Module 6.8 : Singular Value Decomposition

Let us get some more perspective on eigen vectors before moving ahead

- Let $v_1, v_2, \cdots, v_n$ be the eigen vectors of A and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be corresponding eigen values

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- The matrix multiplication reduces to a scalar multiplication if the eigen vectors of A are used as a basis.

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- We will see the answer to this question over the next few slides

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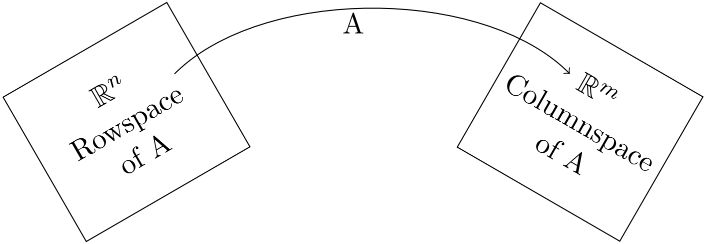
- Once again the matrix multiplication reduces to a scalar multiplication

Let's look at a geometric interpretation of this



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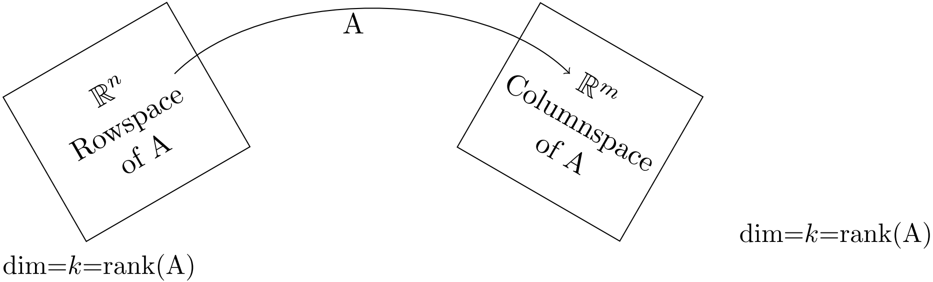

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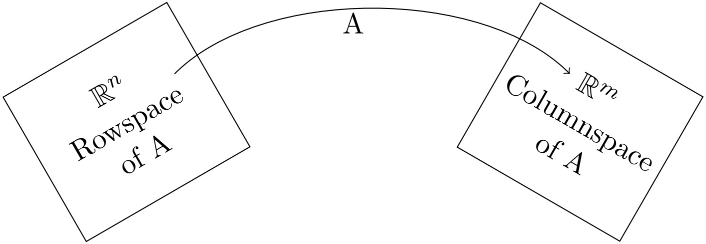

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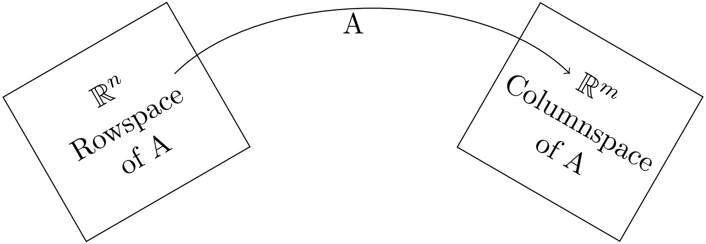

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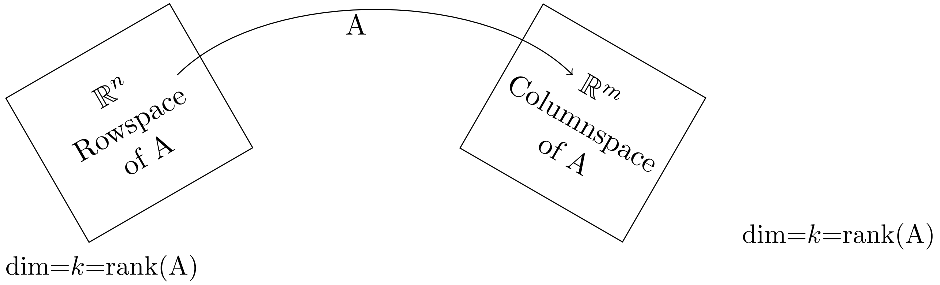

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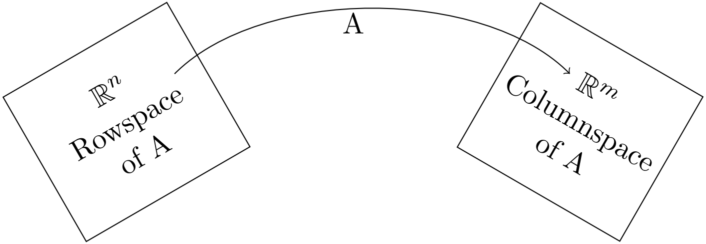

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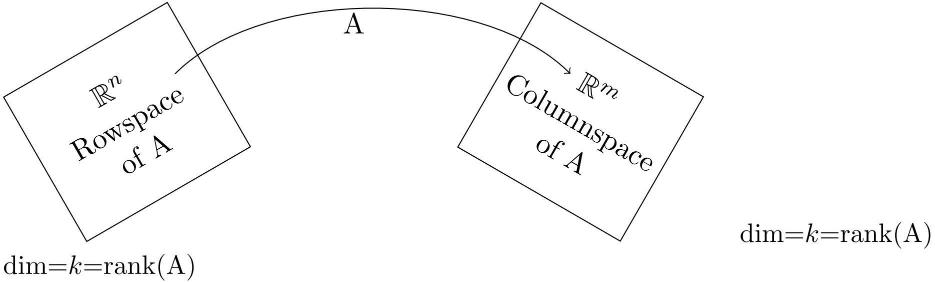

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- such that if the inputs and outputs are represented using this basis then the operation Ax reduces to a scalar operation

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- Hence we need only k dimensions to represent x

$$x = \sum_{i=1}^{k} \alpha_i v_i$$

- Let's look at a way of writing this as a matrix operation

$$Av_1 = \sigma_1 u_1, Av_2 = \sigma_2 u_2, \cdots, Av_k = \sigma_k u_k$$

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- If we have k orthogonal vectors ($V_{n\times k}$) then using Gram Schmidt orthogonalization, we can find $n-k$ more orthogonal vectors to complete the basis for $\mathbb{R}^n$ [We can do the same for U]

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$$U^T AV = \Sigma \qquad [U^{-1} = U^T] \qquad A = U\Sigma V^T \qquad [V^{-1} = V^T]$$

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- Now the question is how do we find V, U and Σ

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- What does this look like?

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- What does this look like? Eigen Value decomposition of $A^T A$
- Similarly we can show that

$$AA^T = U\Sigma^2 U^T$$

- Suppose V, U and Σ exist, then

$$\begin{aligned} A^TA &= (U\Sigma V^T)^T(U\Sigma V^T) \\ &= V\Sigma^T U^T U\Sigma V^T \\ A^TA &= V\Sigma^2 V^T \end{aligned}$$

- What does this look like? Eigen Value decomposition of A^TA
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- Thus U and V are the eigen vectors of AA^T and A^TA respectively and $\Sigma^2 = \Lambda$ where Λ is the diagonal matrix containing eigen values of A^TA

$$
\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m \times n} = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ u_1 & \cdots & u_k \\ \downarrow & \cdots & \downarrow \end{bmatrix}_{m \times k} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}_{k \times k} \begin{bmatrix} \leftarrow & v_1 & \rightarrow \\ & \vdots & \\ \leftarrow & v_k & \rightarrow \end{bmatrix}_{k \times n}
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Theorem:

$\sigma_1 u_1 v_1^T$ is the best rank-1 approximation of the matrix A. $\sum_{i=1}^{2} \sigma_i u_i v_i^T$ is the best rank-2 approximation of matrix A. In general, $\sum_{i=1}^{k} \sigma_i u_i v_i^T$ is the best rank-k approximation of matrix A. In other words, the solution to

$$
\min \|A - B\|_F^2 \quad \text{is given by :}
$$

$$
B = U_{.,k} \Sigma_{k,k} V_{k,.}^T \quad \text{(minimizes reconstruction error of A)}
$$

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