Lecture-10

A r.v. X is discrete if it takes values in a countable set $\{x_1, x_2, \dots\}$. Its distribution function $F(x) = P(X \le x)$ is a jump function.

Def: The mass function of a discrete r.v. X is the function $f : \mathbb{R} \to [0,1]$ given by

$$f(x) = P(X = x)$$

$$F(x) = \sum_{i : x_i \le x} f(x_i)$$

Mass function aka pmf satisfies

(a) $A = \{x \mid f(x) \neq 0\}$ is countable

(b) $\sum_{i \in A} f(x_i) = 1$

Examples of discrete r.v.s

① Binomial distribution

Coin tossed n times, heads w.p. p $(=1-q)$

$$\Omega = \{H, T\}^n$$

$X = \#$ of heads $\in \{0, 1, \dots n\}$

$f(x) = 0$ if $x \notin \{0, 1, \dots n\}$

$$f(k) = \binom{n}{k} p^k q^{n-k}, \quad 0 \le k \le n$$

$$X \sim \text{Binom}(n, p)$$

$$X = Y_1 + \dots + Y_n, \quad Y_i \sim \text{Ber}(p)$$

(2) Poisson distribution

$X \in \{0, 1, 2, \dots\}$ with pmf

$$f(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda > 0, \ k = 0, 1, 2, \dots$$

$$X \sim \text{Poisson}(\lambda)$$

H.W. $f(x) = \frac{c\,2^x}{x!}$, $x = 1, 2, ---$

Find "c" s.t. f is a valid pmf.

Independence

Recall A and B independent if

$$P(A \cap B) = P(A)\,P(B)$$

<u>Def</u>: X, Y r.v.s independent if the events $\{X = x\}$ & $\{Y = y\}$ are independent $\forall x, y$, i.e.,

$X \in \{x_1, x_2, ---\}$ $\qquad Y \in \{y_1, y_2, ---\}$

$A_i = \{X = x_i\}$ $\qquad B_j = \{Y = y_j\}$

X & Y independent if A_i & B_j independent $\forall i, j$.

Note: Can be generalized to $\{X_1 -- X_n\}$

__Example__ Poisson flips

A coin is tossed, heads turn up w.p. p $(=1-q)$.

$X = \#$ heads, $Y = \#$ tails.

X & Y "not" independent

$P(X=Y=1) = 0$, $P(X=1)=p$, $P(Y=1)=q$

Suppose you toss the coin N times,

where $N \sim \text{Poisson}(\lambda)$

Then, X, Y are independent.

To show: $P(X=x, Y=y) = P(X=x)\,P(Y=y)$

$$LHS = P(X=x, Y=y)$$

$$= P(X=x, Y=y \mid N=x+y)\,P(N=x+y)$$

$$= \binom{x+y}{x} p^x q^y \frac{\lambda^{x+y} e^{-\lambda}}{(x+y)!}$$

$$= \frac{(\lambda p)^x (\lambda q)^y e^{-\lambda}}{x!\, y!}$$

$$P(X=x) = \sum_{n=0}^{\infty} P(X=x \mid N=n)\, P(N=n)$$

$$= \sum_{n \geq x} P(X=x \mid N=n)\, P(N=n)$$

$$= \sum_{n \geq x} \binom{n}{x} p^x q^{n-x} \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= \sum_{n \geq x} \frac{\cancel{n!}}{x!\,(n-x)!} \frac{(\lambda p)^x (\lambda q)^{n-x} e^{-\lambda}}{\cancel{n!}}$$

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{n \geq x} \frac{(\lambda q)^{n-x}}{(n-x)!}$$

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} e^{\lambda q} = \frac{(\lambda p)^x e^{-\lambda p}}{x!}$$

Similarly, $P(Y=y) = \dfrac{(\lambda q)^y e^{-\lambda q}}{y!}$

So, $P(X=x)\,P(Y=y) = \dfrac{(\lambda p)^x (\lambda q)^y e^{-\lambda}}{x!\,y!}$

$= P(X=x, Y=y)$

Notes:

① X, Y independent $g, h : \mathbb{R} \to \mathbb{R}$

Then, $g(X)$ and $h(Y)$ are independent as well.

② Can generalize independence to
$\{X_i, i \in I\}$

$$P(X_i = x_i, \ \forall i \in J) = \prod_{i \in J} P(X_i = x_i), \quad \forall x_i$$

finite subset

(3) Conditional independence can be defined along similar lines

(4) Pair-wise independence $\not\Rightarrow$ independence

Binomial to Poisson:

Let X_n be Binomial$\left(n, \frac{\lambda}{n}\right)$, $\lambda > 0$, $n = 1, 2, \ldots$

Then,

$$P(X_n = k) \to \frac{e^{-\lambda} \lambda^k}{k!} \quad \text{as } n \to \infty$$

Pf:

$$P(X_n = k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{-k} \left(1 - \frac{\lambda}{n}\right)^n$$

As $n \to \infty$, $\left(1 - \frac{\lambda}{n}\right)^{-k} \to 1$

$$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}$$

$$\frac{n!}{(n-k)!\,n^k} \longrightarrow 1$$

Hence, $P(X_n = k) \xrightarrow[n\to\infty]{} \dfrac{e^{-\lambda}\lambda^k}{k!}$

Lecture - 11

Expectation, Variance, Correlation

Toss a coin $\frac{S_N}{N} \sim p$ for large N

In general, $X_1, X_2, \ldots, X_N$ of r.v.s, all with same pmf f

$$f(x) = \frac{\# X_i = x}{N}$$

Average $\mu = \frac{1}{N} \sum_x x N f(x)$

$$= \sum_x x f(x)$$

Def Mean / Expectation / Expected Value

of a r.v. X with pmf is defined as

$$EX = \sum_{x: f(x)>0} x f(x) ,$$

whenever this sum is absolutely convergent.

Note: $\sum_{\omega} f(\omega)$ is absolutely convergent if $\sum_{\omega} |f(\omega)| < \infty$

LOTUS:

If X has pmf f and $g: \mathbb{R} \to \mathbb{R}$, then $E(g(X)) = \sum_{x} g(x) f(x)$

Example:

1) $X \to -2, -1, 1, 3$

$f \to \frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}$

$g(x) = x^2$

$E(g(X)) = 4 \times \frac{1}{4} + 1 \times \frac{1}{8} + 1 \times \frac{1}{4}$

$+ 9 \times \frac{3}{8} = \frac{19}{4}$

(or)

$Y = X^2$ $\quad 1, 4, 9$

$f(y) \quad \frac{3}{8}, \frac{1}{4}, \frac{3}{8}$

$E(Y) = 1 \times \frac{3}{8} + 4 \times \frac{1}{4} + 9 \times \frac{3}{8} = \frac{19}{4}$

Def: k = positive integer

kth moment $m_k = \underline{E}(X^k)$

kth central moment $= \underline{E}\left((X - m_1)^k\right)$

Mean $= \mu$

$Var(X) = \sigma_2^2 = E\left((X-\mu)^2\right)$

Std-dev $\sigma = \sqrt{Var(X)}$

Claim: $Var(X) = E(X^2) - (EX)^2$

Pf I: $E\left((X-\mu)^2\right)$

$= E\left(X^2 - 2\mu X + \mu^2 \right)$

$$= E(X^2) - (EX)^2$$

Pf II:

$$\operatorname{Var} X = \sum_x (x-\mu)^2 f(x)$$

$$= \sum_x x^2 f(x) - 2\mu \sum_x x f(x) + \mu^2 \sum_x f(x)$$

$$= EX^2 - (EX)^2$$

$\blacksquare$

Note: $\operatorname{Var} X \geq 0$

Examples (on next page)

① Bernoulli r.v. $X \sim Ber(p)$

$\mu = p \qquad Var(X) = p(1-p)$

② Binomial r.v. $\qquad q = 1-p$

$$f(k) = \binom{n}{k} p^k q^{n-k}, \quad 0 \le k \le n$$

$$\sum_{k=0}^{n} f(k) = 1 \quad (\text{Why?})$$

Fact: $\quad (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

(Taylor Series expansion)

$$x = \frac{p}{q}$$

$$\left(1 + \frac{p}{q}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{p^k}{q^k}$$

$$\frac{1}{q^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{p^k}{q^k}$$

$$\Rightarrow \sum_{k=0}^{n} f(k) = 1$$

$$EX = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$

$$= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^{l} q^{n-l-1}$$

$$= np$$

H.W. $\operatorname{Var} X = npq$

<u>Thm</u>: (Linearity of expectation)

(a) $X \geq 0 \Rightarrow EX \geq 0$

b) If $a, b \in \mathbb{R}$, then
$E(aX + bY) = a\,EX + b\,EY$

c) $E(\mathbb{1}) = 1$, where

$\mathbb{1}$ is a constant r.v. that takes value 1 always.

[Do your own proof]

<u>Lemma:</u>

If X, Y independant, then

$$E(XY) = EX\, EY$$

Converse not true.

(X, Y) independant $Ber\left(\frac{1}{2}\right)$

Check: $X+Y$, $|X-Y|$ dependent

but uncorrelated.

Pf: $A_x = \{X = x\}$, $B_y = \{Y = y\}$

$$XY = \sum_{x,y} x y \, I_{A_x \cap B_y}$$

$$EXY = \sum_{x,y} x y \, P(A_x \cap B_y)$$

$$= \sum_x \sum_y x \, y \, P(A_x) P(B_y)$$

$$= \sum_x x \, P(A_x) \sum_y y \, P(B_y)$$

$$= EX \, EY$$

Def: X, Y uncorrelated

if $EXY = EX\,EY$

Thm: For r.v.s X, Y

(i) $Var(aX) = a^2 Var(X)$

(ii) $Var(X+Y) = Var(X) + Var(Y)$

if X, Y are uncorrelated.

[Do your own proof]

Note: Variance is not a linear operator.

<u>Note!</u> $f(k) = \frac{A}{k^2}, \quad k = \pm 1, \pm 2, \ldots$

Choose A s.t. $\sum f(k) = 1$

Think about $\sum k f(k)$

Lecture-12

Joint distributions

Def: Joint distribution function

$F : \mathbb{R}^2 \to [0,1]$ of X and Y

is given by

$$F(x,y) = P(X \leq x \text{ and } Y \leq y)$$

Their joint mass function

$f : \mathbb{R}^2 \to [0,1]$ is given by

$$f(x,y) = P(X=x \text{ and } Y=y)$$

Note: Can generalize to a bigger collection of r.v.s

If $A_x = \{X=x\}$ and

$B_y = \{Y=y\}$

$$f(x,y) = P(A_x \cap B_y)$$

Lemma: X and Y independent

iff $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$\forall x, y \in \mathbb{R}$

Given $f_{X,Y}(x,y)$, notice that

$$f_X(x) = P(X = x)$$

$$= P\left(\bigcup_y \{X = x\} \cap \{Y = y\}\right)$$

$$= \sum_y P(X = x, Y = y)$$

$$= \sum_y f_{X,Y}(x,y)$$

Similarly,

$$f_Y(y) = \sum_x f_{X,Y}(x,y)$$

Poisson flips - revisited

$X = \#$ heads, $Y = \#$ tails

in $N \sim \text{Poisson}(\lambda)$ flips

of a coin with bias p.

From an earlier lecture, recall that

$$f_{X,Y}(x,y) = \frac{\alpha^x \beta^y e^{-\alpha-\beta}}{x!\,y!},$$

$x, y = 0, 1, 2, \ldots$

$$f_X(x) = \sum_y f_{X,Y}(x,y)$$

$$= \frac{\alpha^x e^{-\alpha}}{x!} \sum_y \frac{\beta^y e^{-\beta}}{y!}$$

$$= \frac{\alpha^x e^{-\alpha}}{x!}.$$

$X \sim \text{Poisson}(\alpha)$

Similarly $Y \sim \text{Poisson}(\beta)$

Observe that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

So, X, Y are independent

LOTUS:

$$E[g(X,Y)] = \sum_{x,y} g(x,y) f_{X,Y}(x,y)$$

Def: Covariance of X and Y is

$$\text{Cov}(X,Y) = E[(X-EX)(Y-EY)]$$

Correlation Coefficient of X and Y is

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var X \; Var Y}}$$

<u>Note:</u>

1. $Cov(X,Y) = EXY - EXEY$

So, X, Y uncorrelated

if $Cov(X,Y) = 0$

Tedious example

	$y=-1$	$y=0$	$y=2$	f_X
$x=1$	$1/18$	$3/18$	$2/18$	$6/18$
$x=2$	$2/18$	0	$3/18$	$5/18$
$x=3$	0	$4/18$	$3/18$	$7/18$
f_Y	$3/18$	$\frac{7}{18}$	$\frac{8}{18}$	

$$E\,XY = \sum_{x,y} xy\, f(x,y) = \frac{29}{18}$$

$$EX = \sum_x x f_X(x) = \frac{37}{18}$$

$$EY = \sum_y y f_Y(y) = \frac{13}{18}.$$

$$\mathrm{Var}\,X = EX^2 - (EX)^2$$

$$= 233/324$$

$$\mathrm{Var}\,Y = 461/324$$

$$\mathrm{Cov}(X,Y) = 41/324$$

$$\rho(X,Y) = 41/\sqrt{107415}$$

Lemma:

$$|\rho(X,Y)| \le 1$$

with equality iff

$$P(aX+bY=c)=1$$

for some $a,b \in \mathbb{R}$,

one of a,b non-zero

Proof:

Cauchy-Schwarz inequality gives $(EXY)^2 \le EX^2\,EY^2$

Applying to $(X-EX)$ & $(Y-EY)$

$$\left(E\left[(X-EX)(Y-EY)\right]\right)^2$$

$$\leq E\left((X-EX)^2\right) E\left((Y-EY)^2\right)$$

$$\text{Cov}(X,Y)^2 \leq \text{Var}X \,\text{Var}Y$$

So, $|\rho(X,Y)| \leq 1$

From Cauchy-Schwarz ineq,

$$(EXY)^2 = EX^2 EY^2$$

iff $P(aX = bY) = 1$

for $a, b \in \mathbb{R}$, at least one of

a, b non-zero

Applying to $X - EX$, $Y - EY$,

$$P\left(a(X - EX) = b(Y - EY)\right) = 1$$

$$P(aX - bY = c) = 1$$

$\blacksquare$

Lecture - 13

Conditional distribution & expectation

We discussed $P(B|A)$

Want to generalize to r.v.s X and Y, i.e.,

conditional distribution of Y given $X=x$

Def: Conditional distribution of Y given $X=x$, denoted by $F_{Y|X}(\cdot|x)$ is

$$F_{Y|X}(y|x) = P(Y \le y \mid X=x)$$

$\forall x$ s.t. $P(X=x) > 0$

Conditional mass function of Y given $X=x$ is

$$f_{Y|X}(y|x) = P(Y=y \mid X=x)$$

$\forall x$ s.t. $P(X=x) > 0$

To remember, we

$$f_{Y|X} = \frac{f_{X,Y}}{f_X}$$

Note: X and Y independent

if $f_{Y|X} = f_Y$

Given $\{X=x\}$, the new distribution of Y has pmf $f_{Y|X}(y|x)$, which is a function of y.

The expected value of this distribution

$$\sum_y y\, f_{Y|X}(y|x) \text{ is}$$

the conditional expectation of Y given $X = x$

Let $\psi(x) = E[Y | X = x]$

$\psi(X)$ = function of r.v. X

Def: Let $\psi(x) = E[Y \mid X=x]$

$\psi(X)$ is called the conditional expectation of Y given X & written as $E[Y \mid X]$

Note: $E[Y \mid X]$ is a r.v.

Lemma: $E(\psi(X)) = EY$,

where $\psi(X) = E[Y \mid X]$

Pf: Using LOTUS,

$$E(\psi(X)) = \sum_x \psi(x) f_X(x)$$

$$= \sum_x \sum_y y f_{Y|X}(y|x) f_X(x)$$

$$= \sum_x \sum_y y f_{X,Y}(x,y)$$

$$= \sum_y y \sum_x f_{X,Y}(x,y)$$

$$= \sum_y y f_Y(y) = EY \qquad \blacksquare$$

So,

$$E\,Y = \sum_{x} E[Y \mid X = x]\, P(X = x)$$

Example

A hen lays N eggs.

$N \sim \text{Poisson}(\lambda)$.

Each egg hatches w.p. p, independent of other eggs.

K = # eggs that hatched

Find (1) $E(K|N)$

(2) $E(K)$ (3) $E(N|K)$

Solution:

$$f_N(n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

$$f_{K|N}(k|n) = \binom{n}{k} p^k q^{n-k}$$

$(q = 1-p)$

$$E(K|N=n) = \sum_k k f_{K|N}(k|n)$$

PTO

$$E(K \mid N = n) = \sum_{k} k \binom{n}{k} p^{k} q^{n-k}$$

$$= np$$

$$E[K \mid N] = Np$$

$$E[K] = E\big[E[K \mid N]\big]$$

$$= E[Np]$$

$$= p\,EN = p\lambda$$

Lets find $f_{N|K}(n|k)$

$$f_{N|K}(n|k)$$

$$= P(N=n \mid K=k)$$

$$= \frac{P(K=k \mid N=n)\, P(N=n)}{P(K=k)}$$

$$= \frac{\binom{n}{k} p^k q^{n-k} \frac{\lambda^n e^{-\lambda}}{n!}}{\sum_{m \geq k} \binom{m}{k} p^k q^{m-k} \frac{\lambda^m e^{-\lambda}}{m!}}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\lambda p}$$

$$P(K = k, N = m)$$

$$= \frac{\dfrac{\cancel{n!}}{\cancel{k!}(n-k)!}\, p^k q^{n-k} \, \dfrac{\lambda^n e^{-\lambda}}{\cancel{n!}}}{\dfrac{(\lambda p)^k e^{-\lambda p}}{\cancel{k!}}}$$

$$= \frac{(\lambda q)^{n-k} e^{-\lambda q}}{(n-k)!}$$

$$E[N \mid K = k]$$

$$= \sum_{n \geq k} n \, f_{N|K}(n \mid k)$$

$$= \sum_{n \geq k} n \frac{(\lambda q)^{n-k} e^{-\lambda q}}{(n-k)!}$$

$$= e^{-\lambda q} \sum_{n \geq k} \frac{(n-k)(\lambda q)^{n-k}}{(n-k)!}$$

$$+ e^{-\lambda q} \sum_{n \geq k} \frac{k(\lambda q)^{n-k}}{(n-k)!}$$

$$= e^{-\lambda q} \lambda q \sum_{n \geq k} \frac{(\lambda q)^{n-k-1}}{(n-k-1)!}$$

$$+ e^{-\lambda q} k \sum_{n \geq k} \frac{(\lambda q)^{n-k}}{(n-k)!}$$

$$= e^{-\lambda q} \lambda q \cdot e^{\lambda q} + e^{-\lambda q} k e^{\lambda q}$$

$$= \lambda q + k$$

$$E[N|k] = \lambda q + k$$

Lecture - 14

30/10/19

Generalizing $E[E[Y|X]] = E[Y]$

Theorem:

Let $\psi(X) = E[Y|X]$

$$E(\psi(X)\, g(X)) = E[Y g(X)],$$

for any g for which both expectations exist.

Pf: $E(\psi(X)\, g(X))$

$$= \sum_{x} \psi(x)\, g(x)\, f_X(x)$$

$$= \sum_{x} \sum_{y} y\, g(x)\, f_{Y|X}(y|x) f_X(x)$$

$$= \sum_{x,y} y\, g(x)\, f_{X,Y}(x,y)$$

$$= E(Y g(X))$$

$\blacksquare$

Properties of $E[Y|X]$:

① $E[aY + bZ | X]$

$$= a E[Y|X] + b E[Z|X]$$

$\forall a, b \in \mathbb{R}$

(2) $E[Y|X] \geq 0$ if $Y \geq 0$

(3) $E[1|X] = 1$

Also,

(4) X, Y independent

$\Rightarrow E[Y|X] = EY$

(5) $E[Y g(X) | X]$

$= g(X) \, E[Y|X]$

for suitable g

6 Tower property

$$E(E(Y \mid X, Z) \mid X)$$

$$= E(Y \mid X)$$

$$= E(E(Y \mid X) \mid X, Z)$$

PTO

Sums of r.v.s

$Z = X + Y$ $\quad f(x,y) \to$ joint pmf

Want to find pmf of Z

$\{X + Y = z\}$

$= \bigcup_{x} \{X = x\} \cap \{Y = z - x\}$

$\uparrow$

disjoint & countable union

$f_Z(z) = P(X + Y = z)$

$$= \sum_x P(X = x, Y = 2 - x)$$

$$= \sum_x f(x, 2 - x)$$

If X and Y are independant,

then

$$f_{X+Y}(2) = \sum_x f_X(x) f_Y(2-x)$$

$$= \sum_y f_X(2-y) f_Y(y)$$

$f_{X+Y} = f_X * f_Y \rightarrow$ Convolution of pmfs of X, Y

Example :

$X_1, X_2 \sim$ geometric(p) &

independant

$$f(k) = (1-p)^{k-1}\, p$$

$$Z = X_1 + X_2$$

$$P(Z = z) = \sum_{k=1}^{z-1} P(X_1 = k)\, P(X_2 = z-k)$$

$$= \sum_{k=1}^{z-1} p(1-p)^{k-1}\, p(1-p)^{z-k-1}$$

$$= (z-1)\, p^2 (1-p)^{z-2},$$

$$z = 2, 3, \ldots$$

In general, Z is the waiting time until "r" heads

$$Z = X_1 + \cdots + X_r, \quad X_i \sim \text{Geom}(p) \; \forall i$$

$$P(Z=z) = \binom{z-1}{r-1} p^r (1-p)^{z-r}$$

Negative binomial r.v.

A brief tour of concentration inequalities

$X \geq 0$ with pmf $f(k)$, $k = 0, 1, 2, \ldots$

$$EX = \sum_{k=1}^{\infty} k f(k)$$

$$\geq \sum_{k=10}^{\infty} k f(k)$$

$$\geq 10 \sum_{k=10}^{\infty} f(k)$$

$$= 10 \, P(X \geq 10)$$

For any positive m,

$$E(X) \geq m\,P(X \geq m)$$

Lecture-15

Markov inequality:

$X \geq 0$, $EX < \infty$. Then, $\forall \epsilon > 0$

$$P(X \geq \epsilon) \leq \frac{EX}{\epsilon}$$

Pf: Fix $\epsilon > 0$

$$X = \underbrace{X I_{\{X<\epsilon\}}}_{=:Y} + \underbrace{X I_{\{X\geq\epsilon\}}}_{=:Z}$$

$Y \geq 0$, $Z \geq 0$

$$EX \geq EZ$$

$$Z = X I_{\{X \geq \epsilon\}} \geq \epsilon I_{\{X \geq \epsilon\}}$$

$$EZ \geq \epsilon\, E\left(I_{\{X \geq \epsilon\}}\right) = \epsilon\, P(X \geq \epsilon)$$

$\blacksquare$

Variants:

(1) $X \geq 0$, $EX^p < \infty$. Then,

$$P(X \geq \epsilon) \leq \frac{EX^p}{\epsilon^p}, \quad \forall \epsilon > 0.$$

(2) X r.v. $\operatorname{Var} X < \infty$

$$P(|X - \mu| > \epsilon) \leq \frac{\operatorname{Var} X}{\epsilon^2}$$

$\boxed{\text{Chebyshev's inequality}}$

Why? Apply Markov inequality to $(X-\mu)^2$

Weak law of large numbers:

$S_n = X_1 + \cdots + X_n$, X_i i.i.d

$EX_i = \mu$, $\operatorname{Var} X_i = \sigma^2 < \infty$

$$\bar{X}_n = \frac{S_n}{n} \qquad \mathbb{E}\bar{X}_n = \mu, \quad \mathrm{Var}\,\bar{X}_n = \frac{\sigma^2}{n}$$

Applying C. ineq,

$$P(|\bar{X}_n - \mu| > \epsilon) \le \frac{\sigma^2}{n\epsilon^2} \quad \forall \epsilon > 0$$

Recall that, for the Bernoulli case, we had

$$P(|\bar{X}_n - \mu| > \epsilon) \le 2\exp\left(-\frac{n\epsilon^2}{4}\right)$$