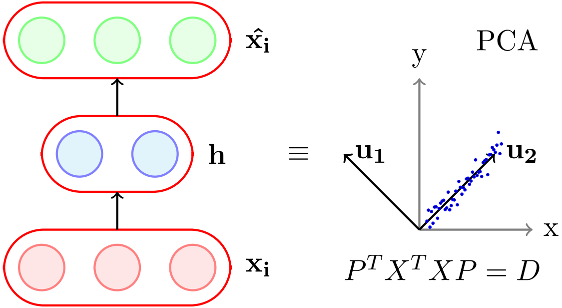
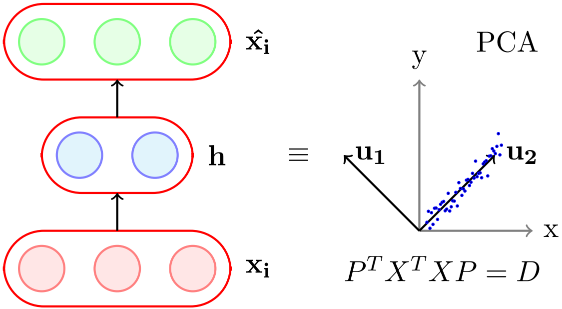
Module 7.2: Link between PCA and Autoencoders



- We will now see that the encoder part of an autoencoder is equivalent to PCA if we



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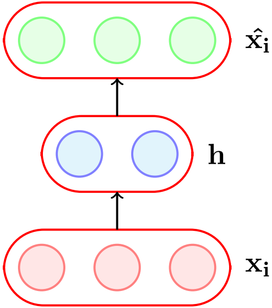

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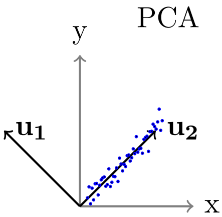

$$P^T X^T X P = D$$

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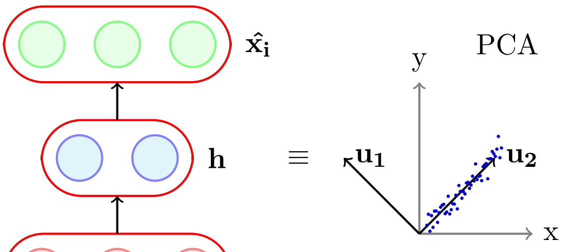
  - use a linear encoder
  - use a linear decoder



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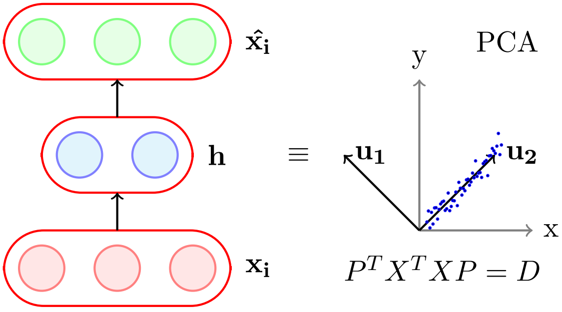
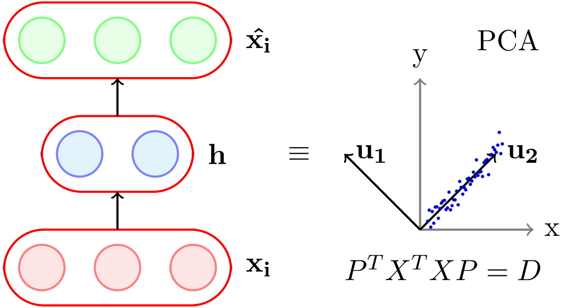


- We will now see that the encoder part of an autoencoder is equivalent to PCA if we
  - use a linear encoder
  - use a linear decoder
  - use squared error loss function
  - normalize the inputs to

$$\hat{x}_{ij} = \frac{1}{\sqrt{m}}\left(x_{ij} - \frac{1}{m}\sum_{k=1}^{m} x_{kj}\right)$$



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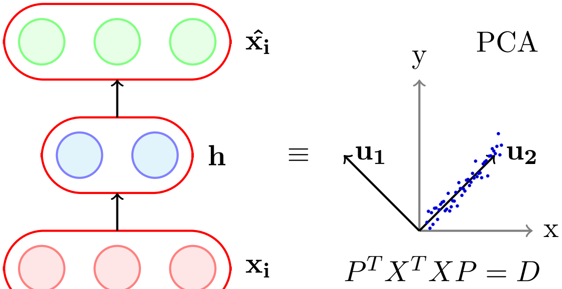

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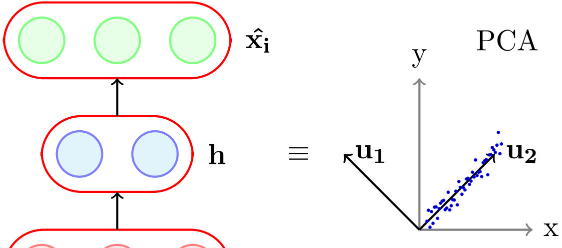

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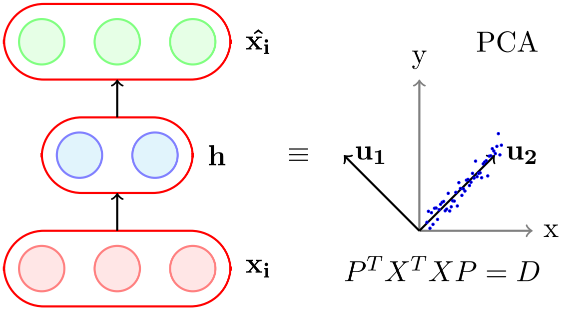
- Let X' be this zero mean data matrix then what the above normalization gives us is $X = \frac{1}{\sqrt{m}}X'$



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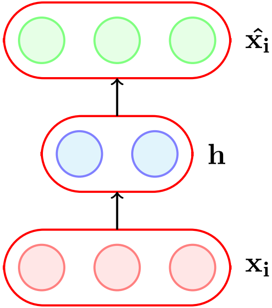
- Let X' be this zero mean data matrix then what the above normalization gives us is $X = \frac{1}{\sqrt{m}}X'$
- Now $(X)^TX = \frac{1}{m}(X')^TX'$ is the covariance matrix (recall that covariance matrix plays an important role in PCA)



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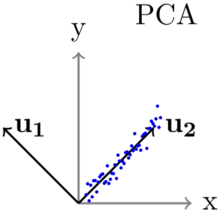

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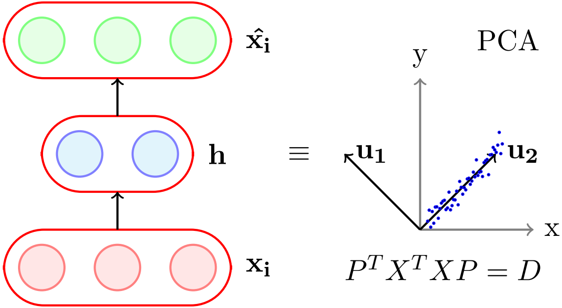

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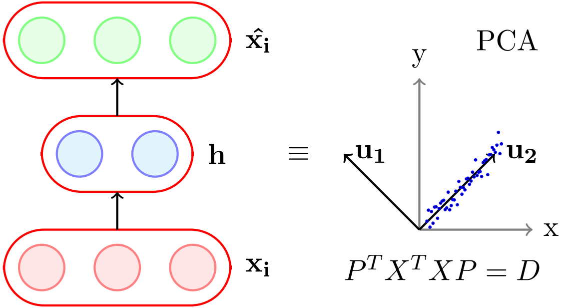

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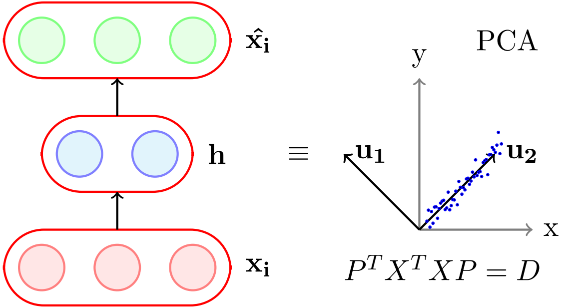

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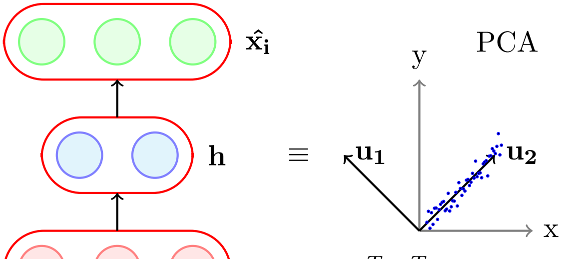

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is obtained when we use a linear encoder.

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- By matching variables one possible solution is

$$H = U_{.,\leq k} \Sigma_{k,k}$$
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Thus H is a linear transformation of X and $W = V_{.,\leq k}$

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- Thus, the encoder matrix for linear autoencoder(W) and the projection matrix(P) for PCA could indeed be the same. Hence proved

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