CS6015: Linear Algebra and Random Processes
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Quiz - 6: Solutions

1. True or False?

   (a) In $B \subset A$ and $\mathbb{P}(B) \neq 0$, then $\mathbb{P}(A \mid B) = 1$.

   Solution: True. Follows by definition of $\mathbb{P}(A \mid B)$ after observing that $A \cap B = B$.

   (b) If $\mathbb{P}(A) = \mathbb{P}(A \mid B) = \mathbb{P}(A \mid C)$, then $\mathbb{P}(A) = \mathbb{P}(A \mid B \cap C)$.

   Solution: False. Toss a fair coin thrice. Let A be the event that the first and second outcomes are different, B the event that the second and third outcomes are different and C the event that the first and third outcomes are different. Then,

   $$\mathbb{P}(A) = \mathbb{P}(A \mid B) = \mathbb{P}(A \mid C) = \frac{1}{2}, \text{ but } \mathbb{P}(A) = \mathbb{P}(A \mid B \cap C) = 0.$$

   (c) Let X be a random variable with distribution F and $a < b$. Then,

   $$\mathbb{P}(X \in (a,b)) = F(b) - F(a).$$

   Solution: False. $\mathbb{P}(X \in (a,b)) = \lim_{y \uparrow b} F(b) - F(a)$.

   (d) If F is a distribution function, then so is G, where $G(x) = 1 - F(x)$, $\forall x \in \mathbb{R}$.

   Solution: False. Notice that $\lim_{x \to \infty} G(x) = 0$.

   (e) If A, B, C are independent events and $\mathbb{P}(C) > 0$, then A and B are conditionally independent given C.

   Solution: True. Since A, C are independent $\mathbb{P}(A \mid C) = \mathbb{P}(A)$. Similarly, $\mathbb{P}(B \mid C) = \mathbb{P}(B)$.

   $$\mathbb{P}(A \cap B \mid C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)}{\mathbb{P}(C)} = \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A \mid C)\mathbb{P}(B \mid C).$$

   (f) If X, Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, then so is $\min\{X, Y\}$.

   Solution: True. Notice that $\{\min\{X,Y\} \leq z\} = \{\omega \mid \min\{X(\omega), Y(\omega)\} \leq z\} = \{\omega \mid X(\omega) \leq z\} \cup \{\omega \mid Y(\omega) \leq z\}$ and the statement follows.

2. Suppose there are $N+1$ urns, each containing a total of N red and white balls. The urn number k contains k red and $N-k$ white balls ($k = 0, 1, \ldots, N$). An urn is chosen at random and n random drawings are made from it, the ball being replaced each time.

   (a) Let A be the event that all n balls turn out to be red. What is $\mathbb{P}(A)$?

Solution: If urn k is chosen, then the probability that n balls are drawn is $\left(\frac{k}{N}\right)^n$. Since each urn has a probability of $\frac{1}{N+1}$ to be chosen, we have

$$\mathbb{P}(A) = \frac{1}{N+1}\left(\left(\frac{1}{N}\right)^n + \ldots + \left(\frac{N}{N}\right)^n\right). \tag{1}$$

(b) Given event A, what is the (conditional) probability that the next drawing will also yield a red ball?

Solution: $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}$, where $\mathbb{P}(A)$ is given by (1) and $\mathbb{P}(A \cap B)$ is calculated as follows:

$$\mathbb{P}(A \cap B) = \frac{1}{N+1}\left(\left(\frac{1}{N}\right)^{n+1} + \ldots + \left(\frac{N}{N}\right)^{n+1}\right). \tag{2}$$

We have used the fact that $A \cap B$ would be the event that $n+1$ balls are drawn from a randomly chosen urn.