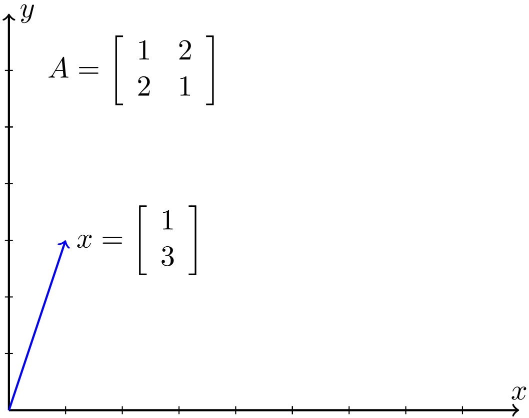
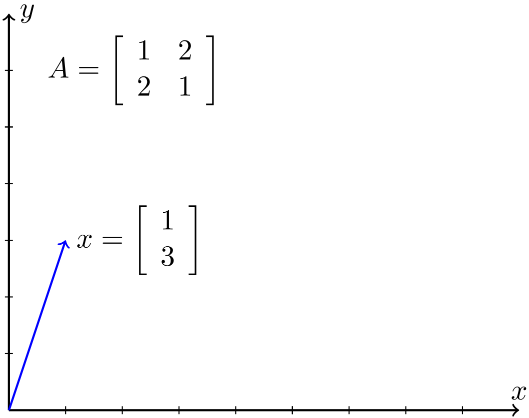
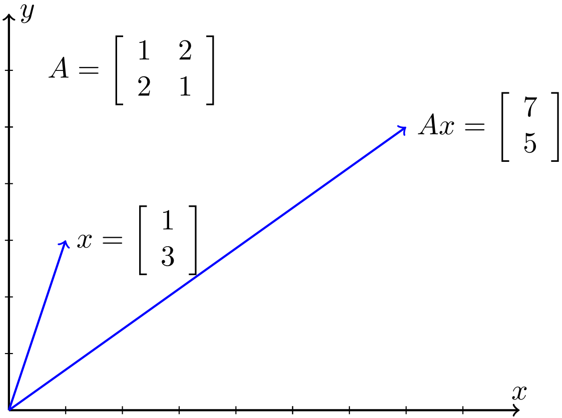
Module 6.1 : Eigenvalues and Eigenvectors



- What happens when a matrix hits a vector?



- What happens when a matrix hits a vector?
- The vector gets transformed into a new vector (it strays from its path)



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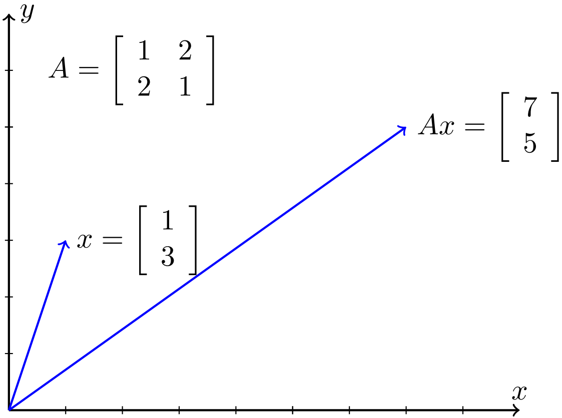
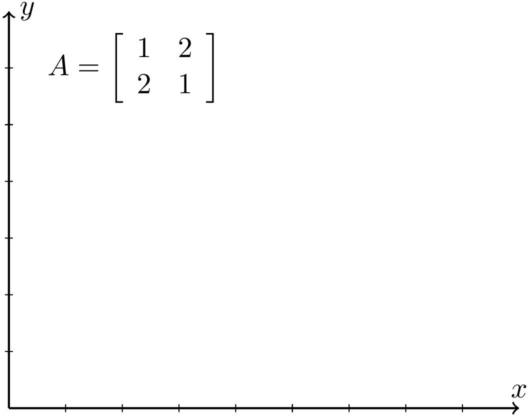
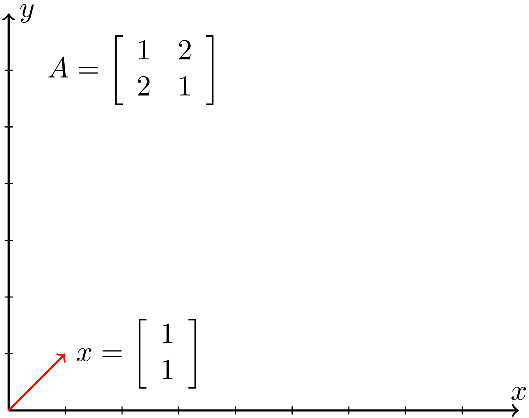


- What happens when a matrix hits a vector?
- The vector gets transformed into a new vector (it strays from its path)
- The vector may also get scaled (elongated or shortened) in the process.



$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

- For a given square matrix A, there exist special vectors which refuse to stray from their path.



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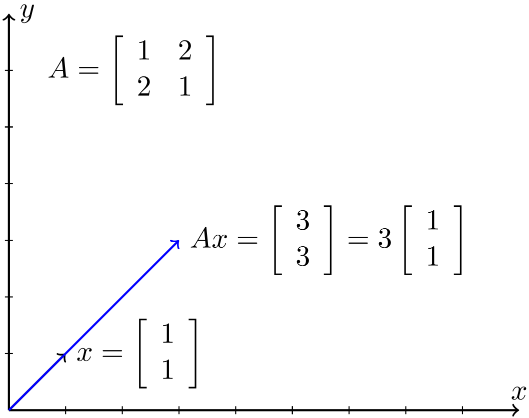

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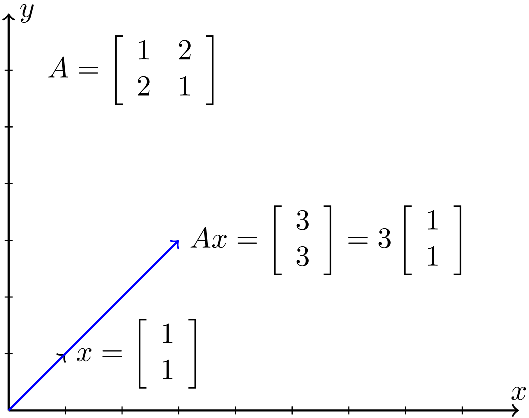

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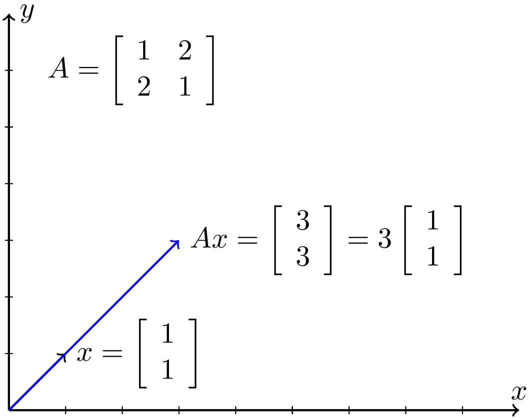
- These vectors are called eigenvectors.



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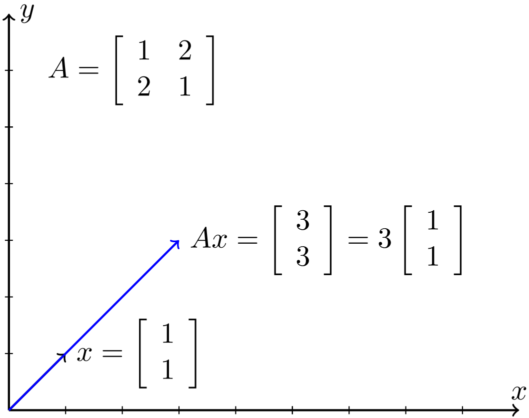
- More formally,

  $Ax = \lambda x$ [direction remains the same]



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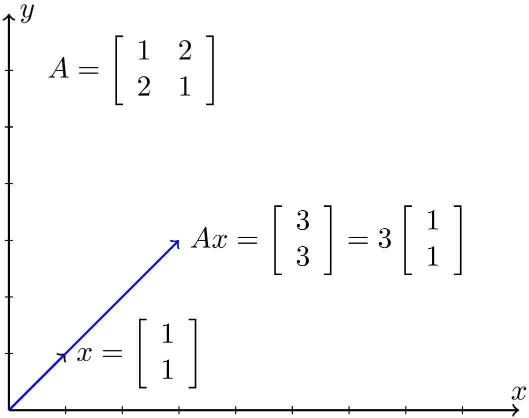
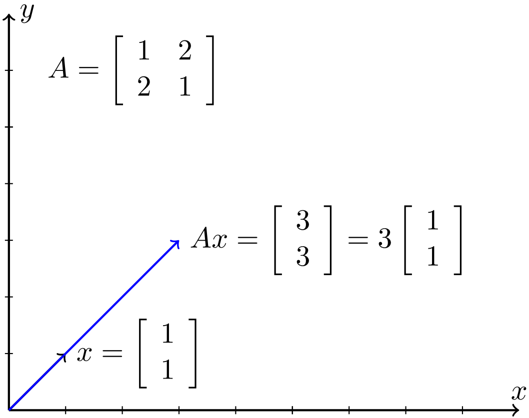
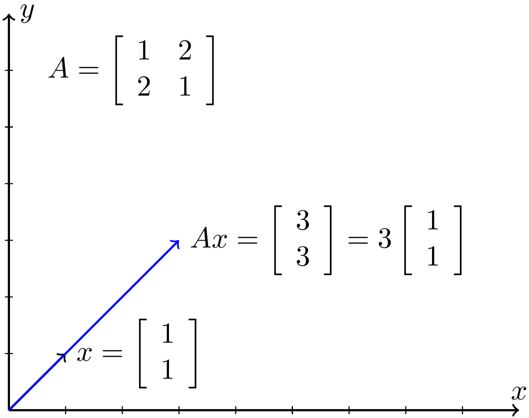
$$Ax = \lambda x \text{ [direction remains the same]}$$

- The vector will only get scaled but will not change its direction.

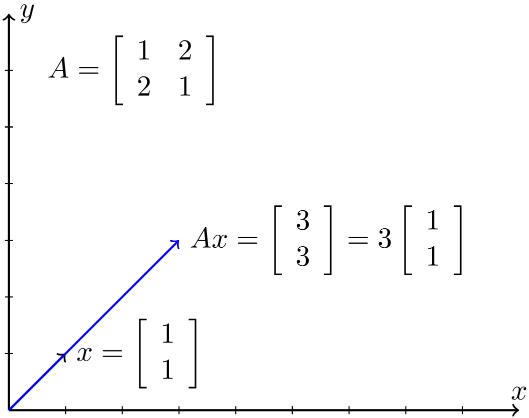

y
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
$Ax = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
x



- So what is so special about eigenvectors?



- So what is so special about eigenvectors?
- Why are they always in the limelight?



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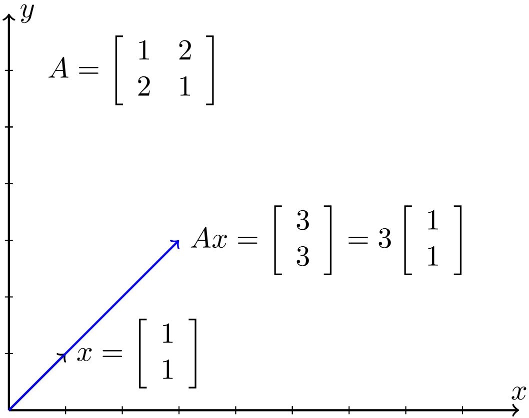
- It turns out that several properties of matrices can be analyzed based on their eigenvalues (for example, see spectral graph theory)



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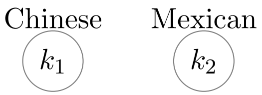
- It turns out that several properties of matrices can be analyzed based on their eigenvalues (for example, see spectral graph theory)
- We will now see two cases where eigenvalues/vectors will help us in this course

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)



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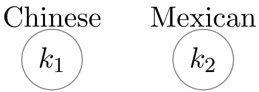

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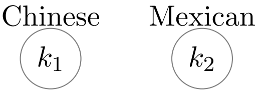

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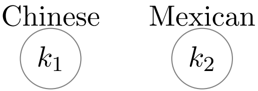

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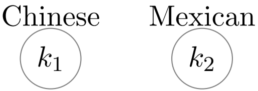

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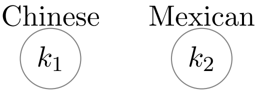

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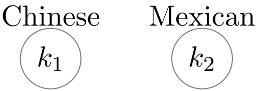

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In general, $v_{(n)} = M^n v_{(0)}$

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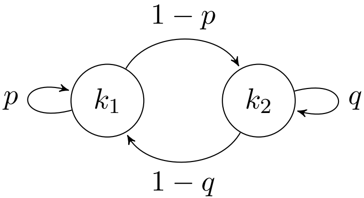
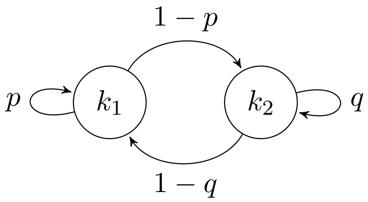
- The number of customers in the two restaurants is thus given by the following series:

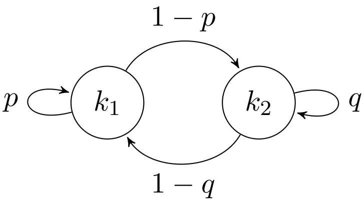
$$v_{(0)}, Mv_{(0)}, M^2 v_{(0)}, M^3 v_{(0)}, \ldots$$

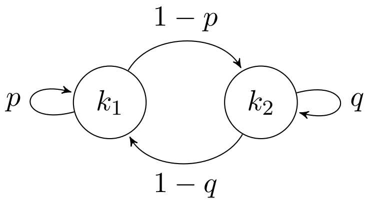

$1 - p$
p
k_1
k_2
q
$1 - q$

- This is a problem for the two restaurant owners.

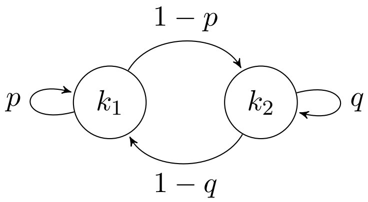


- This is a problem for the two restaurant owners.
- The number of patrons is changing constantly.





- This is a problem for the two restaurant owners.
- The number of patrons is changing constantly.
- Or is it? Will the system eventually reach a steady state? (i.e. will the number of customers in the two restaurants become constant over time?)



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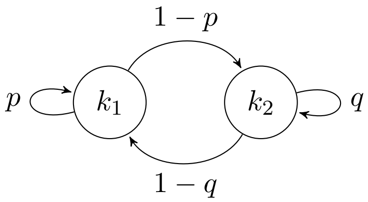

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- Let's see how?

Definition

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvectors of an $n \times n$ matrix A. λ_1 is called the dominant eigen value of A if

$$|\lambda_1| \geq |\lambda_i| \; i = 2, \ldots, n$$

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(Note that the matrix in our example is a stochastic matrix)

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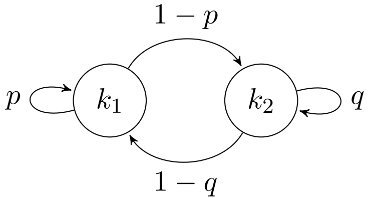
A matrix M is called a stochastic matrix if all the entries are positive and the sum of the elements in each column is equal to 1.
(Note that the matrix in our example is a stochastic matrix)

Theorem

If A is a $n \times n$ square matrix with a dominant eigenvalue, then the sequence of vectors given by Av_0, A^2v_0, $\ldots$, $A^n v_0, \ldots$ approaches a multiple of the dominant eigenvector of A.
(the theorem is slightly misstated here for ease of explanation)

- Let e_d be the dominant eigenvector of M and $\lambda_d = 1$ the corresponding dominant eigenvalue



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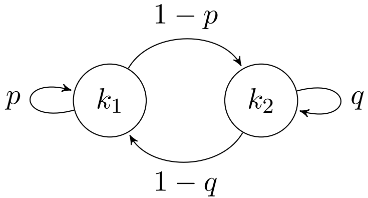
- Given the previous definitions and theorems, what can you say about the sequence $Mv_{(0)}, M^2 v_{(0)}, M^3 v_{(0)}, \ldots$?



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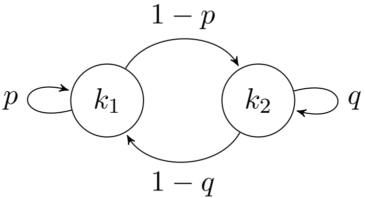
- There exists an n such that

$$v_{(n)} = M^n v_{(0)} = ke_d \text{ (some multiple of } e_d)$$



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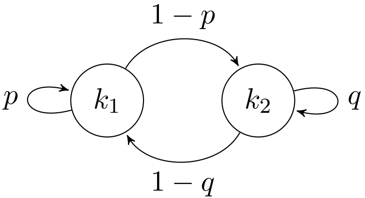
- Now what happens at time step $(n+1)$?

$$v_{(n+1)} = Mv_{(n)} = M(ke_d) = k(Me_d) = k(\lambda_d e_d) = ke_d$$



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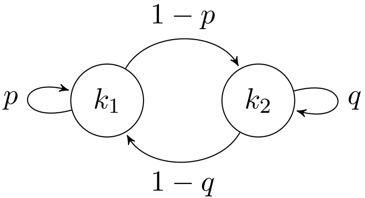
$$v_{(n+1)} = Mv_{(n)} = M(ke_d) = k(Me_d) = k(\lambda_d e_d) = ke_d$$

- The population in the two restaurants becomes constant after time step n.
  See Proof Here



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$$
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\end{aligned}
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- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if

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- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if
  - $|\lambda_d| > 1$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

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- (We will use this in the course at some point)