CS6015; Linear Algebra and Random Processes
Tutorial – 2
Date: 1/9/2017

1. If the matrix of a linear transformation A w.r.t. the basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$, what is the matrix of A w.r.t. the basis $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$? What about the basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$?

Solution: Given

$$A(e_1) = \begin{pmatrix}1\\1\end{pmatrix}, \quad A(e_2) = \begin{pmatrix}1\\1\end{pmatrix}. \tag{1}$$

To find the matrix w.r.t $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, we need to compute $A\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$ and $A\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$.

Observe that $\begin{bmatrix}1\\1\end{bmatrix} = e_1 + e_2$. So, $A\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = A(e_1 + e_2) = A(e_1) + A(e_2) = \begin{bmatrix}2\\2\end{bmatrix}$.

Similarly, $A\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = A(e_1 - e_2) = \begin{bmatrix}0\\0\end{bmatrix}$.

So, matrix of A w.r.t. the basis $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ is $\begin{bmatrix}2 & 0\\2 & 0\end{bmatrix}$.

Along similar lines, matrix of A w.r.t. the basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is $\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}$.

2. If the matrix of a linear transformation w.r.t basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is $\begin{bmatrix}0 & 1 & 1\\1 & 0 & -1\\-1 & -1 & 0\end{bmatrix}$, what is the matrix of A w.r.t $\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$

Solution: Homework!

3. (a) What matrix M transforms $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}r\\t\end{bmatrix}$ & $\begin{bmatrix}s\\u\end{bmatrix}$?

Solution:

$$M = \begin{bmatrix}r & s\\t & u\end{bmatrix} \tag{2}$$

(b) What matrix N transforms $\begin{bmatrix}a\\c\end{bmatrix}$ & $\begin{bmatrix}b\\d\end{bmatrix}$ to $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$?

Solution:

$$N = \begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} \tag{3}$$

(c) What condition on a, b, c, d makes the above transformation impossible?

Solution: $ad = bc$ makes N impossible.

(d) How to combine M, N to yield a matrix that transforms $\begin{bmatrix} a \\ c \end{bmatrix}$ to $\begin{bmatrix} r \\ t \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ to $\begin{bmatrix} s \\ u \end{bmatrix}$? Work out the resulting transformation for the special case where $a = 2, b = 1, c = 5, d = 3, r = 1, s = 0, t = 1, u = 2$.

Solution:

$$MN = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$
$$\text{Specific example: } MN = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix} \tag{4}$$

4. Suppose matrix Q satisfies $Q^T Q = I$ then,
   (a) Columns of Q are "orthonormal"
   (b) Relation between m and n is "$m \geq n$"
   (c) Rank of Q is "n"
   (d) What is the solution $\hat{x}$ to $Qx = b$? "$\hat{x} = Q^T b$"
   (e) Is $P = QQ^T$ a projection matrix?

Solution: P is a projection if P is symmetric & $P^2 = P$. We check these properties for $P = QQ^T$ below.
$P^T = (QQ^T) = (Q^T)^T Q^T = QQ^T = P$.
Further, $P^2 = Q(Q^T Q)Q^T = QQ^T = P$. Hence, P is a projection matrix.

   (f) Is $P = QQ^T$ singular if Q is a $m \times n$ matrix with $m > n$?

Solution: Yes. Q, Q^T have rank n & hence $rank(P) = n$, but P is a $m \times m$ matrix with $m > n$. Hence, P is not full rank, implying singularity.

   (g) Let $A = \begin{bmatrix} 0.1 & 0.5 & 1 \\ 0.7 & 0.5 & 1 \\ 0.1 & -0.5 & 1 \\ 0.7 & -0.5 & 1 \end{bmatrix}$ Check if the first and second columns of A form an orthonormal set of vectors. Use Gram-Schmidt algorithm to convert A into an orthogonal matrix.

Solution: The first two columns, say q_1 and q_2, of A are orthonormal. We apply Gram-Schmidt algorithm to the third column of A, say a_3, to arrive at a vector q_3 that is of unit length and orthogonal to q_1 and q_2.
Let $A_3 = a_3 - (q_1^T \cdot a_3)q_1 - (q_2^T \cdot a_3)q_2$
Notice that $q_2^T a_3 = 0, q_1^T a_3 = 1.6$ and hence, we obtain
$A_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1.6 \begin{pmatrix} 0.1 \\ 0.7 \\ 0.1 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 0.84 \\ -0.12 \\ 0.84 \\ -0.12 \end{pmatrix}$.

Normalizing, $q_3 = \frac{A_3}{||A_3||} = \begin{pmatrix} 0.7 \\ -0.1 \\ 0.7 \\ -0.1 \end{pmatrix}$.