**Linear Algebra and Random Processes**

Tutorial – 3

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1. Let $e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ denote the standard basis vectors in $\mathbb{R}^2$. $\mathcal{B}' = \left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-3\\2\end{pmatrix}\right\}$. For any vector $v \in \mathbb{R}^2$, let $v_{\mathbb{B}}$, $v_{\mathbb{B}'}$ denote its representation using $\mathbb{B}$, $\mathbb{B}'$, respectively. Find an invertible matrix $A$ such that

$$v_{\mathbb{B}'} = Av_{\mathbb{B}}.$$

**Solution:**

$$A = \begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}^{-1} = \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$$

2. Let $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}-7x-15y\\6x+12y\end{bmatrix}$.

   (a) Find a basis $\mathcal{B}'$, such that $\left[T\right]_{\mathcal{B}'} = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$.

   **Solution:** Let $\mathcal{B}' = \{b_1, b_2\}$, where $b_1 = \begin{bmatrix}b_{11}\\b_{12}\end{bmatrix}$, $b_2 = \begin{bmatrix}b_{21}\\b_{22}\end{bmatrix}$.

   From $\left[T\right]_{\mathcal{B}'}$, we have $T(b_1) = 2b_1$, $T(b_2) = 3b_2$

   From $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}-7x-15y\\6x+12y\end{bmatrix}$ & $T(b_1) = 2b_1$, we get

$$\begin{matrix}-7b_{11} - 15b_{12} = 2b_{11}\\ 6b_{11} + 12b_{12} = 2b_{12}\end{matrix} \implies \begin{matrix}-9b_{11} - 15b_{12} = 0\\ 6b_{11} + 10b_{12} = 0\end{matrix} \implies b_{11} = -\frac{5}{3}b_{12} \text{ and } b_{12} \text{ is free.}$$

   Setting $b_{12} = 6$, we get $b_{11} = -10$.

   In a similar fashion,

$$\begin{matrix}-7b_{21} - 15b_{22} = 3b_{21}\\ 6b_{21} + 12b_{22} = 3b_{22}\end{matrix} \implies \begin{matrix}-10b_{21} - 15b_{22} = 0\\ 6b_{21} + 9b_{22} = 0\end{matrix} \implies b_{21} = -\frac{3}{2}b_{22}.$$

   Setting $b_{22} = -4$, we get $b_{21} = 6$.

   So, $\mathcal{B}' = \left\{\begin{bmatrix}-10\\6\end{bmatrix}, \begin{bmatrix}6\\-4\end{bmatrix}\right\}$.

   (b) Find an invertible $S$, such that $\left[T\right]_{\mathcal{B}'} = S^{-1}\left[T\right]_{\mathcal{B}} S$, where $\mathcal{B}$ is the standard basis.

   **Solution:** $S = \begin{bmatrix}-10 & 6\\6 & -4\end{bmatrix}$

3. Recall that the trace of the matrix $A$ is the sum of its diagonal entries. Suppose $A$ is similar to $B$. Then, trace $(A)$ = trace $(B)$.

**Solution:** Notice that trace $(AB) =$ trace $(BA)$ for any $m \times n$ matrix $A$ and $n \times m$ matrix B because

$$\operatorname{trace}(AB) = \sum_i [AB]_{ii} = \sum_i \sum_k a_{ik} b_{ki} = \sum_k \sum_i b_{ki} a_{ik} = \sum_k [BA]_{kk} = \operatorname{trace}(BA).$$

Suppose that $A = C^{-1}BC$. Then,

$$\operatorname{trace}(A) = \operatorname{trace}\left(C^{-1}BC\right) = \operatorname{trace}\left(BCC^{-1}\right) = \operatorname{trace}(B).$$

4. Let $P$ be a projection matrix onto a $r$-dimensional subspace of $\mathbb{R}^n$. Let $\{v_1, \ldots, v_r\}$ be the basis for column-space of $P$ & $\{u_1, \ldots, u_{n-r}\}$ is a basis for null-space of $P$.

(a) Show that $\mathcal{B} = \{v_1 \ldots v_r, u_1 \ldots u_{n-r}\}$ is a basis for $\mathbb{R}^n$?

**Solution:** Suppose

$$\sum_{i=1}^{r} c_i v_i + \sum_{j=1}^{n-r} d_j u_j = 0 \tag{1}$$

$$\implies \sum_{i=1}^{r} c_i P v_i + \sum_{j=1}^{n-r} d_j P u_j = 0 \tag{2}$$

Notice that $u_j \in \mathcal{N}(P) \implies Pu_j = 0$

Further, $v_i \in \mathcal{C}(P)$
$\implies v_i = Px_i$ for some $x_i$
$\implies Pv_i = P^2 x_i = Px_i = v_i$

Plugging these into Equation (2), we get
$\sum_{i=1}^{r} c_i v_i = 0$
So, $c_i = 0, i = 1, \ldots, r$

From Equation (1), we get
$\sum_{j=1}^{n-r} d_j u_j = 0 \implies d_j = 0, \ \forall j$
Hence, $\mathcal{B} = \{v_1 \ldots v_r, u_1 \ldots u_{n-r}\}$ is a linearly independent set, implying $\mathcal{B}$ is a basis for $\mathbb{R}^n$ as set has $n$ elements.

(b) What is the matrix of $P$ w.r.t $\mathcal{B}$?

**Solution:**

$$[P]_{\mathcal{B}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

(c) Suppose $P_1$ and $P_2$ are two $n \times n$ projection matrices of rank $r$. Show that they are similar.

**Solution:** From part(b), we know that there exist a basis $\mathcal{B}$, such that

$$[P_1]_{\mathcal{B}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

& a possible different basis $\mathcal{B}'$, such that

$$[P_2]_{\mathcal{B}'} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Let

$$P = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Then,

$$\begin{aligned} P_1 &= S_1^{-1}\ P\ S_1 \\ P &= S_2^{-1} P_2 S_2 \\ \text{So, } P_1 &= S_1^{-1} S_2^{-1} P_2 S_2 S_1 \\ &= (S_2 S_1)^{-1} P_2 S_2 S_1 \end{aligned}$$

$S_2 S_1$ is invertible and hence $P_1$ is similar to $P_2$.