CS6015: Linear Algebra and Random Processes

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Quiz - 2: Solutions

1. True or False?

   (a) True

   (b) True

   (c) False

   (d) False

   (e) True

   (f) True

   (g) False

2. For each of the matrices below, solve $Ax = 0$ and characterize the null space by finding its basis.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

   Solution: A is invertible and so $N(A) = \{0\}$

   (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

   Solution: Let u_1, u_2, u_3 denote the columns of A. Then, we have

$$u_2 - u_1 = u_3 - u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\Rightarrow \quad u_1 - 2u_2 + u_3 = 0$$
$$\Rightarrow \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \in N(A).$$

   It can be clearly seen that rank of A is 2 and hence, the null space is the line through $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 1 & 2 & \dots & n \\ 2 & 3 & \dots & n+1 \\ \vdots & \vdots & \ddots & \vdots \\ n & n+1 & \dots & 2n-1 \end{bmatrix}$

Solution: Let $u_1, u_2, \ldots, u_n$ denote the columns of A. Then, we have

$$u_2 - u_1 = u_3 - u_2 = \cdots = u_n - u_{n-1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{1}$$

$$\tag{2}$$

From the first equality above, we get

$$u_1 - 2u_2 + u_3 = 0. \text{ Thus, } \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in N(A).$$

Similarly, from the other equalities in (2), it can be seen that

$$\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \in N(A)$$

Since we obtained $n-2$ linearly independent vectors in $N(A)$, the dimension of $N(A)$ is at least $n-2$. On the other hand, u_1 and u_2 being linearly independent imply that the rank of A is at least 2. But, the rank and null space dimension add up to n and hence, the $N(A)$ is characterized by the basis $\mathcal{B}$ defined as

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$