\partial W_{111}} = \frac{\partial \mathcal{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_{L11}} \frac{\partial a_{L11}}{\partial h_{21}} \frac{\partial h_{21}}{\partial a_{21}} \frac{\partial a_{21}}{\partial h_{11}} \frac{\partial h_{11}}{\partial a_{11}} \frac{\partial a_{11}}{\partial W_{111}}$$



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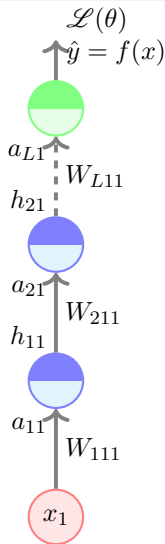
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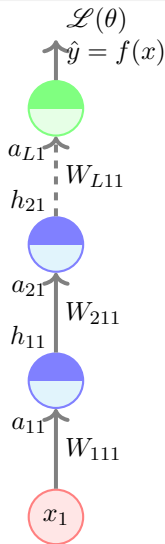
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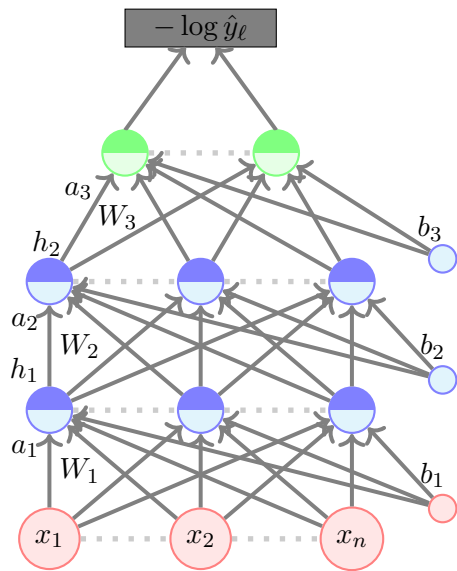
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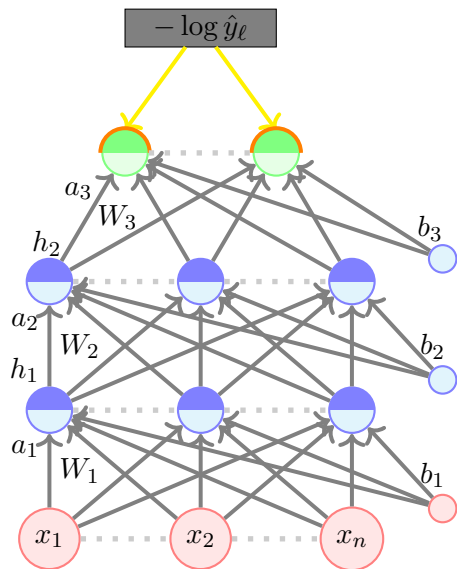


Let us see an intuitive explanation of backpropagation before we get into the mathematical details

- We get a certain loss at the output and we try to figure out who is responsible for this loss

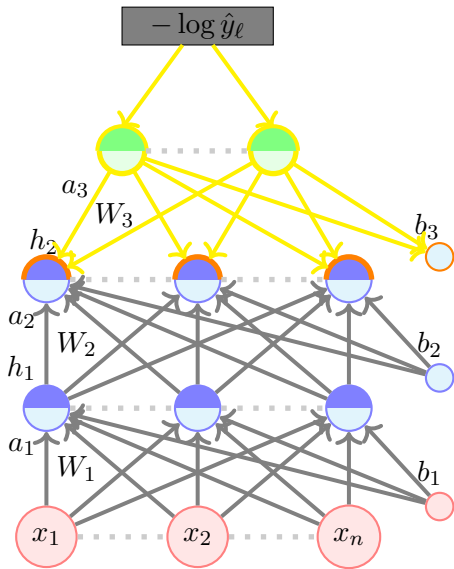


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- So, we talk to the output layer and say “Hey! You are not producing the desired output, better take responsibility”.

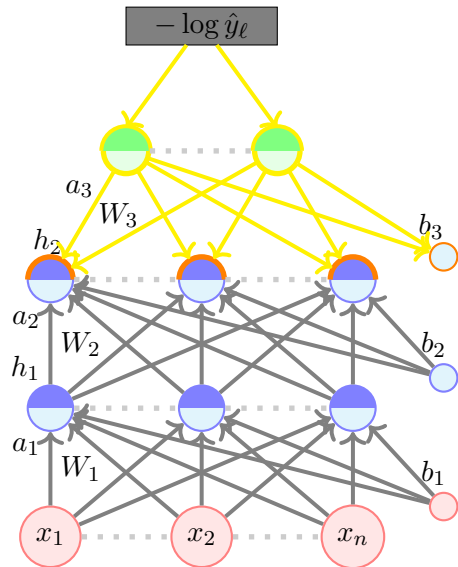


- We get a certain loss at the output and we try to figure out who is responsible for this loss
- So, we talk to the output layer and say “Hey! You are not producing the desired output, better take responsibility”.
- The output layer says “Well, I take responsibility for my part but please understand that I am only as the good as the hidden layer and weights below me”. After all ...

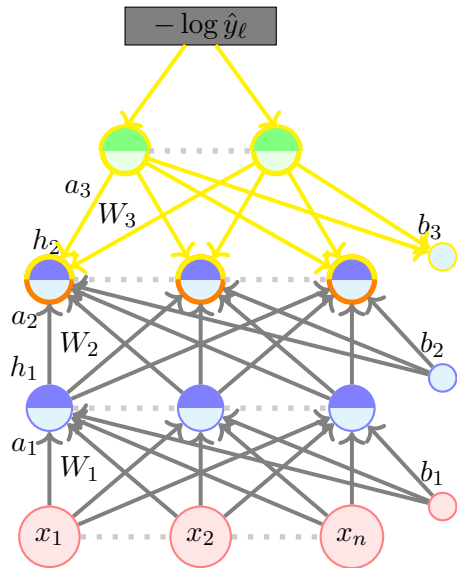
$$f(x) = \hat{y} = O(W_L h_{L-1} + b_L)$$



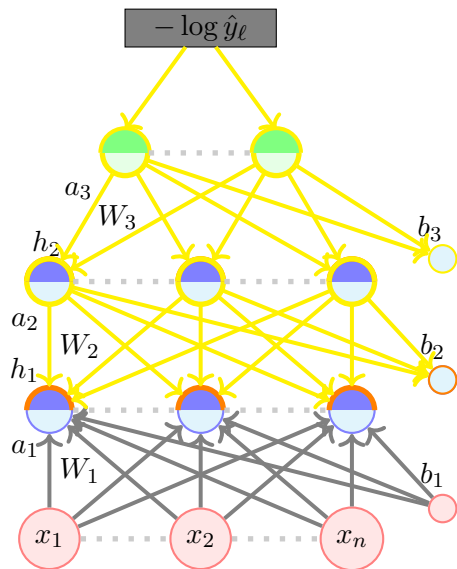
- So, we talk to W_L, b_L and h_L and ask them “What is wrong with you?”



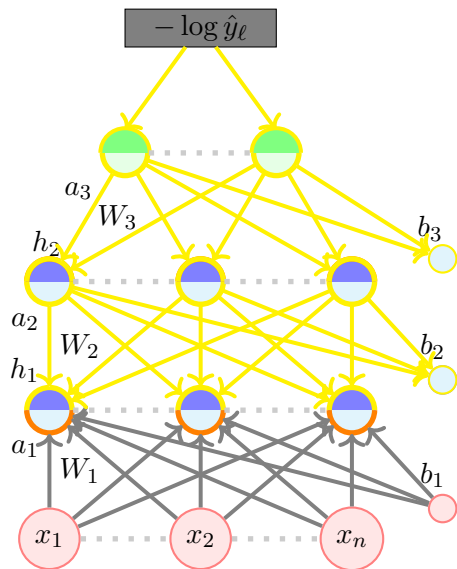
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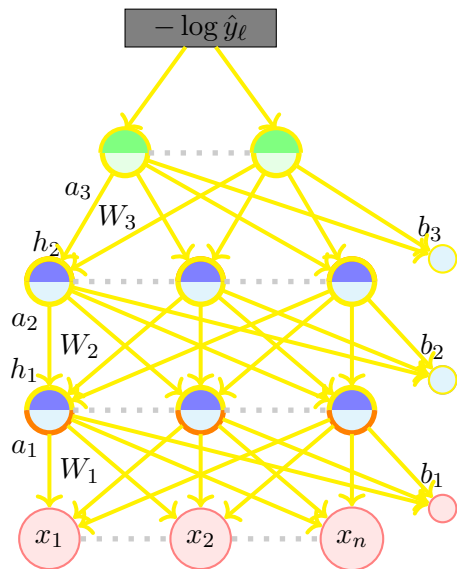
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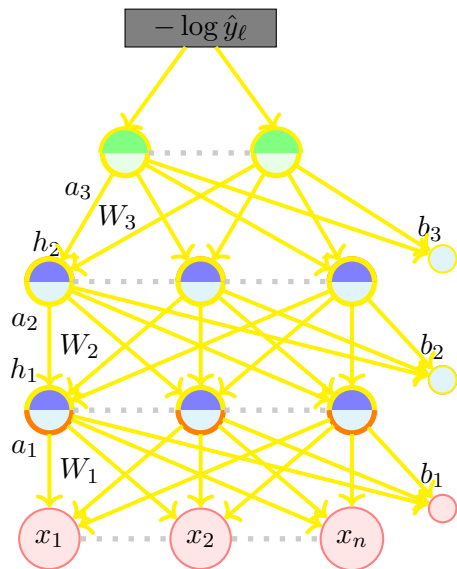


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Quantities of interest (roadmap for the remaining part):

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Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
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- Gradient w.r.t. output units
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- Our focus is on *Cross entropy loss* and *Softmax* output.