Module 6.2 : Linear Algebra - Basic Definitions

- We will see some more examples where eigenvectors are important, but before that let's revisit some basic definitions from linear algebra.

Basis

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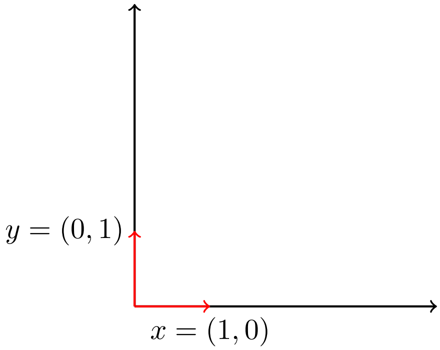
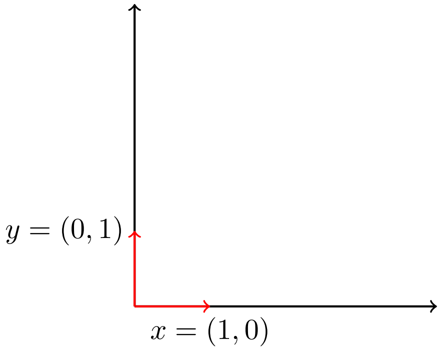
Linearly independent vectors

A set of n vectors $v_1, v_2, \ldots, v_n$ is linearly independent if no vector in the set can be expressed as a linear combination of the remaining $n-1$ vectors.
In other words, the only solution to

$$c_1 v_1 + c_2 v_2 + \ldots c_n v_n = 0 \text{ is } c_1 = c_2 = \cdots = c_n = 0 (c_i\text{'s } \textit{are scalars})$$

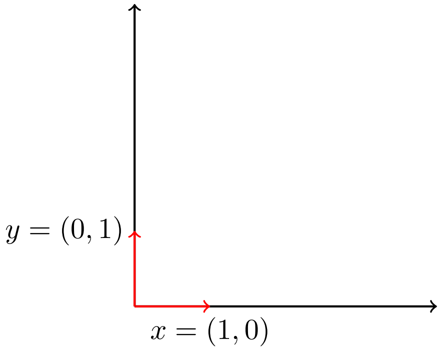


- For example consider the space $\mathbb{R}^2$



- For example consider the space $\mathbb{R}^2$
- Now consider the vectors

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



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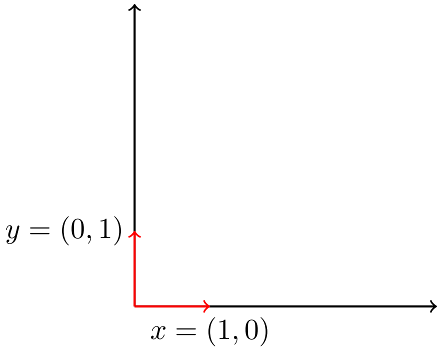
$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Any vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, can be expressed as a linear combination of these two vectors i.e

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



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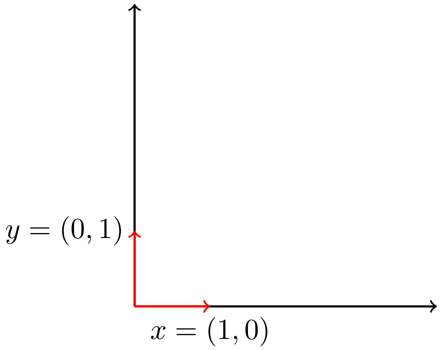
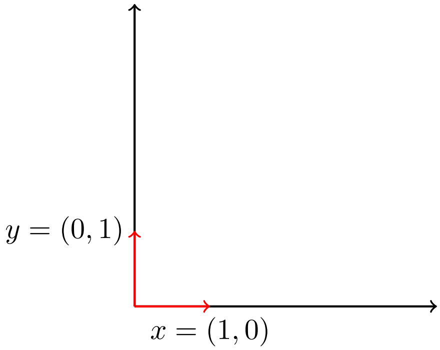
$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

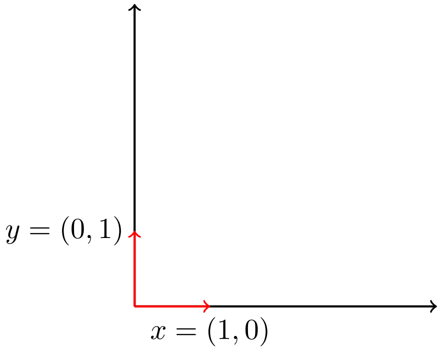
- Further, x and y are linearly independent. (the only solution to $c_1x + c_2y = 0$ is $c_1 = c_2 = 0$)



- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.



- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.



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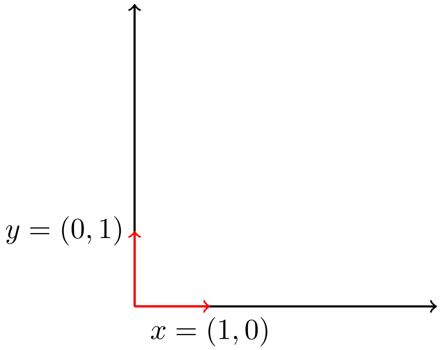

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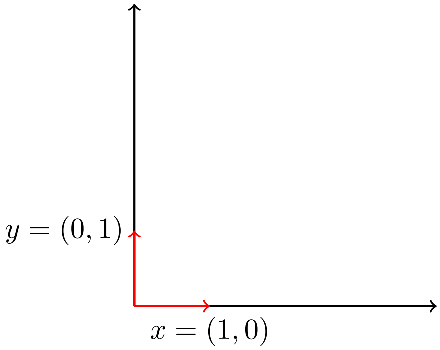
- We could have chosen any 2 linearly independent vectors in $\mathbb{R}^2$ as the basis vectors.



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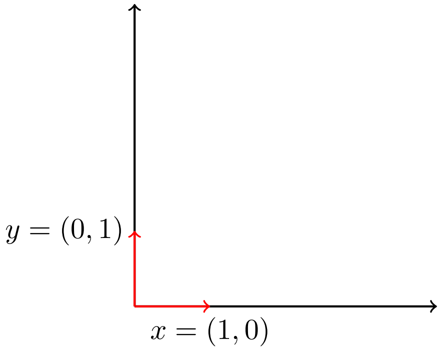
- For example, consider the linearly independent vectors, $[2, 3]^T$ and $[5, 7]^T$. See how any vector $[a, b]^T \in \mathbb{R}^2$ can be expressed as a linear combination of these two vectors.



$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

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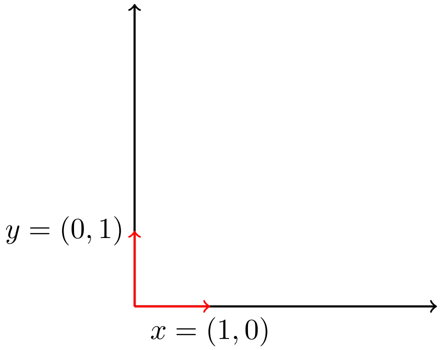

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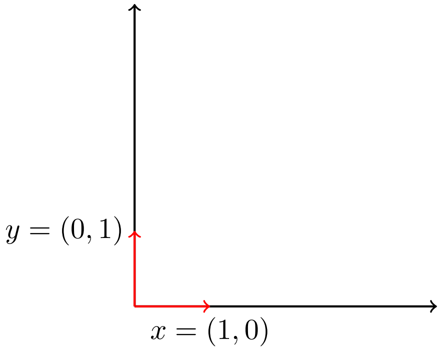

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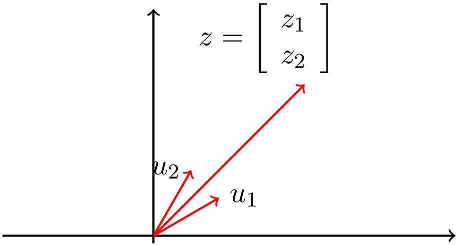
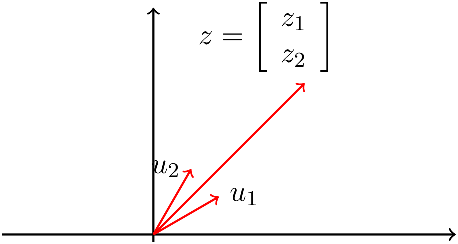


- In general, given a set of linearly independent vectors $u_1, u_2, \ldots u_n \in \mathbb{R}^n$, we can express any vector $z \in \mathbb{R}^n$ as a linear combination of these vectors.



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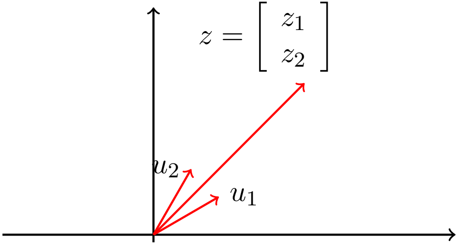

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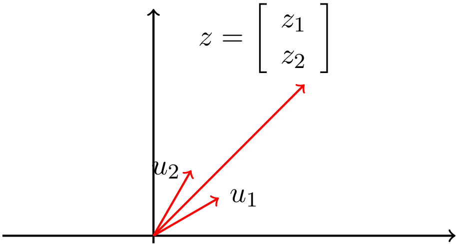
$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \alpha_1 \begin{bmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1n} \end{bmatrix} + \alpha_2 \begin{bmatrix} u_{21} \\ u_{22} \\ \vdots \\ u_{2n} \end{bmatrix} + \dots + \alpha_n \begin{bmatrix} u_{n1} \\ u_{n2} \\ \vdots \\ u_{nn} \end{bmatrix}$$



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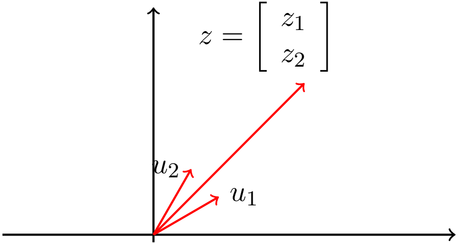
(Basically rewriting in matrix form)



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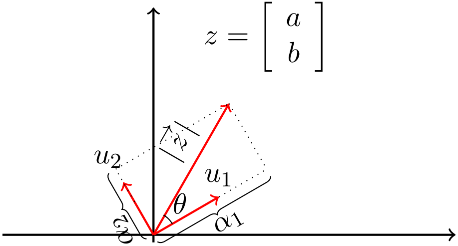
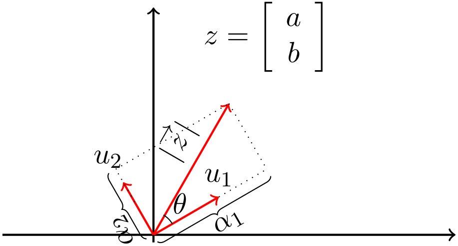
- We can now find the α_is using Gaussian Elimination (Time Complexity: $O(n^3)$)



- Now let us see if we have orthonormal basis.



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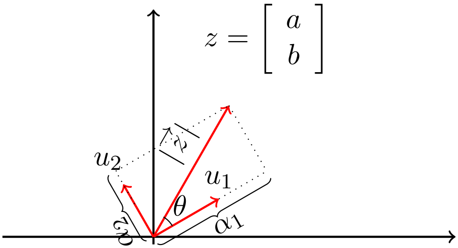

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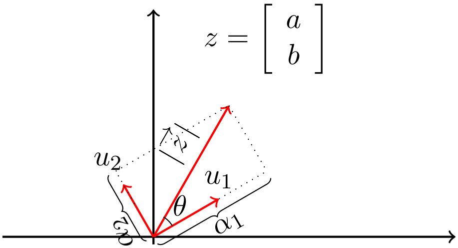

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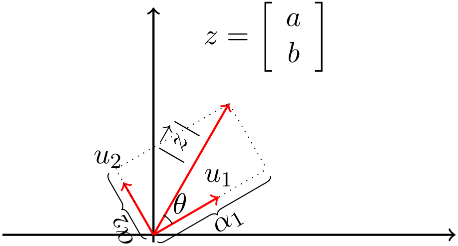

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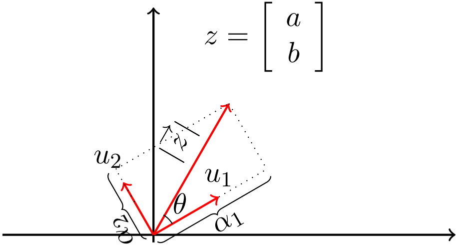
$$



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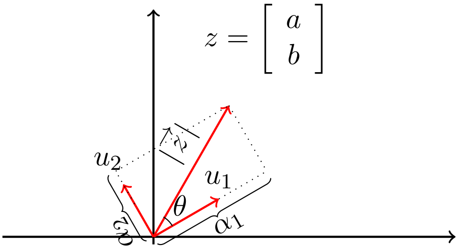

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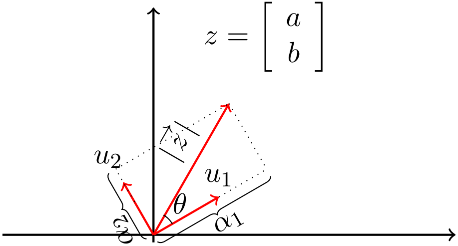

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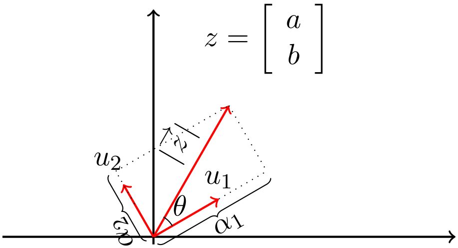

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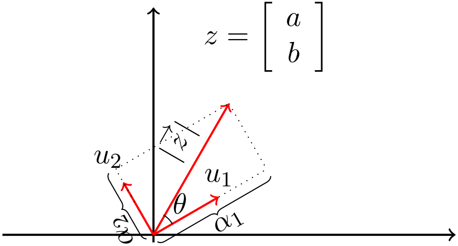

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When u_1 and u_2 are unit vectors along the co-ordinate axes

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Remember

An orthogonal basis is the most convenient basis that one can hope for.

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- Turns out that the eigenvectors can form a basis.

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The eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ having distinct eigenvalues are linearly independent.

Proof: See here

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- We will answer this question soon.