CS6015: Linear Algebra and Random Processes
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Quiz - 4: Solutions

1. True or False?

   (a) If 0 is the eigenvalue of a matrix A, then A cannot be invertible.

   > **Solution:** True. If there exists an $x \neq 0$ such that $Ax = 0$, then A is not invertible.

   (b) For any square matrix, $|-A| = -|A|$.

   > **Solution:** False. Consider any $2n \times 2n$ matrix A with $|A| \neq 0$.

   (c) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^8 \to \mathbb{R}^3$ are linear transformations, then ST is a linear transformation from $\mathbb{R}^8$ to $\mathbb{R}^2$.

   > **Solution:** False.

   (d) Let A, B, C, D be square matrices. Then,
   $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix}$.

   > **Solution:** False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$.

   (e) For any $a, b, c, d, e, f, g, h, i \in \mathbb{R}$,
   $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} c & a & b \\ f & d & e \\ i & g & h \end{bmatrix}$ are similar.

   > **Solution:** False[1]. If the matrix A were to be similar to B, they would have the same characteristic polynomial. However, with $a = 1, e = 2, i = 3$ and rest of the entries zero, the characteristic polynomial of A is $(\lambda - 1)(\lambda - 2)(\lambda - 3)$, while that of B is $\lambda^3 - 6$, leading to a contradiction.

   (f) For a matrix A, let λ_1 and λ_2 be two different eigenvalues with corresponding eigenvectors x_1 and x_2. Then, $x_1 + x_2$ is an eigenvector for A.

   > **Solution:** False. Suppose $x_1 + x_2$ is an eigenvector for A, i.e., $A(x_1 + x_2) = \lambda_3(x_1 + x_2)$. Then,
   >
   > $$\begin{aligned} & A(x_1 + x_2) = \lambda_3(x_1 + x_2) \\ \Longrightarrow\ & Ax_1 + Ax_2 = \lambda_3(x_1 + x_2) \\ \Longrightarrow\ & \lambda_1 x_1 + \lambda_2 x_2 = \lambda_3(x_1 + x_2) \\ \Longrightarrow\ & (\lambda_1 - \lambda_3)x_1 + (\lambda_2 - \lambda_3)x_2 = 0, \end{aligned}$$
   >
   > a contradiction since eigenvectors x_1 and x_2 corresponding to different eigenvalues λ_1 and λ_2 are linearly independent.

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[1]Thanks to Dr. P. Viswanadha Reddy of EE-IITM for the counterexample.

(g) Let A, B be $n \times n$ matrices. Then, $|AB| = |BA|$.

Solution: True.

2. Let $A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y-z \\ -y \\ x+7z \end{bmatrix}$.

(a) What is the matrix of A w.r.t. the basis $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$?

Solution:

$$A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, A\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } A\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix}$$

So, the matrix of A w.r.t $\mathcal{B}$ is $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix}$.

(b) What is the matrix of A w.r.t. the basis $\mathcal{B}' = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$?

Solution:

$$A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So, the first column of $[A]_{\mathcal{B}'}$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The other two columns follow in a similar fashion and we obtain,

$$[A]_{\mathcal{B}'} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{bmatrix}.$$

(c) Relate the matrices, say $[A]_{\mathcal{B}}, [A]_{\mathcal{B}'}$, obtained in the answers to the parts (a) and (b) above using an invertible S such that

$$[A]_{\mathcal{B}'} = S^{-1}[A]_{\mathcal{B}}S.$$

Solution:

Let $\mathcal{B} = \{e_1, e_2, e_3\}$ and $\mathcal{B}' = \{b_1, b_2, b_3\}$

Now,

$$Se_1 = b_1 = e_1,$$
$$Se_2 = b_2 = e_1 + e_2,$$
$$Se_3 = b_3 = e_1 + e_2 + e_3.$$

Hence, $S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.