Module 4.6: Backpropagation: Computing Gradients w.r.t. Hidden Units

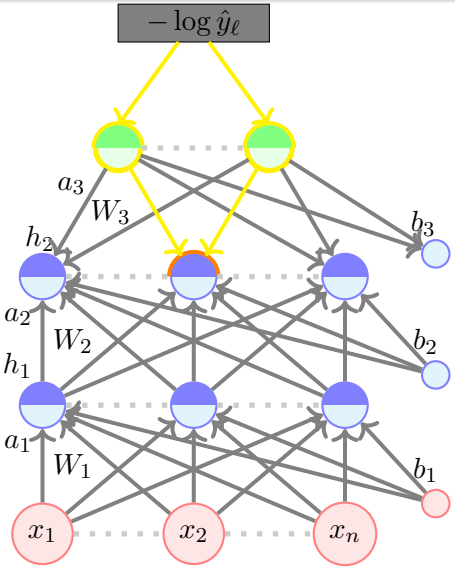
Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{111}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{111}}}_{\text{and now talk to the weights}}$$

- Our focus is on *Cross entropy loss* and *Softmax* output.

Chain rule along multiple paths: If a function $p(z)$ can be written as a function of intermediate results $q_i(z)$ then we have :



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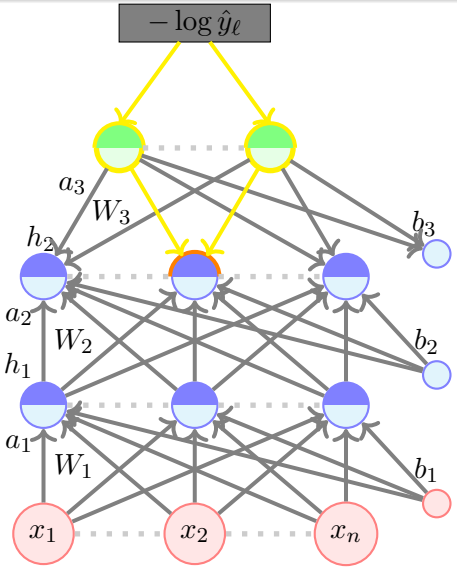

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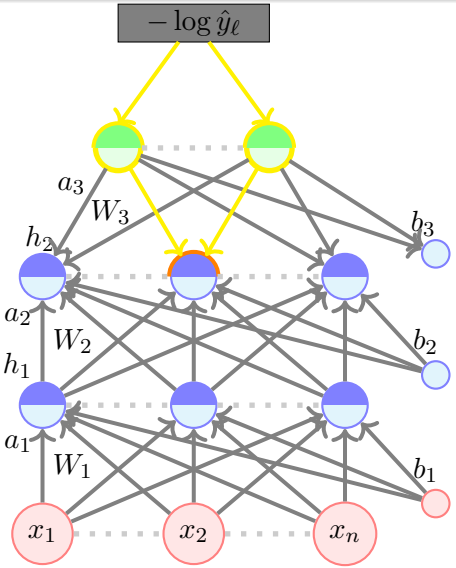
In our case:

- $p(z)$ is the loss function $\mathscr{L}(\theta)$



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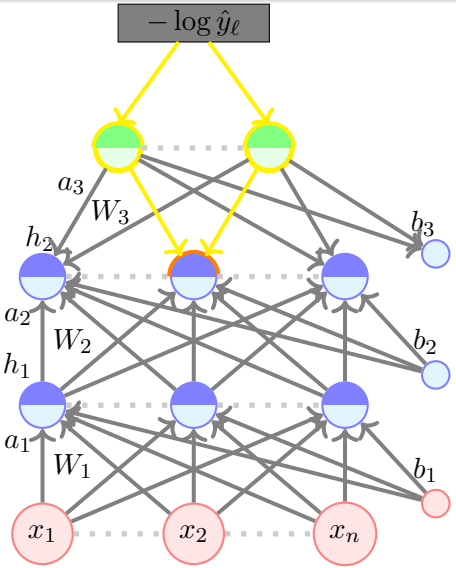
- $z = h_{ij}$



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In our case:

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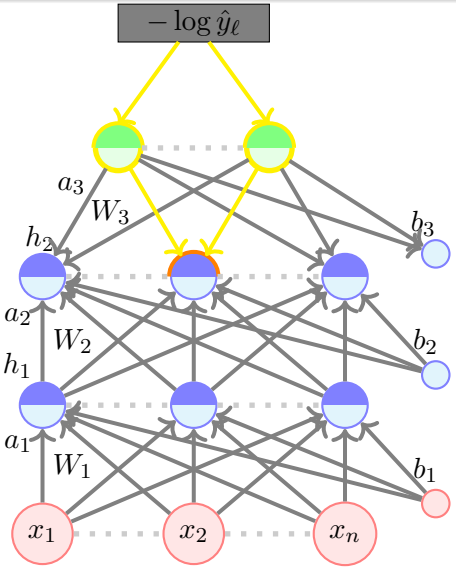
- $z = h_{ij}$
- $q_m(z) = a_{Lm}$



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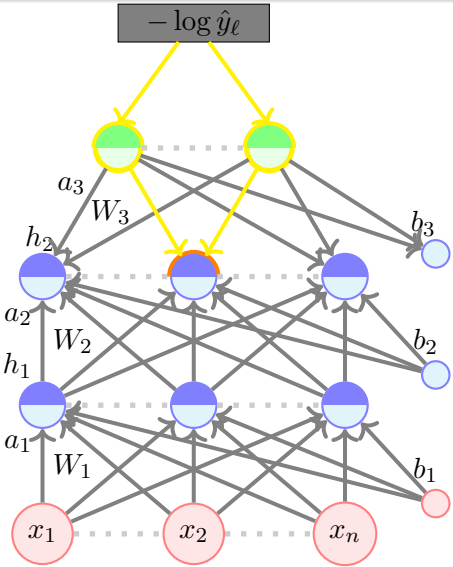
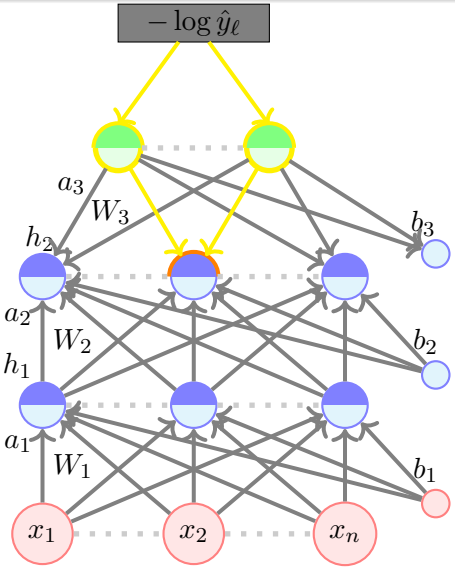
$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}}$



$$a_{i+1} = W_{i+1} h_{ij} + b_{i+1}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} \frac{\partial a_{i+1,m}}{\partial h_{ij}}$$



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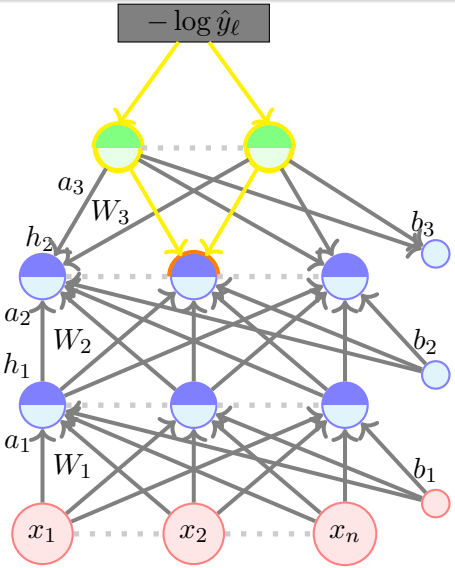

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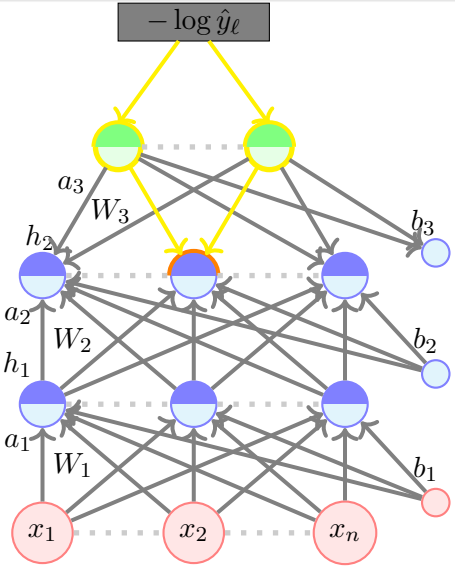
Now consider these two vectors,



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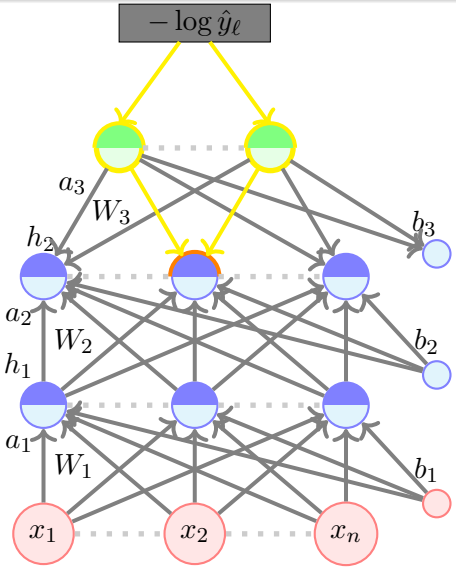

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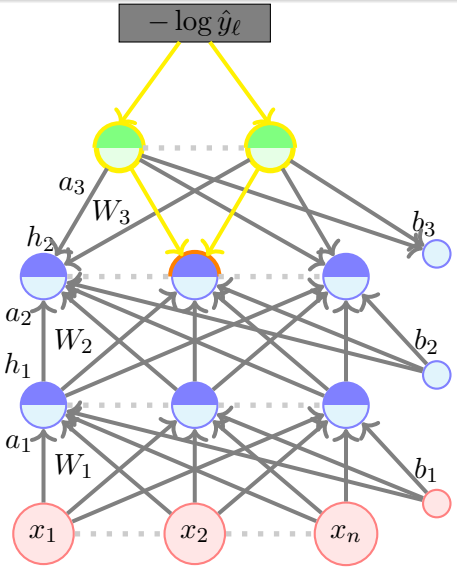

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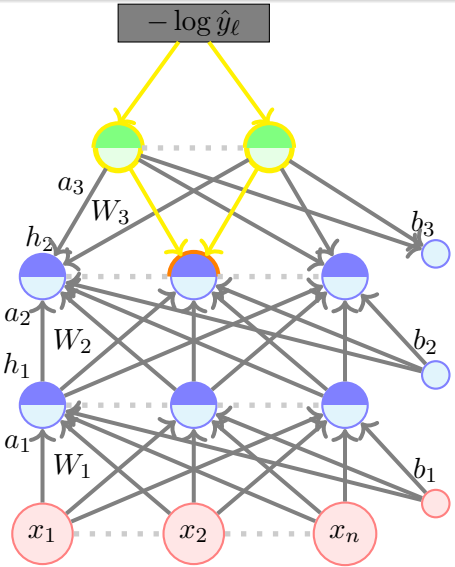

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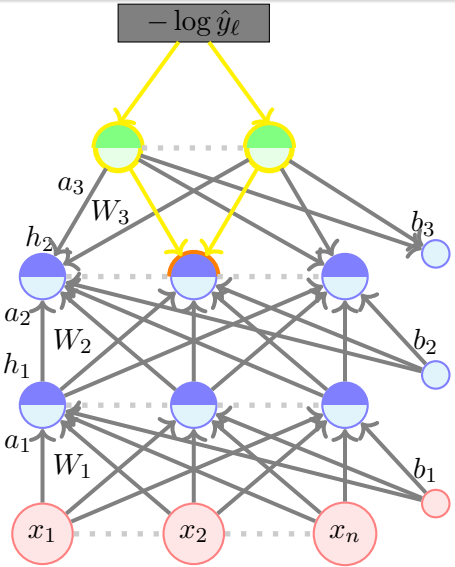

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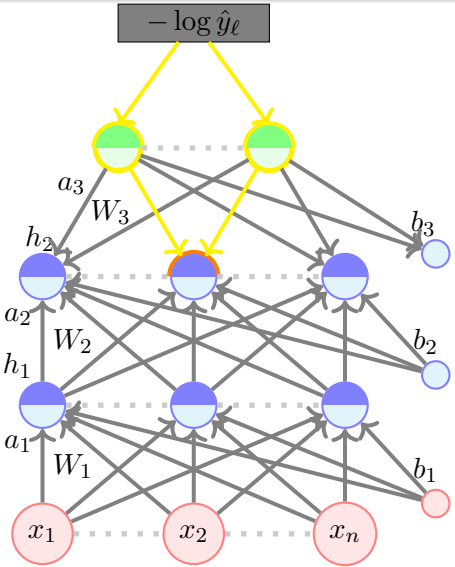

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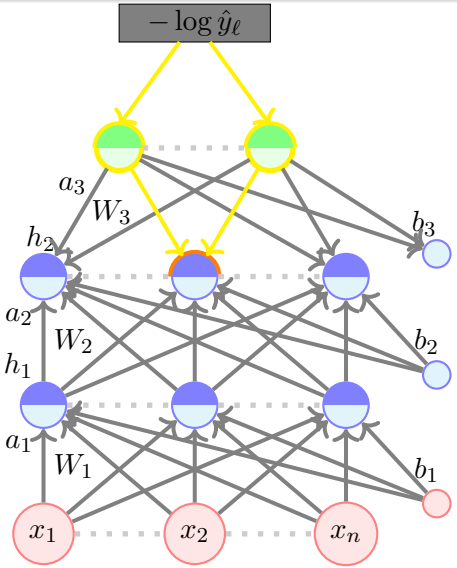

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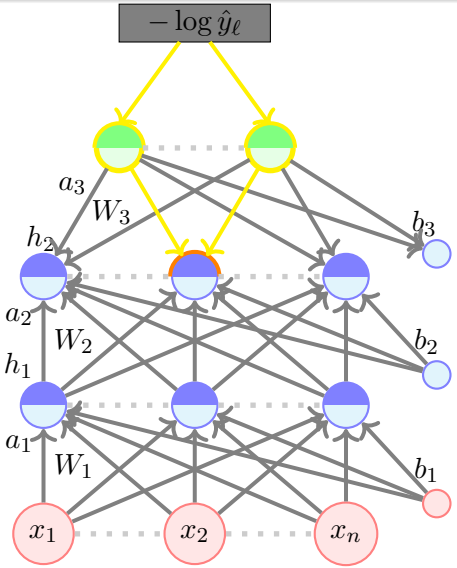

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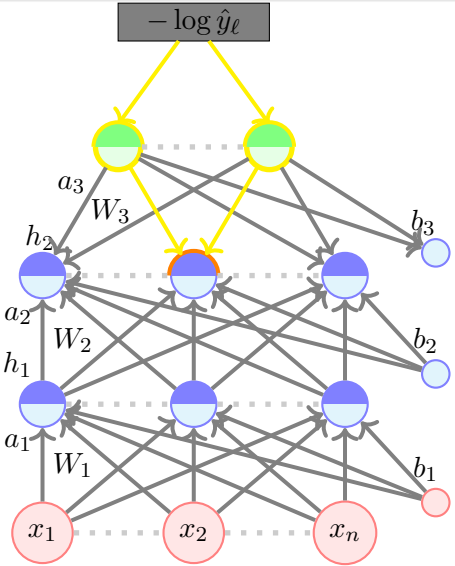

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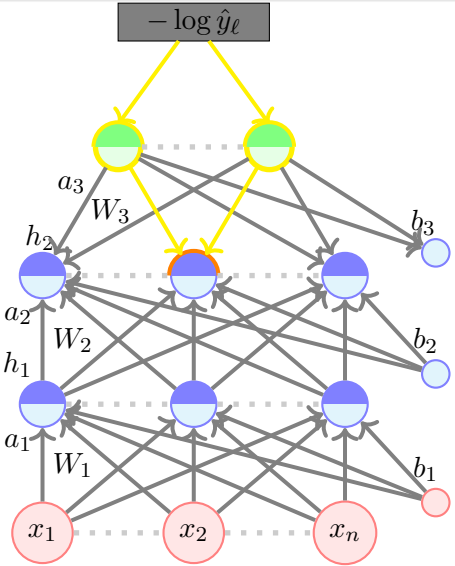

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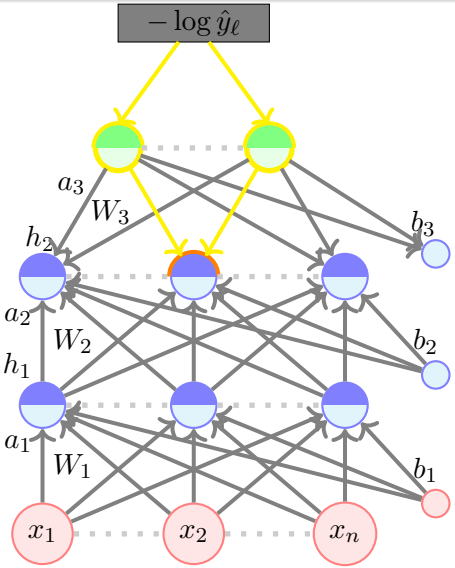
$$(W_{i+1,\cdot,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) =$$



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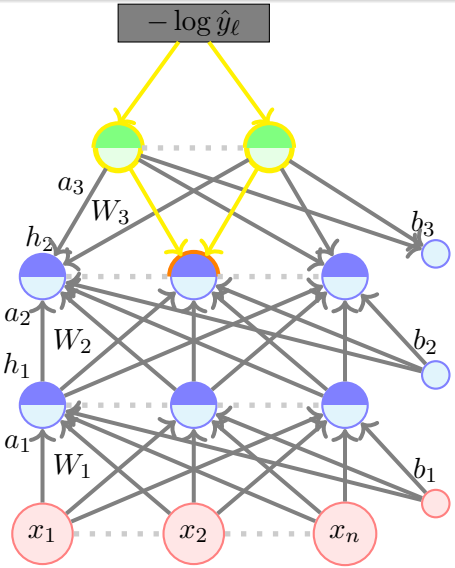

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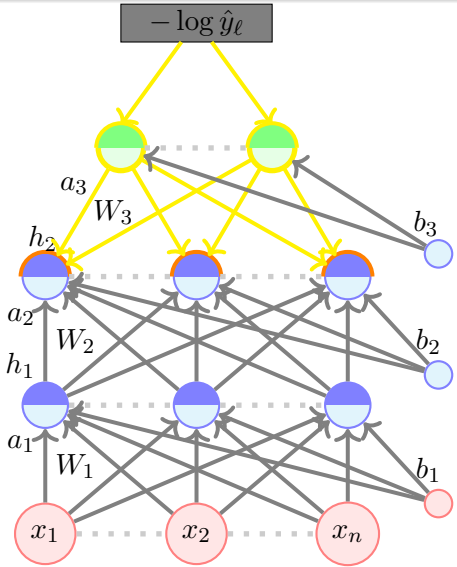

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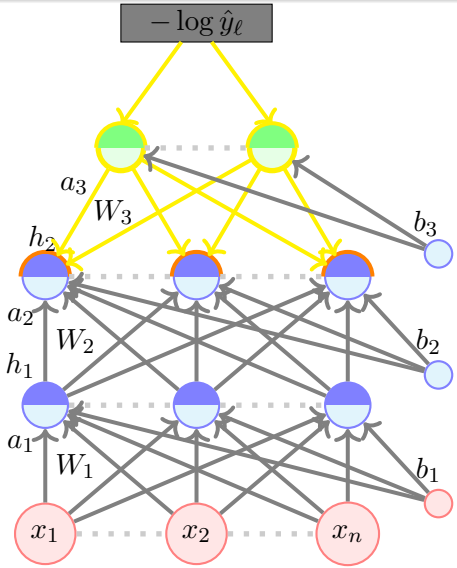
We can now write the gradient w.r.t. h_i

$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta)$



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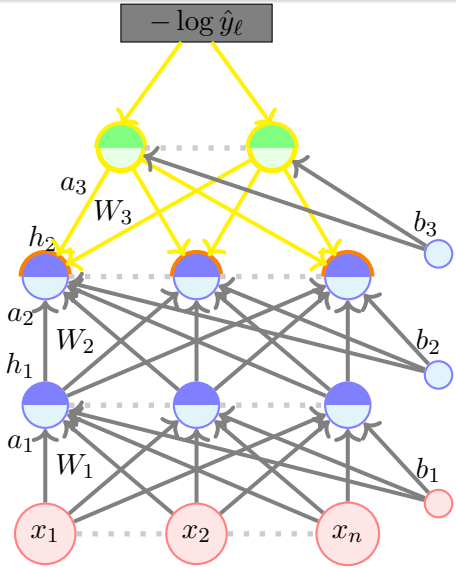

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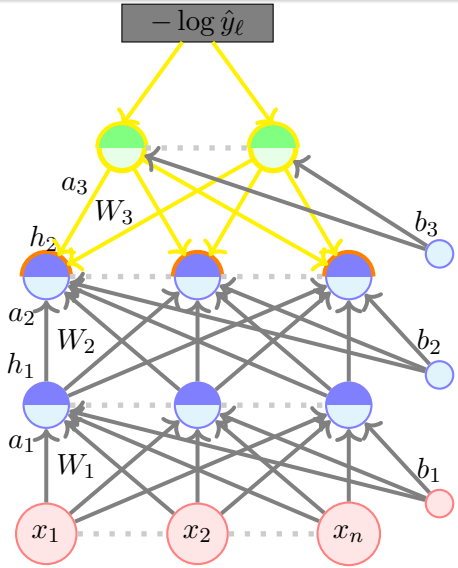

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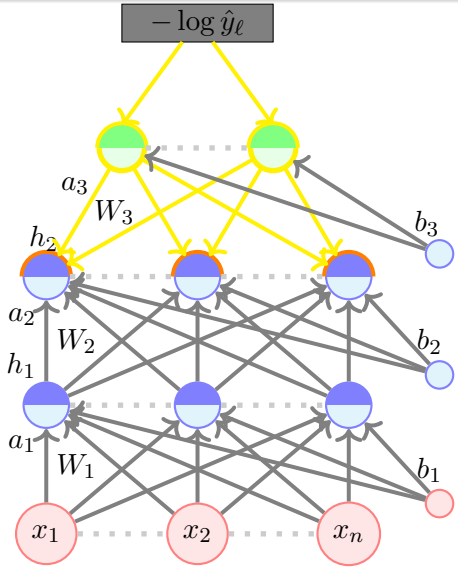

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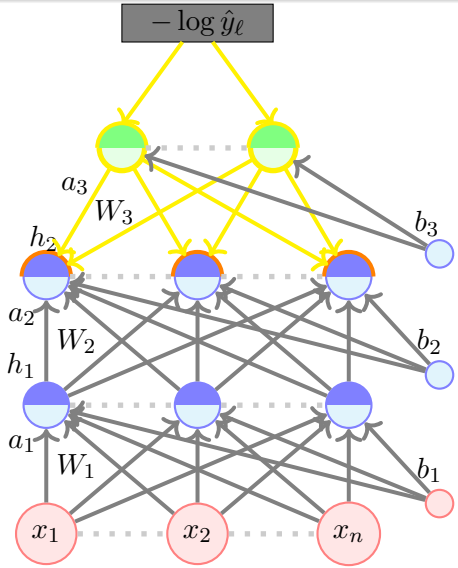

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We can now write the gradient w.r.t. h_i

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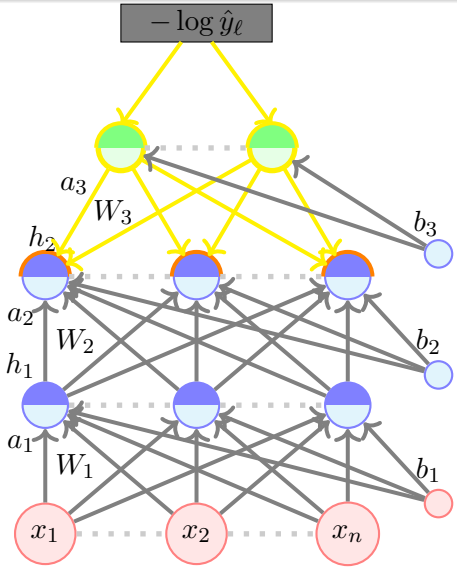

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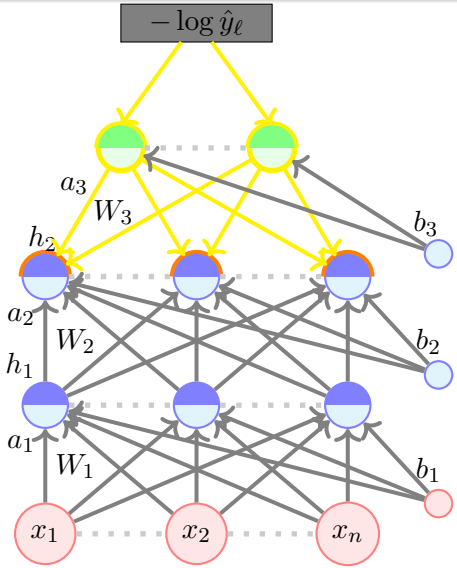

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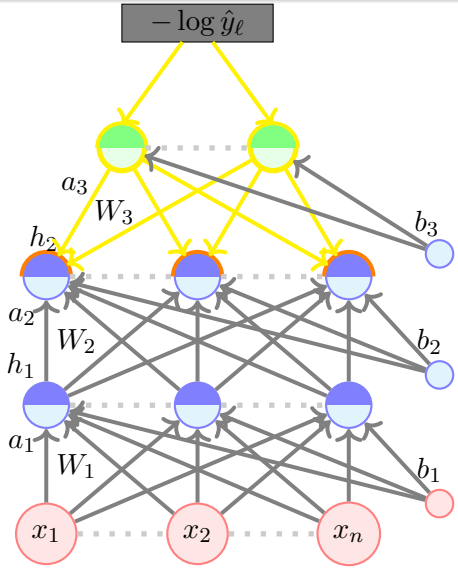

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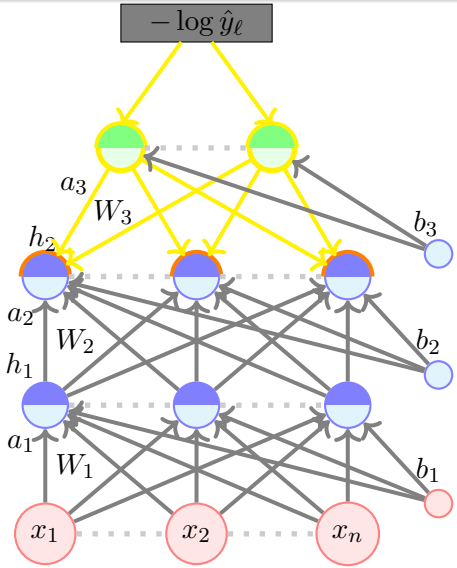

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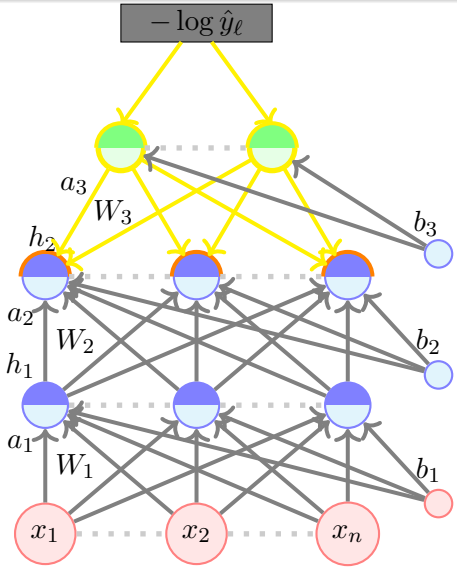

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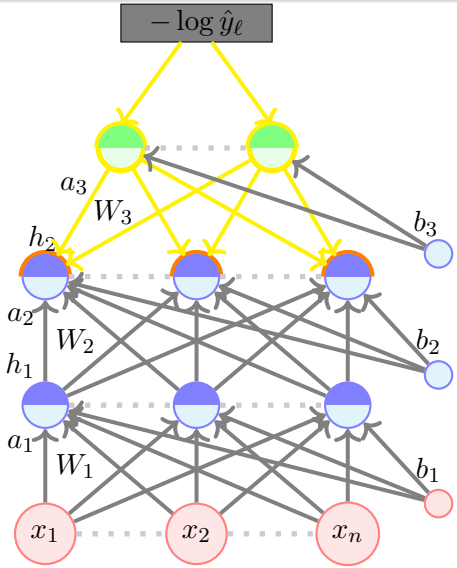

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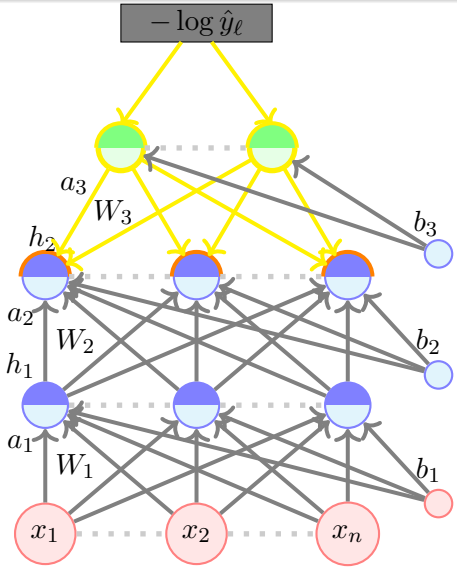
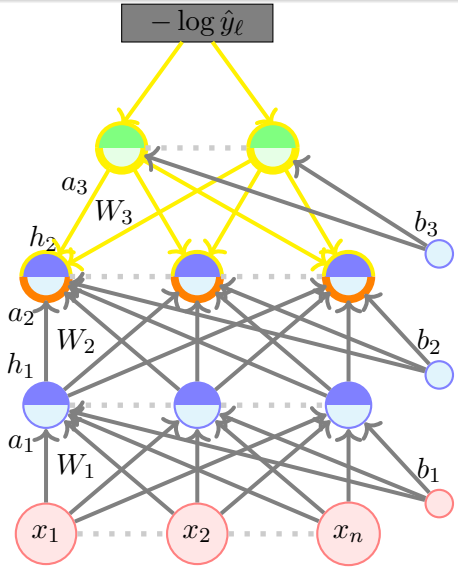
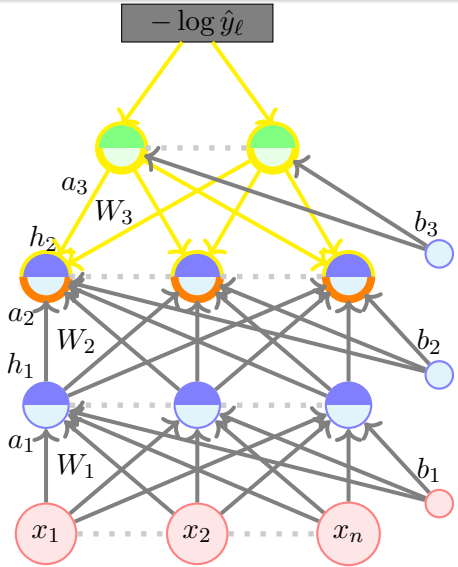


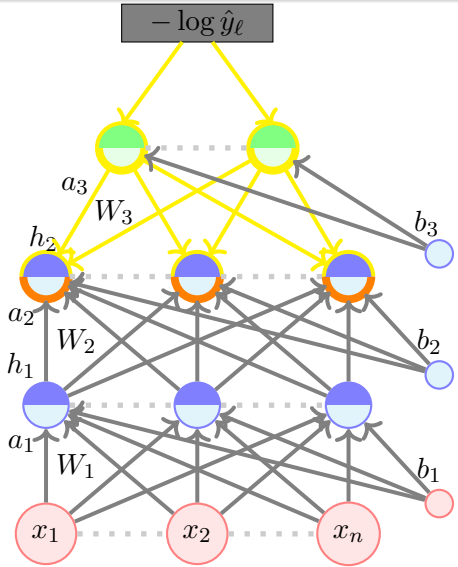
$\nabla_{\mathbf{a_i}} \mathscr{L}(\theta)$



$$\nabla_{\mathbf{a_i}} \mathscr{L}(\theta) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



$$\nabla_{\mathbf{a_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \\ \\ \end{bmatrix}$$



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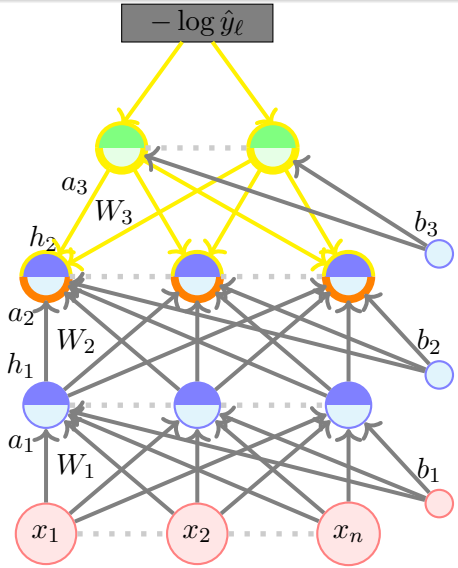

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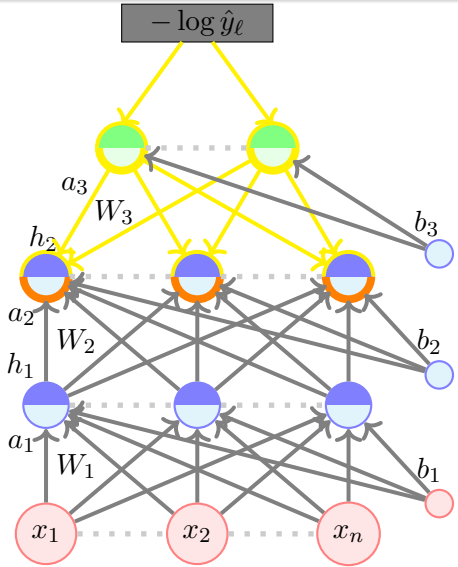

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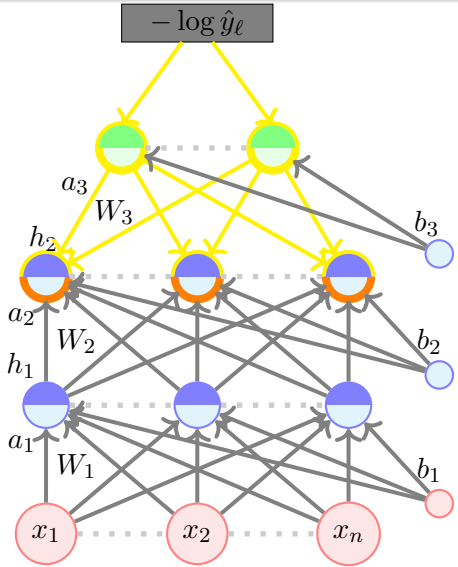

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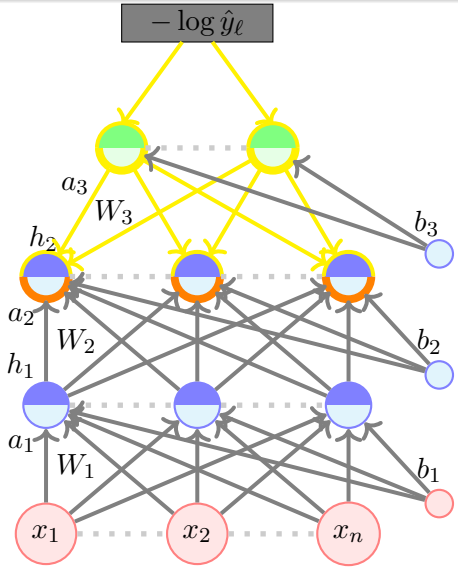

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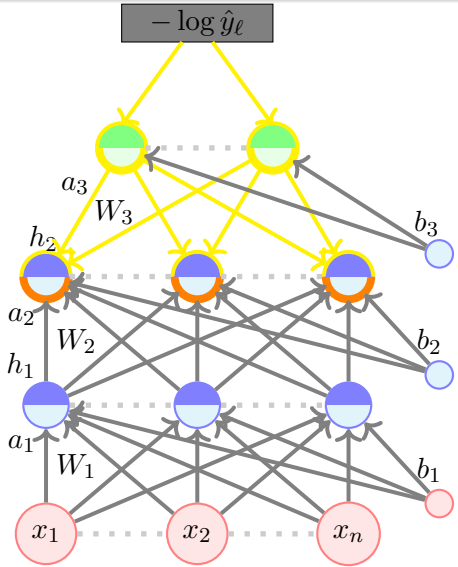

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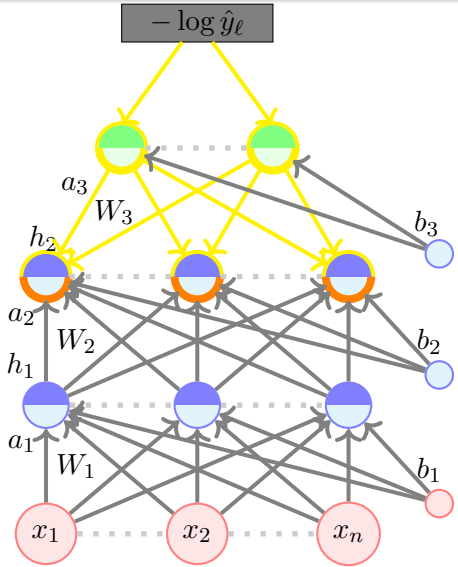

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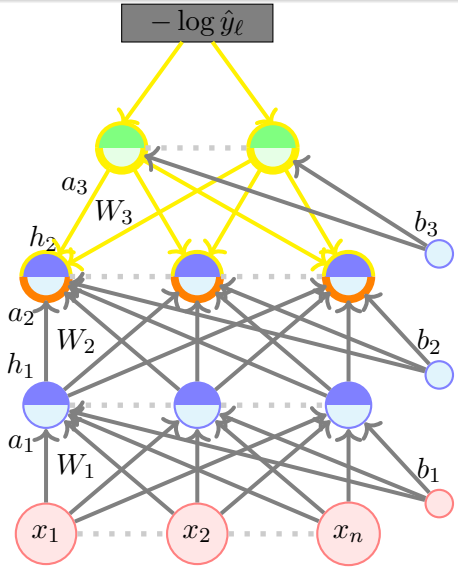

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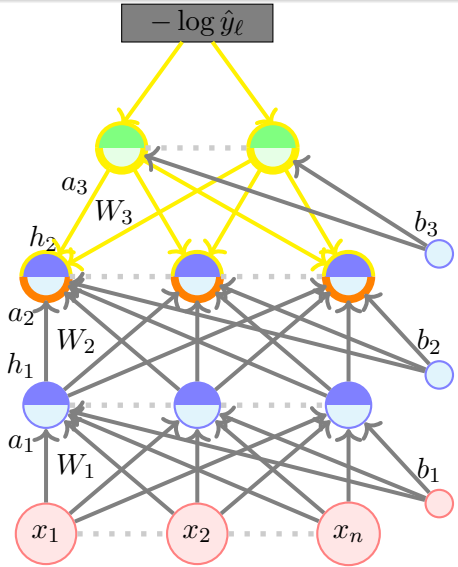

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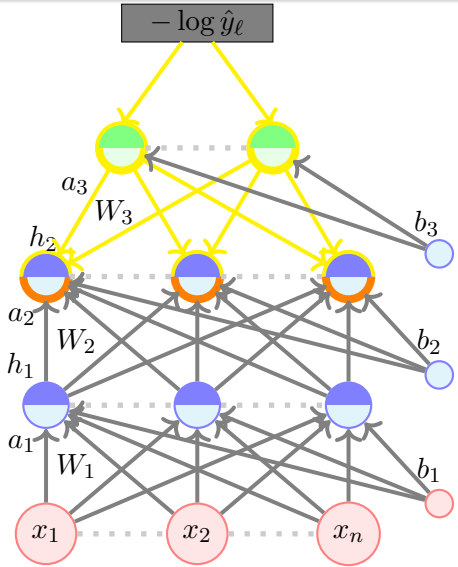

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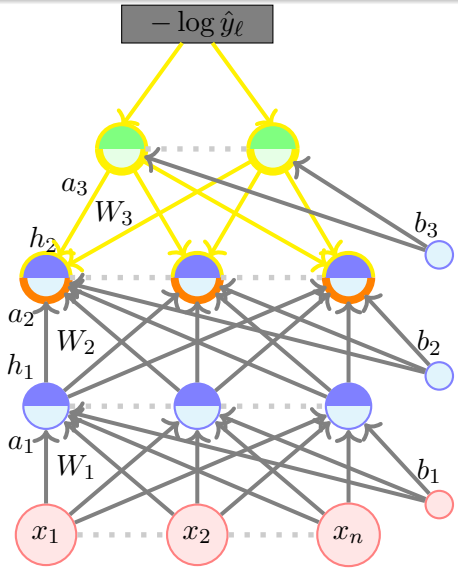

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$$= \nabla_{h_i} \mathscr{L}(\theta) \odot [\dots, g^{'}(a_{ik}), \dots]$$