Continuous r.v.s

Def: A r.v. X is a continuous r.v. if $\exists$ some integrable f_X, called the probability density function (pdf) of X, such that

$$F_X(x) = \int_{-\infty}^{x} f(u)\,du, \quad \forall x \in \mathbb{R}$$

cdf of X

Note: ① $P(X=x) = 0 \quad \forall x$

If $a<b$, $P(a < X \le b) = F_X(b) - F_X(a)$

$$= \int_a^b f_X(u)\,du$$

$P(X=a) = P(X=b) = 0 \Rightarrow$

$$P(a < X \le b) = P(a < X < b) = P(a \le X \le b)$$
$$= P(a \le X < b) = \int_a^b f_X(u)\,du$$

② $\forall (a,b) \in \mathbb{R}^2$, $\int_a^b f_X \geq 0 \Rightarrow f_X \geq 0$

③ $\int_a^b f_X(u)\,du = F_X(b) - F_X(a)$

$$\int_{-\infty}^{\infty} f_X(u)\,du = \lim_{a\to-\infty}\ \lim_{b\to\infty} \left(F_X(b) - F_X(a)\right)$$
$$= 1$$

(I) $P(X=x) = 0$. So, f_X is not a probability

In particular, f_X can take values > 1

But,
$$\lim_{dx\to 0} \frac{F_X(x+dx) - F_X(x)}{dx} = f_X(x)$$
(assuming f is continuous at x)

$$F_X(x+dx) - F_X(x) \approx f_X(x)\,dx \quad \text{for small } dx$$

$$P(x \le X \le x+dx) \approx f_X(x)\,dx$$

Def: Expectation of a continuous r.v. X with pdf f is

$$EX = \int_{-\infty}^{\infty} x f(x)\,dx,$$ whenever the integral exists.

LOTUS:
$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

$$Var(x) = E\left((X - Ex)^2\right) = Ex^2 - (Ex)^2$$

Example:
$$f_X(u) = \begin{cases} A(1-u^2) & -1 \le u \le 1 \\ 0 & \text{else} \end{cases}$$

Find A s.t. f_X is a valid pdf.

$$1 = \int_{-\infty}^{\infty} f_X(u)\,du = \int_{-1}^{1} A(1-u^2)\,du$$

$$\Rightarrow \quad A = \frac{3}{4}$$

Find
$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{\frac{1}{2}}^{3/2} f_X(u)\,du$$

$$= \text{H.W.}$$

cdf
$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \text{H.W.} & -1 < x \le 1 \\ 1 & x > 1 \end{cases}$$

$$EX = \int_{-1}^{1} u\, f_X(u)\,du$$

$$= \int_{-1}^{0} \quad + \int_{0}^{1} \quad = 0.$$

$$\mathrm{Var}\,X = EX^2 = \int_{-1}^{1} u^2 \frac{3}{4}(1-u^2)\,du$$

$$= \text{H.W.}$$

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Lemma: X has pdf f s.t. $f(x)=0$ for $x<0$.

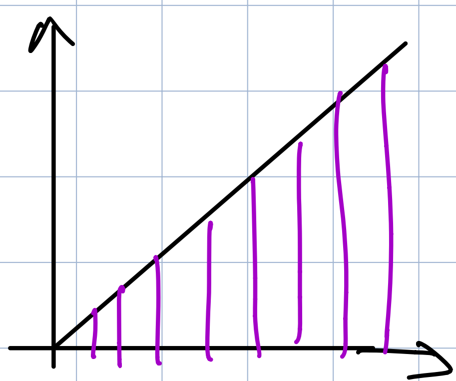
Then,

$$EX = \int_0^\infty P(X>x)\,dx$$

Pf:

$$EX = \int_0^\infty x f(x)\,dx$$

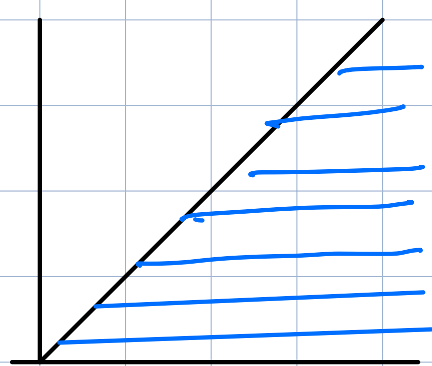
$$= \int_0^\infty \left(\int_0^x dy\right) f(x)\,dx$$



x: 0 to ∞

y: 0 to x

Changing order



y: 0 to ∞

x: y to ∞

So,
$$EX = \int_0^\infty \left(\int_y^\infty f(x)\, dx \right) dy$$

$$= \int_0^\infty P(X > y)\, dy$$

$\blacksquare$

<u>LOTUS:</u> $g \geq 0$

$$\underline{E}(g(X)) = \int_0^\infty P(g(X) > x)\, dx$$

$$= \int_0^\infty \left(\int_B f_X(y)\, dy \right) dx$$

$$B = \{ y \mid g(y) > x \}$$

$$= \int_0^\infty \int_0^{g(y)} dx\, f_X(y)\, dy$$

$$= \int_0^\infty g(y) f_X(y)\, dy$$

$\blacksquare$

More generally,

if X and $g(X)$ continuous r.v.s, then

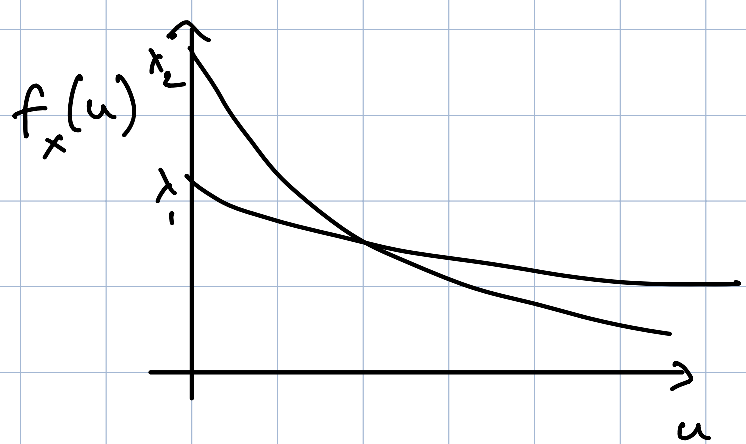
$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Examples

① Exponential distribution

A r.v. X is exponential with parameter $\lambda > 0$ if its pdf is given by

$$f_X(u) = \begin{cases} \lambda e^{-\lambda u}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$



$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{0}^{x} \lambda e^{-\lambda u}\,du = 1 - e^{-\lambda x}$$

$$F_X(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

$$EX = \int_0^{\infty} (1 - F(x))\,dx = \int_0^{\infty} e^{-\lambda x}\,dx = \frac{1}{\lambda}$$

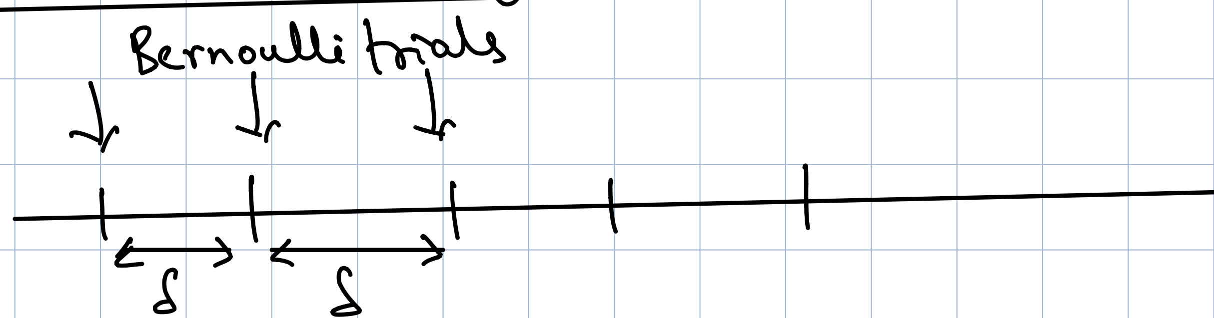
H.W. Calculate $EX^n = \int_0^{\infty} x^n \lambda e^{-\lambda x}\,dx$

Memorylessness:

$$P(X > s+t \mid X > s) = P(X > t)$$

$$\frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

Relation between geometric & exponential r.v-s



$W =$ time for the first head

$$P(W > k\delta) = (1-p)^k$$

$$EW = \frac{\delta}{p}$$

Fix a time "t". By t, we have $\frac{t}{\delta}$ trials.

Let $\delta \downarrow 0$

Want $\lim_{\delta \downarrow 0} P(W > t)$ to be non-trivial

Assume $\frac{p}{\delta} \to \lambda$.

$$P(W > t) = P\left(W > \left(\frac{t}{\delta}\right)\delta\right)$$

$$= (1-p)^{t/\delta}$$

$$\approx (1 - \lambda\delta)^{t/\delta} \longrightarrow e^{-\lambda t}$$

(2) Normal / Gaussian r.v.

2 parameters : μ and σ^2

Density: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty$

Denote: $N(\mu, \sigma^2)$

Notes:

(1) If $\mu = 0, \sigma^2 = 1$, then $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

(2) $X \sim N(\mu, \sigma^2) \qquad Y = \frac{X-\mu}{\sigma}$

$$P(Y \le y) = P(X \le y\sigma + \mu)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y\sigma+\mu} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{v^2}{2}\right) dv,$$

Change of variable $x = v\sigma + \mu$

So, $Y \sim N(0,1)$

$E\,Y = 0$, $Var\,Y = 1$.

$X \sim N(\mu, \sigma^2)$, then $E\,X = \mu$, $Var\,X = \sigma^2$

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③ Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty$$

Reading assignment:-

Check Gamma & Beta distributions.

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Joint distributions

Def: Joint distribution of (jointly continuous) r.v.s X, Y is $F : \mathbb{R}^2 \to [0,1]$ given by

$$F(x,y) = P(X \le x, Y \le y)$$
$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv, \quad \forall x,y \in \mathbb{R}$$

f = joint density function

Note: $P(X \in (a,b), Y \in (c,d)) = \int_{c}^{d}\int_{a}^{b} f(x,y)\,dx\,dy$

Marginal distributions

$$F_X(x) = P(X \le x) = F(x, \infty) = \lim_{y \to \infty} F(x,y)$$

Similarly, $F_Y(y) = P(Y \le y) = F(\infty, y)$

$$F_X(x) = \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f(u,y)\,dy \right) du$$

Marginal pdf $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Similarly, $\quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

LOTUS: $$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy$$

Corollary: $$E(aX+bY) = a\,EX + b\,EY$$

Independence

X and Y independent if

$\{X \le x\}$ and $\{Y \le y\}$ are independent $\forall x, y$.

Alternately, X and Y independent if

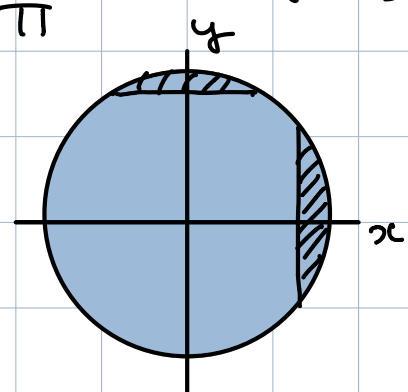
$$F(x,y) = F_X(x)\, F_Y(y) \quad \forall x,y$$

(or) $$f(x,y) = f_X(x)\, f_Y(y)$$

Examples

① (X,Y) r.v.s with $f(x,y) = \frac{1}{\pi} I_{\{x^2+y^2 \le 1\}}$

Are (X,Y) independent?



$$P((X,Y) \in [0.9,1] \times [0.9,1]) = 0$$

But $P(X \in [0.9,1])$ & $P(Y \in [0.9,1])$ are not zero.

② X, Y independent Unif$[0,1]$

(a) Let $U = \min(X,Y)$

$$P(U > u) = P(X > u)\, P(Y > u) = (1-u)(1-u)$$

$$F_U(u) = P(U \le u) = 1 - (1-u)^2, \quad 0 < u < 1$$
$$= 2u - u^2$$

$$f_U(u) = 2 - 2u$$

$$EU = 1/3.$$

(b) $V = \max(X,Y)$

$$P(V \le v) = P(X \le v)\, P(Y \le v) = v^2$$

$$f_V(v) = 2v$$

$$EV = \int_0^1 v\, 2v = \frac{2}{3}$$

(c) $Cov(U, V) = E(UV) - E(U)E(V)$

$E(UV) = E(XY) = EX\,EY = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$Cov(U, V) = \frac{1}{4} - \frac{1}{3} \times \frac{2}{3} = \frac{1}{36}$

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(2) X, Y finite variances, independent

$U = X + Y$, $V = XY$

Let μ_X, μ_Y be means & σ_X^2, σ_Y^2 variances of X, Y.

Find a condition involving $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ that ensures U, V are uncorrelated.

Bivariate normal distribution

Let $f(x,y) = \dfrac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\dfrac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2) \right)$

$-1 < \rho < 1$

Note: f is a valid pdf, i.e.,

$$f(x,y) \geq 0 \quad \forall x,y \quad \& \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

So, f is the joint density of some X, Y.

Marginals: X, Y are $N(0,1)$.

$$\text{Cov}(X,Y) = E(XY) - EX\,EY$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \rho$$

Remark:- If $\rho = 0$,

$$f(x,y) = \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)$$

So, X, Y are independent

In general, uncorrelated $\not\Rightarrow$ independence.

Fact:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$ is a valid pdf

By a change of variable argument,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \text{ is a valid pdf}$$

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$$\mathrm{Cov}(X,Y) = \iint xy\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \left[\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2}\frac{(x-\rho y)^2}{(1-\rho^2)}\right) dx \right] dy$$

$$= \int y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \rho y\, dy$$

$$= \frac{\rho}{\sqrt{2\pi}} \int y^2 e^{-y^2/2}\, dy = \rho$$

The joint density can be written as

$$f(x,y) = \underbrace{\left[\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right]}_{f_Y(y)} \underbrace{\left[\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2}\frac{(x-\rho y)^2}{(1-\rho^2)}\right)\right]}_{f_{X|Y}(x|y)}$$

$$f_Y \leftarrow N(0,1)$$
$$f_{X|Y} \leftarrow N(\rho y, (1-\rho^2))$$

(X,Y) is bivariate normal with means μ_1, μ_2, variances $\sigma_1^2, \sigma_2^2 > 0$ & correlation coefficient $|\rho|<1$, if its joint pdf is given by

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}Q(x,y)\right), \text{ where}$$

$$Q(x,y) = \frac{1}{(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]$$

Show that

1. $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$
2. Correlation coefficient between X, Y is ρ
3. X & Y independent if & only if $\rho = 0$

Conditional distribution & conditional expectation

X, Y r.v.s with joint density f

$P(Y \le y \mid X = x)$ is undefined

If $f_X(x) > 0$, then

$$P(Y \le y \mid x \le X \le x + dx) = \frac{P(Y \le y,\ x \le X \le x + dx)}{P(x \le X \le x + dx)}$$

$$\approx \frac{\int_{-\infty}^{y} f(x, v)\, dx\, dv}{f_X(x)\, dx}$$

$$= \int_{-\infty}^{y} \frac{f(x, v)}{f_X(x)}\, dv$$

As $dx \downarrow 0$, LHS approaches $P(Y \le y \mid X = x)$ & so,

__Def:__ Conditional distribution of Y given $X = x$ is

$$F_{Y|X}(\cdot \mid x) = \int_{-\infty}^{y} \frac{f(x, v)}{f_X(x)}\, dv,$$

for any x s.t. $f_X(x) > 0$

We shall denote the above by $P(Y \le y \mid X = x)$

Def: Conditional density function, denoted by $f_{Y|X}$, is

$$\boxed{f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}}, \quad \forall x \text{ s.t. } f_X(x) > 0$$

Since $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dy}$$

EXAMPLE: X, Y with density

$$f_{X,Y}(x,y) = \frac{1}{x}, \quad 0 \le y \le x \le 1$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x \frac{1}{x}\,dy = 1$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{x}, \quad 0 \le y \le x \le 1$$

So, $X \sim \text{Unif}[0,1]$, $Y|X \sim \text{Unif}[0,x]$
conditioned on $X=x$

$$P(X^2+Y^2 \le 1 \mid X=x)$$

Let $A(x) = \{ y \in \mathbb{R} \mid 0 \le y \le x,\ x^2 + y^2 \le 1 \},\ x > 0$

$$A(x) = \left[0,\ \min\left(x, \sqrt{1-x^2}\right) \right]$$

$$P(X^2 + Y^2 \le 1 \mid X = x) = \int_{A(x)} f_{Y|X}(y|x)\, dy$$

$$= \int_0^{\min(x, \sqrt{1-x^2})} \frac{1}{x}\, dy = \frac{1}{x} \min\left(x, \sqrt{1-x^2}\right)$$

$$P(X^2 + Y^2 \le 1) = \iint_A f(x,y)\, dx\, dy$$

$$A = \{ (x,y) \mid x^2 + y^2 \le 1,\ 0 \le y \le x \le 1 \}$$

$$P(X^2 + Y^2 \le 1) = \int_0^1 f_X(x) \left[\int_{y \in A(x)} f_{Y|X}(y|x)\, dy \right] dx$$

$$= \int_0^1 \frac{1}{x} \min\left(x, \sqrt{1-x^2}\right) dx$$

$$= \int_0^1 \min\left(1, \sqrt{\frac{1}{x^2} - 1}\right) dx$$

$$= \log\left(1 + \sqrt{2}\right) \qquad \blacksquare$$

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Def: Conditional expectation $\psi(X) = E[Y|X]$ is defined by

$$\psi(x) = E[Y|X=x] = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x)\, dy$$

Thm: $\psi(X) = E[Y|X]$ satisfies

$$E[\psi(X)] = EY$$

i.e., $$E[E[Y|X]] = EY$$

So, $$EY = \int_{-\infty}^{\infty} E[Y|X=x] f_X(x)\, dx$$

Example Bivariate normal

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)}_{\substack{f_X(x) \\ = N(0,1)}} \underbrace{\left(\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)\right)}_{\substack{f_{Y|X}(y|x) \\ = N(\rho x,\, 1-\rho^2)}}$$

$$E(Y \mid X=x) = \rho x, \quad E[Y \mid X] = \rho X$$

The other example re-visited

$$f(x,y) = \frac{1}{x}, \quad 0 \le y \le x \le 1$$

$$f_X(x) = 1, \quad 0 \le x \le 1, \quad f_{Y|X}(y \mid x) = \frac{1}{x}, \quad 0 \le y \le x \le 1$$

$$E[Y \mid X=x] = \frac{x}{2}$$

$$E[Y] = \int_0^1 E[Y \mid X=x] f_X(x)\, dx$$

$$= \int_0^1 \frac{x}{2} \cdot 1 \cdot dx = \frac{1}{4}$$

Functions of r.v-s

Let X be a r.v. with density f

$g : \mathbb{R} \to \mathbb{R}$, $Y = g(X)$ is a r.v.

$$P(Y \le y) = P(g(X) \le y)$$

$$= P(g(X) \in (-\infty, y])$$

$$= P\left(X \in g^{-1}(-\infty, y] \right),$$

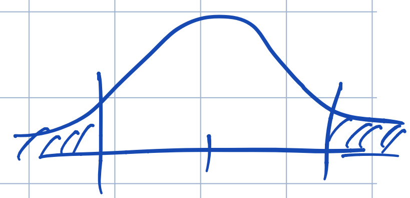
where $g^{-1}(A) = \{ x \in \mathbb{R} \mid g(x) \in A \}$ for any $A \subseteq \mathbb{R}$

So, $$P(Y \leq y) = \int_{g^{-1}(-\infty, y]} f(x)\, dx$$

Example: $X \sim N(0,1)$, $g(x) = x^2$, $Y = X^2$

$$P(Y \leq y) = P(X^2 \leq y)$$

$$= P(-\sqrt{y} \leq X \leq \sqrt{y})$$



$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

$$= 2\Phi(\sqrt{y}) - 1, \text{ since } \Phi(x) = 1 - \Phi(-x)$$

Differentiating on both sides,

$$f_Y(y) = 2 \frac{d}{dy} \Phi(\sqrt{y}) = \frac{1}{\sqrt{y}} \Phi'(\sqrt{y})$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2}, \quad y \geq 0$$

So, X^2 is Gamma$\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\boxed{Z \text{ is Gamma}(\lambda, t),\ \lambda, t > 0 \text{ if } f(z) = \frac{1}{\Gamma(t)} \lambda^t z^{t-1} e^{-\lambda z},\ z \geq 0}$$

where $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$

Another example

$$X, \quad g(x) = ax+b, \quad a, b \in \mathbb{R}, \quad Y = aX+b$$

$$P(Y \le y) = P(aX+b \le y) = \begin{cases} P\left(X \le \frac{y-b}{a}\right) & \text{if } a > 0 \\ P\left(X > \frac{y-b}{a}\right) & \text{if } a < 0 \end{cases}$$

$$f_Y(y) = |a|^{-1} f_X\left(\frac{y-b}{a}\right)$$

Change of variable formula

$X = (X_1, X_2)$ with joint density f

$Y = T(X)$, $\quad Y_1 = T_1(X_1, X_2)$, $Y_2 = T_2(X_1, X_2)$

Assume T is invertible

Question: What is the joint density $g(y)$ of Y_1, Y_2

For e.g., X_1, X_2 are independent $\mathcal{E}xp(\lambda)$

$$T(x_1, x_2) = \left(x_1 + x_2, \ \frac{x_1}{x_1 + x_2}\right)$$

Answer: Change of variable formula on next page

$$g(y) = f(T^{-1}y)\,|J(T^{-1})(y)|$$

(or)

$$g(y_1, y_2) = f(T^{-1}(y_1, y_2))\,|J(T^{-1})(y)|$$

T is invertible, so, given y_1, y_2, we can get

$$x_1 = T_1^{-1}(y_1, y_2), \quad x_2 = T_2^{-1}(y_1, y_2)$$

$$J(T^{-1})(y) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \leftarrow \text{Jacobian determinant}$$

Example: X_1, X_2 $\mathcal{E}xp(\lambda)$ independent

$$T(x_1, x_2) = \left(x_1 + x_2, \frac{x_1}{x_1 + x_2} \right) \quad \text{Range } \mathbb{R}_+ \times (0,1)$$

$$T^{-1}(y_1, y_2) = (y_1 y_2, y_1(1 - y_2))$$

$$J(T^{-1})(y_1, y_2) = \begin{vmatrix} y_2 & 1 - y_2 \\ y_1 & -y_1 \end{vmatrix} = -y_1$$

So, $|J(T^{-1})(y_1, y_2)| = y_1$

Using change of variable formula,

$$g(y_1, y_2) = f(y_1 y_2, y_1(1-y_2)) \cdot y_1$$

$$= \lambda^2 \exp(-\lambda(y_1 y_2 + y_1(1-y_2))) \cdot y_1$$

$$= \lambda^2 y_1 \exp(-\lambda y_1),$$

$$y_1 > 0, \quad y_2 \in (0,1)$$

$Y_1 = X_1 + X_2$ has density

$$g_{Y_1}(y_1) = \int_0^1 \lambda^2 y_1 \exp(-\lambda y_1)\, dy_2$$

$$= \lambda^2 y_1 \exp(-\lambda y_1)$$

$$Y_1 \sim \text{Gamma}(2, \lambda)$$

$Y_2 = \dfrac{X_1}{X_1 + X_2}$ has density

$$g_{Y_2}(y_2) = \int_0^\infty \lambda^2 y_1 \exp(-\lambda y_1)\, dy_1$$

$$= 1 \quad \text{for} \quad y_2 \in (0,1)$$

Y_1, Y_2 are independent since

$$g(y_1, y_2) = g_{Y_1}(y_1)\, g_{Y_2}(y_2)$$

Example $X_1, X_2 \sim \mathcal{E}xp(\lambda)$, independent

$Y_1 = X_1 + X_2$, $\quad Y_2 = X_2$

$T(x_1, x_2) = (x_1 + x_2, x_2)$ $\quad$ Range $\{(y_1, y_2) \mid y_1 \geq y_2 \geq 0\}$

$T^{-1}(y_1, y_2) = (y_1 - y_2, y_2)$

$$J T^{-1}(y_1, y_2) = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$g(y_1, y_2) = f(y_1 - y_2, y_2) \cdot 1$$

$$= \lambda^2 \exp\left(-\lambda(y_1 - y_2 + y_2)\right)$$

if $y_1 \geq y_2 \geq 0$

$$= \lambda^2 \exp(-\lambda y_1)\, I(y_1 > y_2 \geq 0)$$

$$g_{Y_1}(y_1) = \int_0^{y_1} \lambda^2 \exp(-\lambda y_1)\, dy_2$$

$$= \lambda^2 \exp(-\lambda y_1)\, y_1$$

Are Y_1, Y_2 independent? **No!**

$$P(Y_2 \le y_2 \mid Y_1 = y_1)$$

$$= \frac{\int_{-\infty}^{y_2} g(y_1, v)\, dv}{\int_{-\infty}^{\infty} g(y_1, y_2)\, dy_2}$$

$$= \frac{y_2 \lambda^2 e^{-\lambda y_1}}{y_1 \lambda^2 e^{-\lambda y_1}}$$

$$= \frac{y_2}{y_1}, \quad 0 < y_2 < y_1$$

Conditioned on $Y_1 = y_1$, $Y_2 \sim \text{Unif}[0, y_1]$

Example X_1, X_2 with joint density f

$$Y_1 = X_1 - X_2, \quad Y_2 = X_1 + X_2$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$T : x \to Ax, \quad T^{-1} : y \to A^{-1} y$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$(y_1, y_2) \xrightarrow[A^{-1}]{} \left(\frac{y_1 + y_2}{2}, \frac{-y_1 + y_2}{2} \right)$$

$$g_{Y_1, Y_2}(y_1, y_2) = f\left(\frac{y_1 + y_2}{2}, \frac{-y_1 + y_2}{2} \right) \times \frac{1}{2}$$

In general, $g_{Y_1, Y_2}(y) = f(A^{-1}y) \cdot \frac{1}{|A|}$.

Problems from other topics

Lecture-20 (Tutorial)

(1) $X_1, X_2, \ldots$ identically distributed r.v.s with mean μ

N is a r.v. that takes values $1, 2, \ldots$ & is independent of $\{X_i\}$

$$S = X_1 + \cdots + X_N$$

Find $E[S]$.

$E[S \mid N=n] = n\mu$, $E[S \mid N] = N\mu$

So, $E[S] = E[E[S \mid N]] = \mu E[N]$

(2) A fair coin tossed 3 times.

$Y =$ outcome of first toss

$X =$ # heads

Find $E[X|Y]$?

$$E[X|Y=1] = \sum_{k=1}^{3} k \, P(X=k|Y=1)$$

$$= P(X=1|Y=1) + 2P(X=2|Y=1) + 3P(X=3|Y=1)$$

$$= \frac{P(X=1, Y=1) + 2P(X=2, Y=1) + 3P(X=3, Y=1)}{P(Y=1)}$$

$$= \frac{1/8 + 2 \times 1/4 + 3 \times 1/8}{1/2} = ?$$

$E[X|Y=0] =$ H. W.

---

(3) Find $f_{Y|X}(\cdot|x)$ when X, Y have joint density

(a) $f(x,y) = \lambda^2 e^{-\lambda y}, \quad 0 \le x \le y < \infty$

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y} \, dy = \lambda e^{-\lambda x}$$

$$f_{Y|X}(y|x) = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}, \quad 0 \le x \le y < \infty$$

(b) $$f(x,y) = x e^{-x(y+1)}, \quad x, y \ge 0$$

$$f_X(x) = \int_0^\infty x e^{-x(y+1)} \, dy = e^{-x}$$

$$f_{Y|X}(y|x) = \frac{x e^{-x(y+1)}}{e^{-x}} = x e^{-xy}, \quad 0 \le y < \infty$$

---

4 (a) $X \sim \mathcal{E}xp(\lambda)$, $Y \sim \mathcal{E}xp(\mu)$, independent

Find distribution of $U = \min(X, Y)$

$$P(U > u) = P(X > u)\, P(Y > u) = e^{-\lambda u} e^{-\mu u}$$
$$= e^{-(\lambda+\mu)u}$$

$$P(U \le u) = 1 - e^{-(\lambda+\mu)u}$$

$$U \sim \mathcal{E}xp(\lambda + \mu)$$

$$P(X < Y) = \int_0^\infty \int_0^y f_X(x) f_Y(y) \, dx \, dy$$

$$= \int_0^\infty \int_0^y \lambda e^{-\lambda x} \mu e^{-\lambda y} \, dx \, dy$$

$$= \int_0^\infty \mu e^{-\lambda y}\left(1 - e^{-\lambda y}\right) dy$$

$$= \frac{\lambda}{\lambda+\mu}$$

(b) X, Y, Z independent exponential r.v.s with parameters λ, μ, ϑ

$$P(X < Y < Z) = P\left(X < \min(Y, Z)\right) P(Y < Z)$$

$$= \frac{\lambda}{\lambda+\mu+\vartheta} \times \frac{\mu}{\mu+\vartheta}$$

---

5 $\quad f(x,y) = c\,(2x+3y), \quad 0 < x, y < 1$

(i) $c = ?$

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \frac{5}{2} \Rightarrow c = \frac{2}{5}$$

(ii) $$f_X(x) = \int_0^1 \frac{2}{5}(2x+3y)\,dy = \frac{2}{5}\left(2x + \frac{3}{2}\right)$$

(iii) $$f_{Y|X}(y|x) = \frac{2x+3y}{2x+3/2}$$

(iv) $E[Y \mid X = x] = \int_0^1 y \, f_{Y|X}(y|x) \, dy$

$$= \frac{x+1}{2x + 3/2}$$

$$E[Y \mid X] = \frac{X+1}{2X + 3/2}$$

---

6 $\{X_i\}, i = 1 \ldots n$ i.i.d. $EX_i = \mu$, $Var(X_i) = \sigma^2 < \infty$

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Find $E(\overline{X}_n (X_j - \overline{X}_n))$

& $Cov(\overline{X}_n, X_j - \overline{X}_n)$, where $j \in \{1 \ldots n\}$

Lecture-21

Multivariate Normal Distribution

$X \sim N(0,1)$ density $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$X, Y \sim$ Bivariate Normal density $\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}\right)$

Note: quadratic component in the exponent

Generalizing to n-dimensions $X_1, \dots X_n$

Expect the joint density as a re-scaled version of

$$\exp\left(-\sum_{i=1}^{n} x_i^2 - 2\sum_{i<j} b_{ij} x_i x_j\right)$$

Suppose $Q: \mathbb{R}^n \to \mathbb{R}$ $x = (x_1, \dots x_n)^T$

$$Q(x) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j = x^T A x$$

$A = [[a_{ij}]]$

A is real symmetric

$\Rightarrow$ A can be diagonalized, i.e.,

$\exists B$ (orthogonal) s.t. $A = B^T \Lambda B$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad B^T B = I$$

Hence, $Q(x) = y^T \Lambda y$, where $y = Bx$

$$= \sum_{i=1}^{n} \lambda_i y_i^2$$

Suppose A is positive definite, i.e., $x^T A x > 0 \ \forall x \neq 0$

$$A > 0 \iff \lambda_i > 0 \quad \forall i$$

$$\iff Q(x) > 0 \quad \forall x \neq 0$$

We want a density using Q.

Let $f(x) = K \exp\left(-\frac{Q(x)}{2}\right)$, $x \in \mathbb{R}^n$

Want (i) $f(x) \geq 0 \ \forall x$, (ii) $\int_{\mathbb{R}^n} f(x)\,dx = 1$

If $K > 0$, then (i) is satisfied.

For (ii), we shall show that $Q > 0$ is

necessary/sufficient.

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} K \exp\left(-\frac{1}{2} Q(x)\right) dx$$

Change of variable

$$= \int_{\mathbb{R}^n} K \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \lambda_i y_i^2\right) dy \quad (**)$$

Justification for $(**)$:

$T: y = Bx$ or $x = B^T y$

$$f(T^{-1}y) = f(B^T y)$$

$$= K \exp\left(-\frac{1}{2} y^T B A B^T y\right)$$

$$= K \exp\left(-\frac{1}{2} y^T \Lambda y\right)$$

$$= K \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \lambda_i y_i^2\right)$$

Check $|J T^{-1}(y)| = 1$ since $|B^T| = 1$ (B is orthogonal)

Hence,

$$\int_{\mathbb{R}^n} f(x)\,dx = K \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \sum \lambda_i y_i^2\right) dy$$

$$= K \prod_{i=1}^{n} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \lambda_i y_i^2\right) dy_i$$

$$= K \sqrt{\frac{(2\pi)^n}{\lambda_1 \cdots \lambda_n}}$$

$$f(x) = \sqrt{\frac{|A|}{(2\pi)^n}} \exp\left(-\frac{1}{2} x^T A x\right)$$ is a joint density if & only if $A > 0$.

Let $(X_1 \cdots X_n)$ be the random vector with density f

$$f(x) = f(-x)$$

So, $(X_1 \cdots X_n)$ have the same distribution as $(-X_1, \cdots -X_n)$

$$E(X_i) = E(-X_i) \Rightarrow \quad EX_i = 0$$

So, $X = (X_1 \cdots X_n)$ is a multivariate normal with zero means.

$$Y = X + \mu, \quad \mu = (\mu_1 \cdots \mu_n)^T$$

Def: $X = (X_1 \cdots X_n)$ has multivariate normal distribution, denoted $N(\mu, V)$, if its joint density is given by

$$f(x) = \frac{1}{\sqrt{(2\pi)^n |V|}} \exp\left(-\frac{1}{2}(x-\mu)^T V^{-1}(x-\mu)\right)$$

$x \in \mathbb{R}^n$

Note: Switch to V^{-1} is done because

Thm: If X is $N(\mu, V)$, then

$EX = \mu$, and

$V = [[v_{ij}]]$, $v_{ij} = Cov(X_i, X_j)$

Thm: If X is $N(\mu, V)$, then its moment generating function $M(t) = E(e^{t^T X})$ is given by

$$M(t) = \exp\left(t^T\mu + \frac{t^T V t}{2}\right)$$

$\boxed{t = (t_1 - - t_n)^T, \quad t_i \text{ real}}$

Notes: ① Pick $t = (0, - - 0, t_i, 0, - - 0)$

$$M(t) = \exp\left(t_i \mu_i + \frac{v_{ii} t_i^2}{2} \right)$$

$$\Rightarrow \quad X_i \sim N(\mu_i, v_{ii})$$

(2) Pick $t = (0 \ldots 0, t_i, 0 \ldots 0, t_j, 0 \ldots 0)$ $i \neq j$

$$M(t) = \exp\left(t_i \mu_i + t_j \mu_j + \frac{v_{ii} t_i^2 + 2 v_{ij} t_i t_j + v_{jj} t_j^2}{2} \right)$$

$\Rightarrow$ $(X_i, X_j) \sim$ Bivariate normal

with means μ_i, μ_j, variances v_{ii}, v_{jj}

& covariance v_{ij}

---

(3) For $X = (X_1, \ldots X_n) \sim N(\mu, V)$,

where $\mu = (\mu_1, \ldots \mu_n)$ is the mean vector &

$V = [[v_{ij}]]_{i,j=1}^{n}$ is the covariance matrix,

i.e., $(v_{11}, \ldots v_{nn})$ are the variances,

and $v_{ij}, i \neq j$ is the covariance.

Letting $\sigma_i^2 = v_{ii}$, & ρ_{ij} denote the correlation coefficient, $i \neq j$,

$$v_{ij} = \rho_{ij} \sigma_i \sigma_j$$

(4) If $v_{ij} = 0 \ \forall i \neq j$, then V is diagonal, implying $X_1, \dots X_n$ are independent.

Converse also holds, leading to

$(X_1, \dots X_n)$ jointly normal independent if & only $v_{ij} = 0 \ \forall i \neq j$

Linear Combinations of Gaussians:

Let $(X_1, \dots X_n)$ be jointly normal with mean zero.

Let $(Y_1, \dots Y_k)$ be linear functions of X_j, $j = 1 \dots n$, for $k \leq n$, i.e.,

$$Y_m = \sum_{j=1}^{n} a_{mj} X_j, \quad m = 1, \dots k$$

Then, $(Y_1, \dots Y_k)$ is jointly normal.

Pf: $\quad EY_m = 0, \; m = 1 \dots k$

$$\mathrm{Cov}(Y_l, Y_m) = \sum_{i,j=1}^{n} a_{li}\, a_{mj}\, v_{ij},$$

where $\quad v_{ij} = E(X_i X_j), \; i,j = 1 \dots n$

MGF of $(Y_1 \dots Y_k)$ is

$$M(t_1 \dots t_k)$$

$$= E\left(\exp\left(t_1 Y_1 + \dots + t_k Y_k\right)\right)$$

$$= E\left(\exp\left[t_1 \sum_{j=1}^{n} a_{1j} X_j + \dots + t_k \sum_{j=1}^{n} a_{kj} X_j\right]\right)$$

$$= E\left(\exp\left[\sum_{i=1}^{n} u_i X_i\right]\right), \text{ where } u_i = \sum_{m=1}^{k} t_m a_{mi}$$

$$= \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} v_{ij}\, u_i\, u_j\right)$$

[for $X \sim N(0, V)$, MGF $M(u) = \exp\left(\frac{u^T V u}{2}\right)$]

$$= \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} v_{ij} \sum_{l,m=1}^{k} t_l t_m a_{li} a_{mj} \right)$$

$$= \exp\left(\frac{1}{2} \sum_{l,m=1}^{k} t_l t_m \sum_{i,j=1}^{n} a_{li} a_{mj} v_{ij} \right)$$

$$= \exp\left(\frac{1}{2} \sum_{l,m=1}^{k} t_l t_m \operatorname{Cov}(Y_l, Y_m) \right)$$

MGF of a jointly normal vector.

Hence, $(Y_1 \ldots Y_k)$ is jointly normal.

<u>Corollary:</u>

① Every marginal distribution of a multivariate normal is univariate normal

(2) Any linear function of $X_1, \ldots, X_n$ is univariate normal.

<u>Some problems</u>

(1) Let $\rho(X,Y)$ be the correlation coefficient of X and Y. Then, $\forall a,b,c,d,\ a>0,\ c>0$, show that

$$\rho(aX+b,\ cY+d) = \rho(X,Y)$$

<u>Soln</u> $\mathrm{Cov}(aX+b,\ cY+d) = ac\,\mathrm{Cov}(X,Y)$

$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$,

$\mathrm{Var}(cY+d) = c^2\,\mathrm{Var}(Y)$

(2) Let (X,Y) be bivariate normal with means μ_1, μ_2, variances σ_1^2, σ_2^2 & correlation co-efficient ρ.

Let $U = aX + b$, $V = cY + d$, $a, c > 0$

Find joint distribution of (U, V).

Soln: (U, V) is bivariate normal with

means $a\mu_1 + b$, $c\mu_2 + d$,

variances $a^2\sigma_1^2$, $c^2\sigma_2^2$ &

correlation ρ.

$\boxed{///}$

(3) X and Y bivariate normal with equal variance.

Find $a > 0$ s.t. $(X + aY)$ is independent of $(X - aY)$.

Soln: $(X + aY, X - aY)$ is bivariate normal

$$\mathrm{Cov}(X + aY, X - aY) = \mathrm{Var}(X) - a^2\,\mathrm{Var}(Y)$$

$$= 0 \text{ when } a = +1 \text{ or } -1$$

So, when $a = +1$, $(X + aY)$ is indep. of $(X - aY)$.

$\boxed{///}$

4) Suppose X, Y indep. r.v.s with densities

$$f_X(x) = \lambda^2 x e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = \frac{\lambda^3}{2} y^2 e^{-\lambda y}, \quad y \geq 0$$

Let $U = X + Y$, $\quad V = \dfrac{X}{X+Y}$

Find (1) Joint density $g_{U,V}(u, v)$

Sol'n

$$T : (x, y) \to \left(x + y, \frac{x}{x+y}\right)$$

$$T^{-1} : (u, v) \to (uv, u(1-v))$$

$$|J(T^{-1}(u,v))| = u$$

$$f_{X,Y}(x, y) = \frac{\lambda^5}{2} x y^2 e^{-\lambda x} e^{-\lambda y}, \quad x, y \geq 0$$

$$g_{U,V}(u, v) = \frac{\lambda^5}{2} u^4 v (1-v)^2 e^{-\lambda u},$$

$$u \geq 0, \quad 0 < v < 1$$

H.W.

1 Find marginal densities $g_U(\cdot)$ & $g_V(\cdot)$

3 Check if U and V are independent.