CS6015: Linear Algebra and Random Processes
Course Instructor : Prashanth L.A.
Quiz - 3: Solutions

1. True or False?

(a) $T(u) = v$ for some (fixed) $v \neq 0$ is a linear transformation.

> **Solution:** False, since $T(0) \neq 0$.

(b) Suppose P_1 and P_2 are projection matrices. Then

$$(P_1 - P_2)^2 + (I - P_1 - P_2)^2 = I.$$

> **Solution:** True. Using $P_1^2 = P_1$ and $P_2^2 = P_2$, we get
>
> $$\begin{aligned}&(P_1 - P_2)^2 + (I - P_1 - P_2)^2\\ &= (P_1^2 + P_2^2 - P_1P_2 - P_2P_1) + (I + P_1^2 + P_2^2 - 2P_1 - 2P_2 + P_1P_2 + P_2P_1)\\ &= (P_1 + P_2) + (I + P_1 + P_2 - 2P_1 - 2P_2) = I.\end{aligned}$$

(c) Let $\{q_1, \ldots, q_n\}$ be an orthonormal subset of vectors in $\mathbb{R}^n$ and T be a linear transformation that satisfies

$$T(q_i)^\intercal T(q_i) = q_i^\intercal q_i, \; i = 1, \ldots, n.$$

Then, the set $\{T(q_1), \ldots, T(q_n)\}$ is orthonormal.

> **Solution:** False. In $\mathbb{R}^2$, notice that the standard basis $\{e_1, e_2\}$ is orthonormal as well. Define a linear transformation T as follows: $T(e_1) = e_2$ and $T(e_2) = e_2$. Then, $T(e_1)^\intercal T(e_2) = e_2^\intercal e_2 = 1$ and hence, the set $\{T(e_1), T(e_2)\}$ is not orthonormal.

(d) If a linear transformation $T : \mathbb{R} \to \mathbb{R}$ satisfies $T(4) = 24$, then $T(x) = 6x$, for all $x \in \mathbb{R}$.

> **Solution:** True. $T(4) = 4T(1)$ implies $T(1) = 6$. For any $x \in \mathbb{R}$, $T(x) = T(x \cdot 1) = xT(1) = 6x$.

(e) If $v_1, \ldots, v_n$ are linearly independent vectors in V and T is a linear transformation from V to V, then $T(v_1), \ldots, T(v_n)$ are linearly independent.

> **Solution:** False. Consider $T(v_i) = v_i$, for $i = 1, \ldots, n-1$ and $T(v_n) = v_1 + \ldots + v_{n-1}$. As another (trivial) example, the transformation $T(v_i) = 0$ for $i = 1, \ldots, n$ works as well.

(f) If $T(v_1), \ldots, T(v_n)$ are linearly independent vectors in V, where T is a linear transformation from V to V, then $v_1, \ldots, v_n$ are linearly independent.

> **Solution:** True. If $v_1, \ldots, v_n$ are linearly dependent, then $c_1v_1 + \ldots + c_nv_n = 0$ for some scalars c_i, not all of them zero. This implies, $c_1Tv_1 + \ldots + c_nTv_n = 0$, a contradiction.

(g) Every orthonormal set of vectors in $\mathbb{R}^4$ must be a basis for $\mathbb{R}^4$.

Solution: False. Consider the set $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

2. Consider a subspace S of $\mathbb{R}^4$ spanned by the following vectors:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Using the usual dot product on $\mathbb{R}^4$, do the following:

(a) Convert $\{u_1, u_2, u_3\}$ to an orthonormal basis for S.

Solution: Applying Gram Schmidt algorithm, one gets the following orthonormal basis:

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) For $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, find the "least-squares approximation of b" in S.

Solution:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

(c) Explain why Gram Schmidt algorithm fails when the input set of vectors is linearly dependent.

Solution: Let the input set be $\{v_1, \ldots, v_n\}$. Suppose that (w.l.o.g.) v_k is a linear combination of $v_1, \ldots, v_{k-1}$, with the latter set linearly independent. After turning $v_1, \ldots, v_{k-1}$ into an orthonormal set of vectors, when Gram Schmidt algorithm acts on v_k, it would result in a zero vector (why?) and normalizing this vector would lead to the algorithm's failure!