Linear Transformations

A linear transformation T is a correspondence that assigns $v \in V$ to $Tv \in V$ s.t.

$$\left.\begin{array}{ll} \text{(i)} & T(v+w) = Tv + Tw \\ \text{(ii)} & T(\alpha w) = \alpha Tw \end{array}\right\} \quad \forall\, v, w \in V \text{ \& } \alpha \in \mathbb{R}$$

Examples:- 1 $T: \mathbb{R} \to \mathbb{R}$ $\quad T(x) = 2x$

2 $T: \mathbb{R}^2 \to \mathbb{R}^2$. Fix $b \in \mathbb{R}^2$

$$T(v) = v^T b$$

3 $T: \mathbb{R}^2 \to \mathbb{R}^2$. Fix $v_0 \in \mathbb{R}^2$ $\Big\}$ Not a lin. tr.

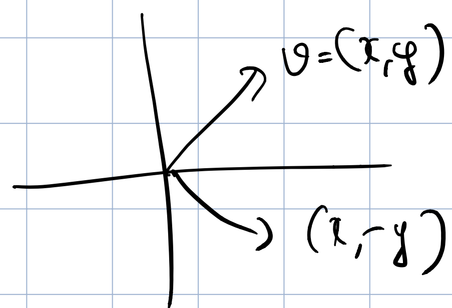
$$T(v) = v + v_0$$

4 Projection $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

5 $T(v) = Av$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Reflection

6 $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2 \end{bmatrix}$$

not a lin. tr.

$T(0) \neq 0$

7 $T(v) = \|v\|$ $\to$ not a lin. tr.

$T(-v) \neq -T(v)$.

Fact: T preserves linear combinations

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = c_1 T v_1 + \cdots + c_k T v_k$$
$$\forall\, v_1 \dots v_k,\ c_1 \dots c_k$$

Linear transformation can be represented by matrices.

$\{u_1, \dots, u_n\} \to$ basis of V

If we know $Tu_i\ \forall i$, then we know $Tx\ \forall x \in V$.

Why?
$$x = c_1 u_1 + \cdots + c_n u_n$$
$$Tx = c_1 T u_1 + \cdots + c_n T u_n$$

Examples: ① Scaling: $T(x,y) = (cx, cy)$

$T(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$, where $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = cI$

② Rotation (orthogonal)

$T(x,y) = (-y, x)$. $\quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

③ Reflection

$T(x,y) = (y, x) \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

④ Projection $T(x,y) = (x, 0)$ $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Matrix of a linear transformation:

Let $e_1 \dots e_n$ is a standard basis of $\mathbb{R}^n$

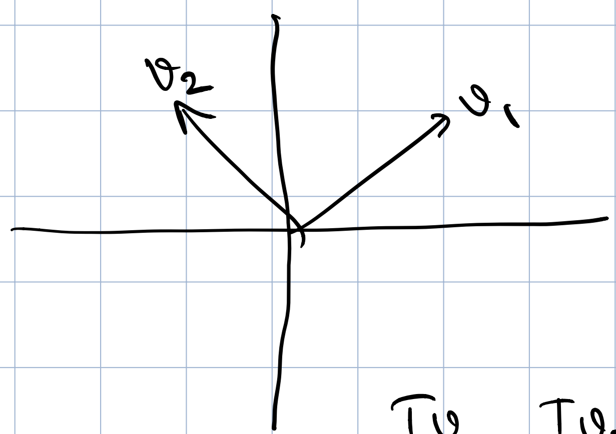
$T: \mathbb{R}^n \to \mathbb{R}^m \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$x = x_1 e_1 + \cdots + x_n e_n$$

$$T(x) = x_1 Te_1 + \cdots + x_n Te_n$$

$$= \begin{bmatrix} | & & | \\ Te_1 & \cdots & Te_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Example:- Projection $T : \mathbb{R}^2 \to \mathbb{R}$



Basis $= \{v_1, v_2\}$

$v = c_1 v_1 + c_2 v_2$

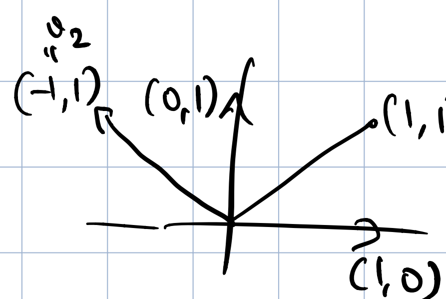
$T(v) = c_1 T v_1 + c_2 T v_2$

$T v_1 = v_1, \quad T v_2 = 0$

$$A = \begin{array}{c c} & \begin{array}{cc} Tv_1 & Tv_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Suppose we change to the standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Te_1 =$



$$e_1 = x_1 v_1 + x_2 v_2$$
$$= \frac{1}{2} v_1 - \frac{1}{2} v_2$$

$$Te_1 = \frac{1}{2} v_1 = \frac{1}{2} e_1 + \frac{1}{2} e_2$$

$$Te_2 = \frac{1}{2} v_1 = \frac{1}{2} e_1 + \frac{1}{2} e_2$$

Matrix of T wrt (e_1, e_2) is

$$\begin{array}{c c} \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ & \begin{array}{cc} Te_1 & Te_2 \end{array} \end{array}$$

What is the projection matrix for projecting onto a line through $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Coming next:- Matrix multiplication

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend left, "ST"] & W \arrow[r, "S"] & U \end{tikzcd}$$

$f(x) = x^2 \qquad g(x) = 3x - 1$

$(f \circ g)(x) = f(g(x)) = (3x-1)^2$

Lecture-15: Linear transformations

Matrix of a linear transformation

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$

$\{v_1 \cdots v_n\} \to$ basis for $\mathbb{R}^n$

$\{w_1 \cdots w_m\} \to$ basis for $\mathbb{R}^m$

$$Ax = \begin{bmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Col 1 of $A = Tv_1 = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$

& so on

$$\begin{array}{cc} & \begin{array}{ccc} Tv_1 & \cdots & Tv_n \\ \downarrow & & \downarrow \end{array} \\ \begin{array}{c} w_1 \\ \vdots \\ w_n \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array}$$

A = matrix of T wrt $\{w_1 \cdots w_m\}$

Matrix multiplication (revisited):

$f(x) = x^2, \quad g(x) = 3x - 1$

$(f \circ g)(x) = f(g(x)) = (3x-1)^2$

$(g \circ f)(x) = g(f(x)) = 3x^2 - 1 \neq (f \circ g)(x)$

Composing linear transformations:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & W \arrow[r, "S"] & U \end{tikzcd}$$

$(ST)(v) = S(T(v))$

<u>Example:-</u>

$$\mathbb{R}^n \xrightarrow[B]{T} \mathbb{R}^m \xrightarrow[A]{S} \mathbb{R}^p$$

matrices for T and S

$e_1, \ldots, e_n \to$ basis for $\mathbb{R}^n$

$(ST)(e_1) = S(T(e_1))$

$T(e_1) = b_1 \leftarrow$ Col 1 of B

$(ST)(e_1) = S(b_1) = Ab_1$

Matrix for ST is (on next page)

$$[M]_{ST} = \begin{bmatrix} Ab_1 & - & - & - & - & Ab_n \end{bmatrix} = A \begin{bmatrix} b_1 & - & - & b_n \end{bmatrix}$$

$$= AB$$

H.W. $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Change of basis:-

$B_1 = \{v_1, \dots, v_n\}$ $B_2 = \{w_1, \dots, w_n\}$

$B_1, B_2 \to$ basis of $\mathbb{R}^n$

Q1>

$$v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i w_i$$

What is the relation between $(\alpha_1 \dots \alpha_n)$ & $(\beta_1 \dots \beta_n)$

Q2> Fix $(\alpha_1 \dots \alpha_n)$

(H.W.) $v = \sum_{i=1}^{n} \alpha_i v_i$, $w = \sum_{i=1}^{n} \alpha_i w_i$

How is v related to w?

Answer to Q17

Define a linear transformation A as

$$A v_i = w_i, \quad i = 1 \ldots n$$

$$A\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i w_i$$

$$[M]_A = \begin{array}{c} \\ v_1 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{ccc} Av_1 & \cdots & Av_n \\ \downarrow & & \downarrow \end{array} \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array}$$

$$Av_1 = w_1 = a_{11} v_1 + \cdots + a_{n1} v_n$$

$$\vdots$$

$$Av_n = w_n = a_{1n} v_1 + \cdots + a_{nn} v_n$$

Notice that $v = \sum_{j=1}^{n} \beta_j w_j = \sum_{j=1}^{n} \beta_j \left(\sum_{i=1}^{n} a_{ij} v_i\right)$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \beta_j\right) v_i$$

$$= \sum_{i=1}^{n} \alpha_i v_i$$

So, $\alpha_i = \sum_{j} a_{ij} \beta_j$

SIMILARITY OF TRANSFORMATIONS

Q3) B is a linear transform on $\mathbb{R}^n$

What is the relation between its matrix

$[[\beta_{ij}]]$ wrt $\mathcal{B}_1$ and $[[\gamma_{ij}]]$ wrt $\mathcal{B}_2$

Q4) If $[[\beta_{ij}]]$ is a given matrix, what is the relation between two linear transformations B and C, defined by

$$Bv_j = \sum_{i=1}^{n} \beta_{ij} v_i \quad \text{and} \quad Cw_j = \sum_{i=1}^{n} \beta_{ij} w_i$$

Q3: One transform, 2 matrices

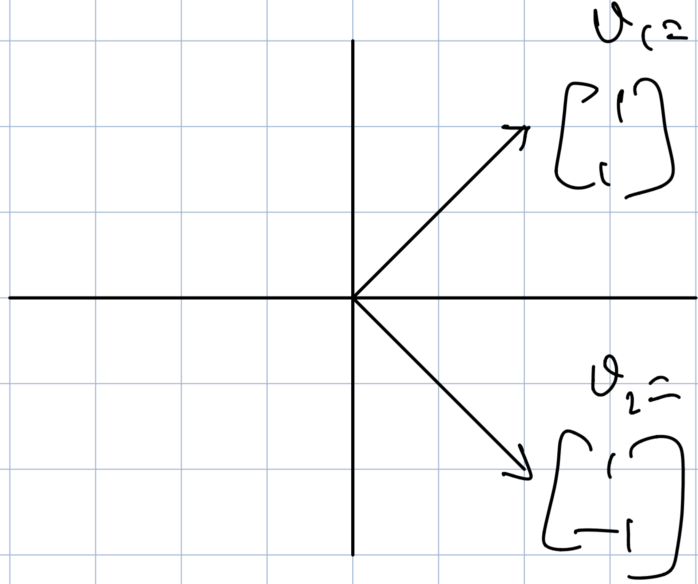
Q4: One matrix, 2 transforms

Def: ① Two matrices $[B]$, $[C]$ similar if there exists an invertible $[A]$ s.t.

$$[C] = [A]^{-1}[B][A]$$

② Two linear transforms B, C are similar if $\exists$ invertible A s.t. $C = ABA^{-1}$

Example:-



$B_1 = \{v_1, v_2\}$

$$[M]_{B_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Suppose $w_1 = e_1,\ w_2 = e_2$

$$[M]_{B_2} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}}_{[A]^{-1}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{[M]_{B_1}} \underbrace{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{[A]} = \underbrace{\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & \frac{1}{2} \end{bmatrix}}_{[M]_{B_2}}$$

Lecture-16 (linear transformations)

Change of basis

Example

$$B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -2x+3y \end{bmatrix}$$

$$B_1 = \{ v_1, v_2 \}$$
$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What is the matrix of B wrt B_1?

$$\begin{array}{c c} & \begin{array}{cc} Bv_1 & Bv_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \end{array}$$

Consider $B_2 = \{ w_1, w_2 \}$

$$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$[M_B]_{B_2} = \begin{array}{c} \\ w_1 \\ w_2 \end{array} \!\! \begin{array}{cc} Bw_1 & Bw_2 \\ \left[\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}\right. & \left.\right] \end{array}$$

<u>Question></u> B is a linear transformation on $\mathbb{R}^n$.

$[[\beta_{ij}]]$ is the matrix of B wrt. B_1,

$[[\gamma_{ij}]]$ —— " —— " —— " —— B_2,

where $B_1 = \{v_1 \text{--} v_n\}$, $B_2 = \{w_1 \text{--} w_n\}$

<u>Relation between these two matrices:</u>

$A v_i = w_i, \quad i = 1 \text{--} n$

$$\begin{array}{c} \\ v_1 \\ \vdots \\ v_n \end{array} \begin{array}{c} w_1 \quad \cdots \quad w_n \\ \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix} \end{array} \leftarrow \text{matrix } A$$

$$Bv_j = \sum_i \beta_{ij} v_i$$

$$Bw_j = \sum_i \gamma_{ij} w_i$$

$$Bw_j = BAv_j = B\left(\sum_k a_{kj} v_k\right)$$

$$= \sum_k a_{kj} Bv_k$$

$$= \sum_k a_{kj} \sum_i \beta_{ik} v_i$$

① $$\boxed{Bw_j = \sum_i \left(\sum_k \beta_{ik} a_{kj}\right) v_i}$$

Notice that $Bw_j = \sum_k \gamma_{kj} w_k$

$$= \sum_k \gamma_{kj} Av_k$$

$$= \sum_k \gamma_{kj} \sum_i a_{ik} v_i$$

② $$\boxed{B w_j = \sum_i \left(\sum_k a_{ik} \gamma_{kj} \right) v_i}$$

From ① & ②,

$$\sum_k a_{ik} \gamma_{kj} = \sum_k \beta_{ik} \alpha_{kj}$$

$$[M_A]_{B_1} [M_B]_{B_2} = [M_B]_{B_1} [M_A]_{B_1}$$

Finally,
$$[M_B]_{B_2} = [M_A]_{B_1}^{-1} [M_B]_{B_1} [M_A]_{B_1}$$

Back to the example:-

$$[M_B]_{B_1} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \qquad B_1 = \left\{ \begin{matrix} v_1, & v_2 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix} \right\}$$

$B_2 = \{ w_1, w_2 \}$, where $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A v_1 = w_1, \quad A v_2 = w_2$

$$A = \begin{array}{c|cc} & w_1 & w_2 \\ \hline v_1 & 1 & 1 \\ v_2 & 1 & 2 \end{array} \qquad A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M_B]_{B_2} = A^{-1} \, [M_B]_{B_1} \, A$$

$$= \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

H.W.: (I) $B\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y-z \\ -y \\ x+7z \end{bmatrix}$

$B_1 = \{e_1, e_2, e_3\}$

(1) $[M_B]_{B_1}$?

② $B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$[M_B]_{B_2}$?

③ Find a matrix C s.t.

$$[M_B]_{B_2} = C^{-1} [M_B]_B \, C,$$

H.W 11 $\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -7x - 15y \\ 6x + 12y \end{bmatrix}$

Find a basis B' s.t. $[T]_{B'} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Find a invertible C s.t.

$$[T]_{B'} = C^{-1} [T]_B \, C,$$

where B is the standard basis.

Lecture - 17

Linear transformation (wrap-up)

Solutions to HW problems (Sketch):-

(I) $$[M_B]_{B_1} = \begin{array}{ccc} Be_1 & Be_2 & Be_3 \\ \hline \end{array}$$
$$\left[\begin{array}{ccc} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{array}\right]$$

$$[M_B]_{B_2} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{bmatrix}$$

$$[M_B]_{B_2} = C^{-1}[M_B]_{B_1} C$$

$B_2 = \{w_1, w_2, w_3\}$ $B_1 = \{e_1, e_2, e_3\}$

$Ce_1 = w_1, \quad Ce_2 = w_2, \quad Ce_3 = w_3$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(II)

H.W. (II) 11 $\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -7x - 15y \\ 6x + 12y \end{bmatrix}$

Find a basis B' s.t. $[T]_{B'} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Find a invertible C s.t.

$$[T]_{B'} = C^{-1} [T]_B C,$$

where B is the standard basis.

Soln) $B' = \{ b_1, b_2 \}$ $b_1 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix}$ $b_2 = \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix}$

$$T b_1 = 2 b_1$$

$$-7 b_{11} - 15 b_{12} = 2 b_{11}$$
$$6 b_{11} + 12 b_{12} = 2 b_{12}$$
$\rightarrow$ Solve this!

Use $T b_2 = 3 b_2$ to obtain b_2.

Question 4) If $[[\beta_{ij}]]$ is a matrix, what is the relation between transforms B and C, defined by

$$B v_j = \sum_i \beta_{ij} v_i, \qquad C w_j = \sum_i \beta_{ij} w_i$$

Answer:- As before, $A v_j = w_j$, $j = 1 \ldots n$

$$Cw_j = CAv_j$$

$$\sum_i \beta_{ij} w_i = \sum_i \beta_{ij} A v_i$$

$$= A\left(\sum_i \beta_{ij} v_i\right)$$

$$= ABv_j$$

$$CA = AB$$

$$C = ABA^{-1}$$

So, B and C are similar.