Discrete-time Markov Chains (DTMCs)

Ref: Chapter 2 of Kulkarni's book

Def: A stochastic process $\{X_n, n \geq 0\}$ with a countable state space S is a DTMC if

(i) $X_n \in S, \forall n \geq 0$

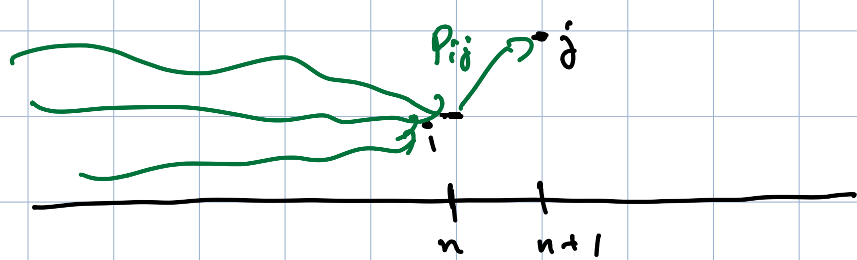
(ii) $\forall n \geq 0, \quad i, j \in S,$

Markov property

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots, X_0 = i_0)$$

$$= P(X_{n+1} = j \mid X_n = i)$$

"The future is conditionally independent of the past, given the present".



Def: A DTMC with countable State space S is time-homogeneous if

$$P(X_{n+1} = j \mid X_n = i) = P_{i,j} \quad \forall n \geq 0, \forall i, j \in S$$

transition probability

Rest of this chapter: Focus on time-homogeneous DTMCs

If the state space is finite, we can form the "transition probability matrix" P as follows:

Let $|S| = m$

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & \cdots & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \begin{bmatrix} P_{1,1} & & P_{1,m} \\ \vdots & & \\ \vdots & & \\ P_{m,1} & \cdots & P_{m,m} \end{bmatrix} \end{array}$$

Stochastic matrix: A square matrix P is stochastic if

(i) $P_{i,j} \geq 0 \quad \forall i,j \in S$

(ii) $\sum_{j \in S} P_{i,j} = 1 \quad \forall i \in S$

Remark: The transition probability matrix of a DTMC with finite state space is a "stochastic matrix".

"Finite dimensional distributions of a DTMC":

Q: Is the transition probability matrix (t.p.m.), say P, enough to derive the finite dimensional distributions, i.e.,

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) \text{ ? "NO"}$$

Simple case: $P(X_0 = i_0, X_1 = i_1)$

$$= P(X_1 = i_1 \mid X_0 = i_0)\, P(X_0 = i_0)$$

$$= p_{i_0, i_1} \times P(X_0 = i_0)$$

Suppose we are given

$$\left. a_i = P(X_0 = i), \quad \forall i \in S \right] \text{Initial distribution}$$

<u>Theorem:</u> A DTMC $\{X_n, n \geq 0\}$ is completely specified by the initial distribution "a" & t.p.m. P.

<u>Pf:</u> Want to find $P(X_0 = i_0, \ldots, X_n = i_n)$

Let $a_{i_0} = P(X_0 = i_0)$, $i_0 \in S$

Proof by induction:

$$P(X_0 = i_0, X_1 = i_1) = a_{i_0} p_{i_0, i_1} \quad \leftarrow \text{Base case}$$

Induction hypothesis: Suppose the following equality holds:

$$P(X_0 = i_0, \ldots, X_k = i_k) = a_{i_0} p_{i_0, i_1} \cdots\cdots p_{i_{k-1}, i_k}$$

for $k = 1, \ldots, n-1$

Induction step:

$$P(X_0 = i_0, \ldots\ldots, X_n = i_n)$$

$$= P(X_n = i_n \mid X_{n-1} = i_{n-1}, \ldots\ldots X_0 = i_0)$$

$$\times P(X_0 = i_0, \ldots\ldots X_{n-1} = i_{n-1})$$

$$= \underbrace{P(X_n = i_n \mid X_{n-1} = i_{n-1})}_{\text{Markov property}} \underbrace{a_{i_0} P_{i_0, i_1} \cdots P_{i_{n-2}, i_{n-1}}}_{\text{induction hypothesis}}$$

$$= \underbrace{P_{i_{n-1}, i_n}}_{\text{time-homogeneity}} \times a_{i_0} P_{i_0, i_1} \cdots P_{i_{n-2}, i_{n-1}}$$

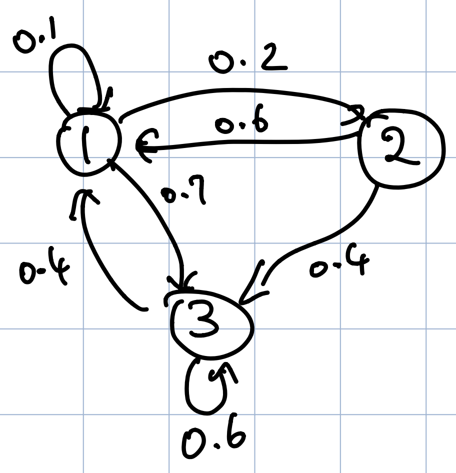
$$= a_{i_0} P_{i_0, i_1} \cdots P_{i_{n-1}, i_n} .$$

$\blacksquare$

EXAMPLES:

① $S = \{1, 2, 3\}$

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0 & 0.6 \end{bmatrix}$$



$a = (1, 0, 0)$

$$P(X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 1)$$

$$= P(X_0 = 1) P_{1,1} \times P_{1,1} \times P_{1,2}$$

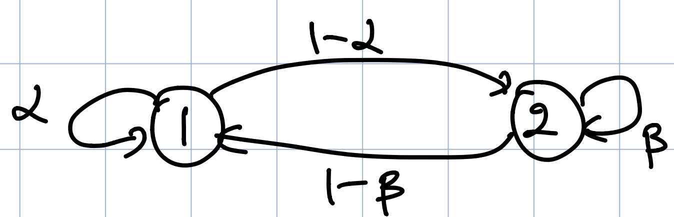
$$= 1 \times 0.1 \times 0.1 \times 0.2$$
$$= 0.002$$

$$P(X_2 = 2,\ X_0 = 1)$$
$$= \sum_{i=1}^{3} P(X_2 = 2,\ X_1 = i,\ X_0 = 1)$$
$$= \text{H.W.}$$

② Two-state DTMC

$$S = \{1, 2\}$$

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} \qquad 0 \le \alpha, \beta \le 1$$



③ Clinical trials

drug A − p_1
drug B − p_2
$\Big\}$ p_1 & p_2 capture the effectiveness of drugs A & B

p_1 & p_2 "unknown".

Q: How to conduct trials using drugs A & B?

Consideration: Do not want to administer a poor drug on a large # of patients

Approach: Pick the drug for patient 0 uniformly at random

& subsequently, if "nth" patient is given drug "i" & if it is effective, continue with drug i, else switch to the other drug.

X_n = drug administered to patient n

$$P(X_{n+1} = 1 \mid X_n = 1, X_{n-1}, \dots, X_0)$$

$$= p_1$$

$$P(X_{n+1} = j \mid X_n = i, \langle \text{history} \rangle)$$

$$= \begin{bmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{bmatrix}$$

"Two-state DTMC"

If $p_1 > p_2$, then state 1 is visited often.

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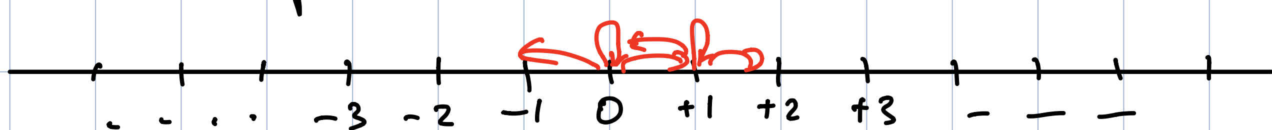
4 State-dependent random-walk



State space $= \{0, \pm 1, \pm 2, \ldots\}$

Particle location: i

Next state: $i-1, i, i+1$

Transition probabilities:

$$P_{i,i+1} = p_i, \quad P_{i,i-1} = q_i, \quad P_{i,i} = r_i, \quad \forall i \in S$$

$$p_i + q_i + r_i = 1$$

Special case: 1 $p_i = p, \; q_i = q, \; r_i = 0$

"Space-homogeneous random walk"

2

Suppose $q_0 = 0$, $P(X_0 \geq 0) = 1$

State space $= \{0, 1, 2, \ldots\}$

"Reflecting barrier": $r_0 = 0 \Rightarrow p_0 = 1$

3 "Absorbing barrier"

If $q_0 = p_0 = 0 \Rightarrow r_0 = 1$

"Hit zero $\Rightarrow$ stays at zero".

Remark: One could have a reflecting/absorbing barrier at any point n.

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5 Gambler's ruin

Two gamblers A & B

Combined fortune: N rupees

Game proceeds as follows:

(i) Toss a coin with bias p

If heads, A gets a rupee from B

else, B gets a rupee from A

(ii) Repeat (i) until one of them goes broke.

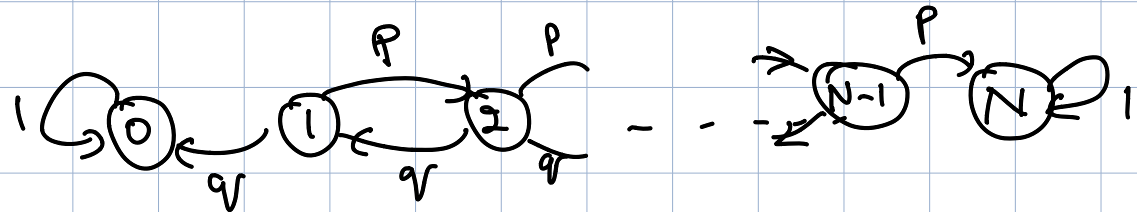
X_n = fortune of A after nth toss

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } p \\ X_n - 1 & \text{w.p. } q = 1-p \end{cases}$$

if $0 < X_n < N$

& $X_{n+1} = X_n$, else i.e., X_n is either $\underline{0}$ or $\underline{N}$

A is broke   B is broke



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⑥ Success runs

Game: Toss a coin with bias p in an iid fashion

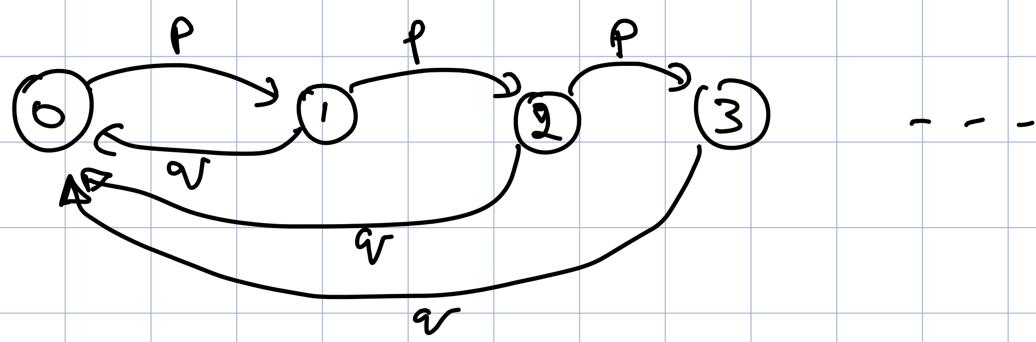
If heads, the player gets a rupee

If tails, all of player's money is "gone".

X_n : player's fortune after n tosses

$S = \{0, 1, 2, \ldots\}$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } p \\ 0 & \text{w.p. } q = 1-p \end{cases}$$



Non-homogeneous case: $P_{i,0} = q_i$, $P_{i,i+1} = 1 - q_i$, $\forall i \in S$

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⑦ Google Search & PageRank algorithm

<u>Model:</u> Suppose there are N pages in the internet

Can be seen as a directed graph with N nodes

Edge from node i to j if there is a link on page i leading to page j.

User starts at page X_0

X_n = page visited by the user after "n" clicks.

Assume $\{X_n, n \geq 0\}$ is a DTMC

Model 1: If k links on a page, user clicks each link with the same probability

Issue! What if there are no links on a page?

Model 2: If stuck with the issue above, the user moves to one of the N pages w.p. $\frac{1}{N}$

Issue: Page 1 links to page 2 &
Page 2 links to page 1, leading to a loop.

Model 3: User visits next page using Model 2 w.p. "d" (damping factor)

& w.p. 1-d, user picks a random page from the set of N pages.

Coming later! Using DTMC to rank the pages, in particular, finding the "long term frequency" of visits for each page & then rank.

Marginal distributions

Let $\{X_n, n \geq 0\}$ be a DTMC with state space $S = \{0, 1, 2, \ldots\}$, t.p.m. "P", & initial distribution "a"

Want: $a_j^{(n)} = P(X_n = j)$ ← distribution of X_n

$$
\begin{aligned}
P(X_n = j) &= \sum_{i \in S} P(X_n = j \mid X_0 = i)\, P(X_0 = i) \\
&= \sum_{i \in S} P(X_n = j \mid X_0 = i)\, a_i \\
&= \sum_{i \in S} a_i\, P_{i,j}^{(n)},
\end{aligned}
$$

where $\underbrace{P_{i,j}^{(n)} = P(X_n = j \mid X_0 = i)}_{\text{n-step transition probability}}$, $\forall i, j \in S, n \geq 0$

In particular,

$$P_{i,j}^{(0)} = P(X_0 = j \mid X_0 = i) = \delta_{i,j}, \quad i, j \in S,$$

where $\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$

$$P_{i,j}^{(1)} = P(X_1 = j \mid X_0 = i) = P_{i,j}, \quad i, j \in S$$

Chapman-Kolmogorov equations

$$P_{i,j}^{(n)} = \sum_{r \in S} P_{i,r}^{(k)} \, P_{r,j}^{(n-k)} \quad , \; i,j \in S$$

where $0 \le k \le n$

$$i \xrightarrow{\text{"k steps"}} r \xrightarrow{\text{"n-k steps"}} j$$

Pf: Fix "k" $\in \{0, \dots, n\}$

$$P_{i,j}^{(n)} = P(X_n = j \mid X_0 = i)$$

$$= \sum_{r \in S} P(X_n = j , X_k = r \mid X_0 = i)$$

$$= \sum_{r \in S} P(X_n = j \mid X_k = r , X_0 = i) \, P(X_k = r \mid X_0 = i)$$

Markov property

$$= \sum_{r \in S} P(X_n = j \mid X_k = r) \, P(X_k = r \mid X_0 = i)$$

Time-homogeneity

$$= \sum_{r \in S} P(X_{n-k} = j \mid X_0 = r) \, P(X_k = r \mid X_0 = i)$$

$$= \sum_{r \in S} P_{r,j}^{(n-k)} \; P_{i,r}^{(k)}$$

$\blacksquare$

Suppose $|S| = m < \infty$ (finite state space)

$$P^{(n)} = \begin{bmatrix} p^{(n)}_{1,1} & - - - - - & p^{(n)}_{1,m} \\ | & & \\ \vdots & & \\ p^{(n)}_{m,1} & - - - - - & p^{(n)}_{m,m} \end{bmatrix}$$

$$P^{(n)} = \left[p^{(n)}_{i,j} \right]_{i,j=1\ldots m}$$

Chapman-Kolmogorov equation:

$$P^{(n)} = P^{(k)} \, P^{(n-k)} \quad , \quad 0 \le k \le n$$

Lecture - 9

<u>Claim</u>: $P^{(n)} = \underline{P^n}$

nth power of t.p.m. P

<u>Pf</u>: Induction (on n)

$$P^{(0)} = I = P^0$$

$$P^{(1)} = P = P^1$$

Now, assume the claim for k. ← induction hypothesis

$$P^{(k+1)} \underset{\text{Chapman-Kolmogorov}}{=} P^{(k)} \, P^{(1)} = P^k \, P = P^{k+1}$$

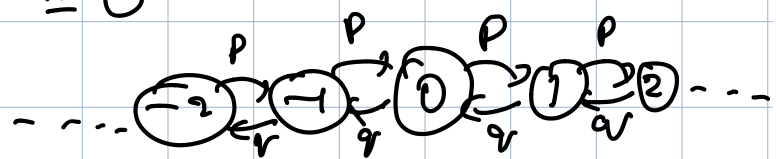
■

EXAMPLE: Random walk

$$P_{i,i+1} = p, \quad P_{i,i-1} = q \ (= 1-p), \quad -\infty < i < \infty, \quad 0 < p < 1$$

Find $P_{0,0}^{(n)} = P(X_n = 0 \mid X_0 = 0)$

If n is odd, $P_{0,0}^{(n)} = 0$



If n is even, say $n = 2k$, then one has to take k steps to right & k steps to left, in any order.

$$P_{0,0}^{(2k)} = \underbrace{\frac{(2k)!}{k!\,k!}}_{\substack{\#\text{ ways (distinct)}\\ \text{of } k \text{ right } \& \\ k \text{ left steps}}} \ \underbrace{p^k \, q^k}_{\text{prob of } k \text{ left} \times \text{prob } k \text{ right steps}}$$

H.W. Show that

$$P_{i,j}^{(n)} = \begin{cases} \binom{n}{b} p^a q^b, & \text{if } n+j-i \text{ is even, where } a = \frac{n+j-i}{2}, \ b = \frac{n+i-j}{2} \\ 0 & \text{else} \end{cases}$$

From an earlier proof, we have

$$a_j^{(n)} = P(X_n = j) = \sum_{i \in S} a_i \; p_{i,j}^{(n)} \quad \text{—— } (*)$$

$$a^{(n)} = [\, a_1^{(n)} \; - - - - \; a_m^{(n)} \,], \quad |S| = m$$

$$a^{(0)} = a = [\, a_1 \; - - - - \; a_m \,]$$ ← Initial distribution

Then, $(*)$ becomes

$$a^{(n)} = a^{(0)} \; P^{(n)}$$

$$a^{(n)} = a \; P^n$$

EXAMPLE:

DTMC with $S = \{1, 2, 3, 4\}$

$$a = \{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\}$$

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.25 & 0.25 & 0.5 & 0 \\ 0.5 & 0 & 0.1 & 0.4 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$$

$$a^{(4)} = a \; P^4 = \text{H.W.}$$

Occupancy times

$\{X_n, n \geq 0\}$ DTMC with state space S, t.p.m. P

$V_j^{(n)}$ = # visits to "j" upto time n (including 0)

Note: $\quad V_j^{(0)} = 1$ if $X_0 = j$

Define $\quad M_{i,j}^{(n)} = E\left(V_j^{(n)} \mid X_0 = i\right), \quad i,j \in S, \; n \geq 0$

Occupancy time of "j" up to "n", starting in "i"

$$M^{(n)} = \left[M_{i,j}^{(n)} \right]_{i,j \in S}, \qquad |S| < \infty$$

Claim:

$$M^{(n)} = \sum_{r=0}^{n} P^r, \quad n \geq 0$$

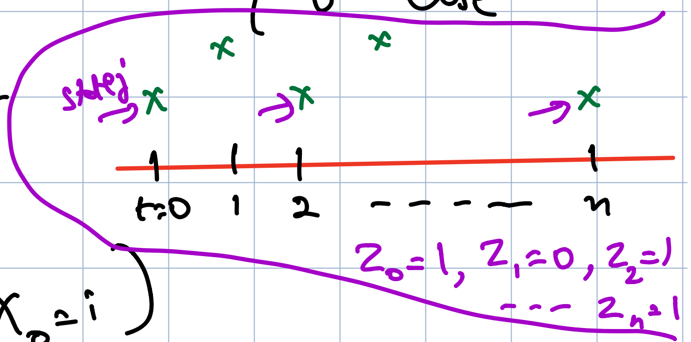
$P^0 = I$

$P^1 = P$

Pf: Fix $j \in S$.

Define an indicator variable $Z_r = \begin{cases} 1 & \text{if } X_r = j \\ 0 & \text{else} \end{cases}$

$$V_j^{(n)} = \sum_{r=0}^{n} Z_r$$



$$M_{i,j}^{(n)} = E\left(V_j^{(n)} \mid X_0 = i\right)$$

$$= E\left[\sum_{r=0}^{n} Z_r \mid X_0 = i\right)$$

$$= \sum_{r=0}^{n} E\left(Z_r \mid X_0 = i\right)$$

$$= \sum_{r=0}^{n} P\left(X_r = j \mid X_0 = i\right)$$

$E(I(A)) = P(A)$

indicator ie,

$I(A) = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{else} \end{cases}$

$$= \sum_{r=0}^{n} p_{i,j}^{(r)}$$

$$M_{i,j}^{(n)} = \sum_{r=0}^{n} \left[P^{(r)} \right]_{i,j} = \sum_{r=0}^{n} \left[P^{r} \right]_{i,j}$$

So,

$$M^{(n)} = \sum_{r=0}^{n} P^{r}$$

$\blacksquare$

EXAMPLE:

3-State DTMC, $S = \{A, B, C\}$

$$P = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \end{array}$$

$$M^{(9)} = \sum_{r=0}^{9} P^{r} = \begin{bmatrix} 2.14 & 2.74 & 5.12 \\ 1.26 & 3.95 & 4.78 \\ 1.15 & 2.85 & 6 \end{bmatrix}$$

$M_{(A,A)}^{(9)} = 2.14 =$ if customer buys A on day 1 (initial distribution), then the expected # of purchases of A $= 2.14$ over 10 days

$$M_{(A,C)}^{(9)} = 5.12$$

Lecture - 10

Computing matrix powers:

$A \to m \times m$ matrix

A is diagonalizable if $\exists\ D, X$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$$

$\lambda_i \to$ eigenvalues of A

and invertible X such that

$$A = X D X^{-1},$$

$$X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_m \\ | & & | \end{bmatrix}$$

$$A x_i = \lambda_i x_i \quad \forall i$$

Claim: If A is diagonalizable, then

$$A^n = X D^n X^{-1}$$

Why? $A^n = \underbrace{(XDX^{-1})(XDX^{-1}) \cdots (XDX^{-1})}_{n \text{ times}}$

$$= X D^n X^{-1}$$

$$= \begin{bmatrix} | & & | \\ x_1 & \cdots & x_m \\ | & & | \end{bmatrix} \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_m^n \end{pmatrix} \begin{bmatrix} - & y_1 & - \\ - & y_2 & - \\ & \vdots & \\ - & y_m & - \end{bmatrix}$$

$$= \sum_{j=1}^{m} \lambda_j^n x_j y_j$$

■

<u>Remark!</u> The t.p.m P of a DTMC is not "necessarily" diagonalizable.

Eigenvalues of P (= stochastic matrix)

<u>Claim!</u> P is a $m \times m$ t.p.m. with eigenvalues $\lambda_1, \ldots, \lambda_m$

(i) At least one of the eigenvalues is one

$$\begin{bmatrix} P_{11} & \cdots & P_{1m} \\ & & \\ P_{m1} & \cdots & P_{mm} \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\text{call this vector "e"}} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$Pe = e$$

(ii) $|\lambda_i| \le 1 \quad \forall i$

Let λ be an eigenvalue, with x as the corresponding eigenvector.

For any m-vector x, define $\|x\| = \max\{|x_i|, i = 1 \ldots m\}$

If x is a eigenvector of P, then $\|x\| > 0$

$$Px = \lambda x$$

$$\sum_{j=1}^{m} P_{i,j}\, x_j = \lambda x_i\,, \quad i = 1, \dots m$$

Let $\|x\| = |x_k|$ for some $k \in \{1, \dots m\}$

So,
$$\sum_{j=1}^{m} P_{k,j}\, x_j = \lambda x_k$$

$$|\lambda x_k| = \left| \sum_{j=1}^{m} P_{k,j}\, x_j \right|$$

triangle ineq $\leq \sum_{j=1}^{m} |P_{k,j}\, x_j|$

$P_{k,j} \geq 0 \ \forall j$ $= \sum_{j=1}^{m} P_{k,j}\, |x_j|$

since $\|x\| = |x_k|$ $\leq \sum_{j=1}^{m} P_{k,j}\, |x_k|$

$\Rightarrow |\lambda x_k| \leq |x_k|$

$\Rightarrow |\lambda|\, |x_k| \leq |x_k|$

$\Rightarrow |\lambda| \leq 1$ since $|x_k| > 0$

$\blacksquare$

EXAMPLE: 3-state DTMC with $P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$

$0 < p, q < 1$, $p + q = 1$

Eigenvalues $-1, 0, 1$ ⇐ distinct eigenvalues
So, P is diagonalizable

Eigenvectors → $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} p \\ 0 \\ -q \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} 1 & p & 1 \\ -1 & 0 & 1 \\ 1 & -q & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} \quad \end{bmatrix}$$

$$P^n = X D^n X^{-1}$$

H.W. Check $P^{2n} = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}$ &

$$P^{2n+1} = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

DTMCs: First passage times

Ref: Chap. 3 of Kulkarni's book

Let $\{X_n, n \geq 0\}$ be a DTMC on $S = \{0, 1, 2, \dots\}$

$$\underbrace{T = \min\{n \geq 0 \mid X_n = 0\}}_{\text{First passage time into "0".}}$$

Quantities related to T:

1. Complementary cdf $P(T > n)$, $n \geq 0$ ✓
2. Probability of eventually hitting "0" $P(T < \infty)$ ✓
3. Factorial moments $E(T^{(k)})$, $k \geq 1$

   where $T^{(k)} = T(T-1)\dots(T-k+1)$

   "We will look at $E(T)$"
4. Generating function $E(z^T)$ → skipped

Some definitions

$$v_i(n) = P(T > n \mid X_0 = i)$$

$$u_i = P(T < \infty \mid X_0 = i)$$

$$m_i(k) = E(T^{(k)} \mid X_0 = i)$$

$$\phi_i(z) = E(z^T \mid X_0 = i)$$

We have

$$P(T > n) = \sum_{i \in S} a_i v_i(n)$$

$a_i \to P(X_0 = i)$

$$P(T < \infty) = \sum_{i \in S} a_i u_i$$

$$E(T^{(k)}) = \sum_{i \in S} a_i m_i(k)$$

$$E(z^T) = \sum_{i \in S} a_i \phi_i(z)$$

CDF of T:

Idea: first-step analysis, i.e., conditioning on X_1

$$v(n) = \begin{bmatrix} v_1(n) \\ \vdots \\ v_m(n) \end{bmatrix}, \quad n \geq 0$$

$$B = \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{m,m} \end{bmatrix}$$

$\leftarrow$ excluding the row/col for "0".

for the case $|S| = m$

In general, B is the sub-matrix of P with " — ".

i.e., $B = [P_{i,j}]_{i,j \geq 1}$

<u>Claim</u>: $v(n) = B^n e, \quad n \geq 0$

<u>Pf</u>: Fix $n \geq 1$, $i \geq 1$

$$v_i(n) = P(T > n \mid X_0 = i)$$

$$= \sum_{j=0}^{\infty} P(T > n \mid X_1 = j, X_0 = i)\, P(X_1 = j \mid X_0 = i)$$

$$= \sum_{j=0}^{\infty} P_{i,j}\, P(T > n \mid X_1 = j, X_0 = i)$$

$$= P_{i,0}\, \underbrace{P(T > n \mid X_1 = 0, X_0 = i)}_{\substack{=0 \text{ since} \\ X_0 = i \,\&\, X_1 = 0 \\ \Rightarrow T = 1}} + \sum_{j=1}^{\infty} P_{i,j}\, P(T > n \mid X_1 = j, X_0 = i)$$

Markov property

$$= \sum_{j=1}^{\infty} P_{i,j}\, P(T > n \mid X_1 = j)$$

Time-homogeneity

$$= \sum_{j=1}^{\infty} P_{i,j}\, P(T > n-1 \mid X_0 = j)$$

$$v_i(n) = \sum_{j=1}^{\infty} P_{i,j}\, v_j(n-1)$$

In vector-matrix notation,

$$v(n) = B\, v(n-1)$$

$$= B^2\, v(n-2)$$

$$v(n) = B^n v(0) \qquad (*)$$

$$v_i(0) = P(T > 0 \mid X_0 = i) \qquad i \geq 1$$
$$= 1 \quad \forall i \geq 1$$

$$v(0) = e$$

So, using the above in $(*)$, we obtain

$$v(n) = B^n e, \quad n \geq 1$$

For $n=0$, $B^n = B^0 = I$, $v(0) = e$.

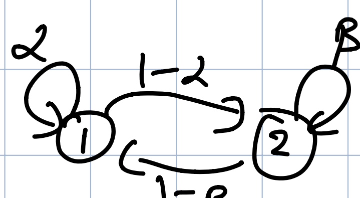
So, $\boxed{v(n) = B^n e} \quad \forall n \geq 0$

$\blacksquare$

Lecture - 11

EXAMPLE:

(1) Two-state DTMC



$$P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$

$$B = [\beta]$$

$$T = \min\{n \geq 0 \mid X_n = 1\}$$

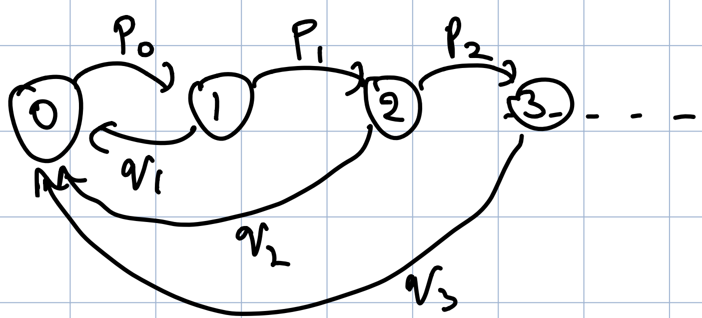
$$v_2(n) = P(T > n \mid X_0 = 2)$$
$$= \beta^n$$

$$P(T=n \mid X_0=2) = v_2(n-1) - v_2(n)$$
$$= \beta^{n-1}(1-\beta)$$

Geometric distribution

---

② Success runs:



$$P_{i,0} = q_i, \quad P_{i,i+1} = p_i, \quad i = 0, 1, 2, \ldots$$

T = first passage time to 0

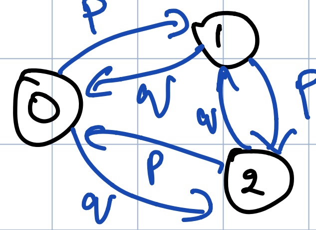
$$v_1(0) = P(T>0 \mid X_0=1) = 1$$

$$v_1(n) = P(T>n \mid X_0=1)$$
$$= P(X_1=2, X_2=3, \ldots, X_n=n+1 \mid X_0=1)$$
$$= P(X_1=2 \mid X_0=1)\, P(X_2=3 \mid X_1=2) \cdots P(X_n=n+1 \mid X_{n-1}=n)$$

$$v_1(n) = p_1 p_2 \cdots p_n$$

---

③ Simple random walk



$$P = \begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}$$

$0 < p, q < 1 \quad p + q = 1$

X_n = position of walker at time n

T = first passage time to 0

$= \min\{ n \geq 0 \mid X_n = 0 \}$

$$v(n) = B^n e \qquad B = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

$$v(n) = \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix}$$

$$B^2 e = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} pq \\ pq \end{bmatrix} \begin{array}{l} \to v_1(2) \\ \to v_2(2) \end{array}$$

$$B^3 e = \begin{bmatrix} p^2 q \\ p q^2 \end{bmatrix} \begin{array}{l} \to v_1(3) \\ \to v_2(3) \end{array}$$

$$v_1(2n) = p^n q^n$$

$$v_1(2n+1) = p^{n+1} q^n$$

Absorption probabilities

$$u_i = P(T < \infty \mid X_0 = i) \leftarrow$$ Absorption probability if $P_{0,0} = 1$

$$u_i = 1 - v_i \text{, where } v_i = \lim_{n \to \infty} v_i(n)$$

$v_i = \lim_{n\to\infty} v_i(n)$ $\rightarrow$ (*) $v_i(1) \geq v_i(2) \geq v_i(3)$ - - -
"monotonically decreasing"

(**) $\{v_i(n)\}$ $\rightarrow$ bounded below by "0".

(*) & (**) $\Rightarrow$ $\lim_{n\to\infty} v_i(n)$ exists

Let $v = [v_i]_{i\in S}$, i.e.,

$$v = \lim_{n\to\infty} \{v(n) := [v_i(n)]_{i\in S}\}$$

Claim: The vector v is given by the "largest" solution to

$$v = Bv \quad \text{such that} \quad v \leq e.$$

"Largest" intrepretation:
For any w satisfying $w = Bw$ with $w \leq e$,
we have, $w \leq v$, where $v = \lim_{n\to\infty} v(n)$

Pf:

$$v(n) = B\, v(n-1)$$ $\leftarrow$ from an earlier proof

$$\lim_{n\to\infty} v(n) = \lim_{n\to\infty} B\, v(n-1)$$

$$v = Bv \quad \& \quad v \leq e$$

Suppose w is another solution, i.e.,

$$w = Bw \quad \& \quad w \le e$$

"Induction proof":

Base Case: $v(0) = e \geq w$

Induction hypothesis: Assume $v(k) \geq w$ for $k = 0, 1, \dots, n$

$$v(n+1) = B\,v(n) \geq B\,w = w$$

So, $v(n+1) \geq w \quad \forall n$

Taking limits, $\lim_{n \to \infty} v(n+1) = v \geq w$

$\blacksquare$

EXAMPLE 1 "Gambler's ruin"

$$\left.\begin{aligned} P_{i,i+1} &= p \\ P_{i,i-1} &= q \ (=1-p) \end{aligned}\right\} \quad i \in \{1, \dots N-1\}$$

$$P_{0,0} = P_{N,N} = 1$$

$$T = \min\{n \geq 0 \mid X_n = 0\}$$

For, $1 \le i \le N-1$,

$$(*) \begin{cases} v_i = p\,v_{i+1} + q\,v_{i-1} \\ v_0 = 0, \quad v_N = 1 \end{cases}$$

← First-step analysis

Solving $(*)$ for $p = q = \frac{1}{2}$

$$v_i = \frac{1}{2} v_{i+1} + \frac{1}{2} v_{i-1}$$

$$b_i = v_i - v_{i-1}, \quad b_{i-1} = v_{i-1} - v_{i-2}$$

Claim: $b_i = b_{i-1}$

So, $b_i = b_{i-1} = \cdots\cdots = b_1$

$$v_i = b_i + v_{i-1}$$
$$= b_1 + v_{i-1}$$
$$= 2b_1 + v_{i-2}$$
$$\vdots$$
$$v_i = i b_1 + v_0$$

Hence, $v_N = N b_1 + v_0$

$$b_1 = \frac{1}{N}$$

$$\boxed{v_i = \frac{i}{N}} \quad \text{for } 1 \le i \le N-1$$

Case $p \neq 1/2$: $v_i = p v_{i+1} + q v_{i-1}$

Guess a solution! Suppose $v_i = \alpha^i$

$$\alpha^i = p\,\alpha^{i+1} + q\,\alpha^{i-1} \quad \text{———} \quad (\ast\ast)$$

$$\alpha = p\alpha^2 + q$$

$$p\alpha^2 - \alpha + q = 0$$

Root $\alpha = \dfrac{1 \pm \sqrt{1-4pq}}{2p}$

$= \dfrac{1 \pm \sqrt{(p-q)^2}}{2p}$ using $p+q=1$

$= \dfrac{1+p-q}{2p}$ & $\dfrac{1-(p-q)}{2p}$

$= 1$ & $\dfrac{q}{p}$

General solution of $(\ast\ast)$: $v_i = c + d\left(\dfrac{q}{p}\right)^i$

$v_0 = 0 \Rightarrow \quad c + d = 0$

$v_N = 1 \Rightarrow \quad c + d\left(\dfrac{q}{p}\right)^N = 1$

$c = -d = \dfrac{1}{1-\left(\dfrac{q}{p}\right)^N}$, leading to

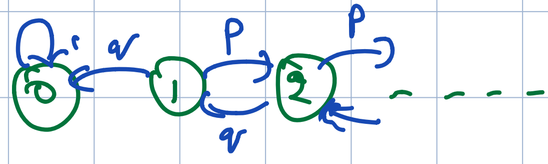
$$v_i = \left(1-\left(\frac{q}{p}\right)^i\right) \Big/ \left(1-\left(\frac{q}{p}\right)^N\right), \quad 1 \le i \le N-1$$

Lecture - 12

Example 2: Random walk (homogeneous)

$$P_{i,i+1} = p, \quad P_{i,i-1} = q; \quad i \geq 1$$

$$P_{0,0} = 1$$



T = first passage time to "0"

$$v_i = q\, v_{i-1} + p\, v_{i+1}$$

$$v_0 = 0$$

General solution: $v_i = \alpha + \beta\left(\frac{q}{p}\right)^i, \; i \geq 0$

Using $v_0 = 0$, we get $\beta = -\alpha$

So, $$v_i = \alpha\left(1 - \left(\frac{q}{p}\right)^i\right)$$

Find α using "v is the largest solution satisfying $v \leq e$"

Case $q < p$: $\alpha = 1$ $\quad v_i = 1 - \left(\frac{q}{p}\right)^i$

Case $q \geq p$: $\alpha = 0$ $\quad v_i = 0$ ← "In this case, the random walk is drifting towards zero"

Example 3: General Random Walk

$$P_{i,i+1} = p_i \;, \quad P_{i,i-1} = q_i \qquad (p_i + q_i = 1)$$

$$P_{0,0} = 1$$

$$v_i = q_i v_{i-1} + p_i v_{i+1} \quad \text{— } (*)$$

$$v_0 = 0$$

Let $\quad b_i = v_i - v_{i-1} \quad$ — $(**)$

$(*) \Leftrightarrow \quad (p_i + q_i) v_i = q_i v_{i-1} + p_i v_{i+1}$

$$q_i b_i = p_i b_{i+1}$$

$$b_{i+1} = \frac{q_i}{p_i} b_i$$

$$= \underbrace{\left(\frac{q_1 \cdots q_i}{p_1 \cdots p_i} \right)}_{\text{Call this } \alpha_i} b_1$$

$$v_{j+1} = \left(\sum_{i=0}^{j} \alpha_i \right) v_1 \quad \leftarrow \text{by summing } (**)$$

$v_1 \rightarrow$ largest value s.t. $0 \le v_j \le 1, \; \forall j$

① Suppose $\quad \sum_{i=0}^{\infty} \alpha_i < \infty$

$$v_1 = \frac{1}{\sum_{i=0}^{\infty} \alpha_i}$$

② If $\sum \alpha_i = \infty$, then $v_1 = 0$

Thus, $v_i = \begin{cases} \dfrac{\sum_{j=0}^{i-1} \alpha_j}{\sum_{j=0}^{\infty} \alpha_j} & \text{if } \sum \alpha_j < \infty \\ 0 & \text{else} \end{cases}$ [Think intuitively about this case, i.e., $\sum \alpha_j = \infty$]

Expectation of T

$\{X_n, n \geq 0\}$ DTMC

$T = \min\{n \geq 0 \mid X_n = 0\}$

$u_i = P(T < \infty \mid X_0 = i)$

$m_i = E(T \mid X_0 = i)$

A1: Assume $u_i = 1 \ \forall i$ If not, i.e., if $u_i < 1$, then $m_i = \infty$.

$$m = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ \vdots \end{bmatrix} \leftarrow \text{col. vector excluding the entry for zero}$$

$B \leftarrow$ Same as before ($=$ t.p.m $-$ "zero" row/col)

Claim: Assume (A1)

Then, m is the smallest non-negative solution to

$$m = e + Bm$$

Pf:

$$m_i = E(T \mid X_0 = i), \quad \forall i \geq 1$$

$$= \sum_{j=0}^{\infty} E(T \mid X_0 = i, X_1 = j)\, P(X_1 = j \mid X_0 = i)$$

$$= \sum_{j=0}^{\infty} P_{i,j}\, E(T \mid X_0 = i, X_1 = j)$$

$$E(T \mid X_0 = i, X_1 = 0) = 1$$

For $j \geq 1$,

$$E(T \mid X_0 = i, X_1 = j) = 1 + E(T \mid X_0 = j)$$

$$= 1 + m_j$$

$$m_i = P_{i,0} + \sum_{j=1}^{\infty} P_{i,j}(1 + m_j)$$

$$= 1 + \sum_{j=1}^{\infty} P_{i,j}\, m_j$$

$$m = e + Bm$$

Proof of "m" being the smallest solution: "homework" $\blacksquare$

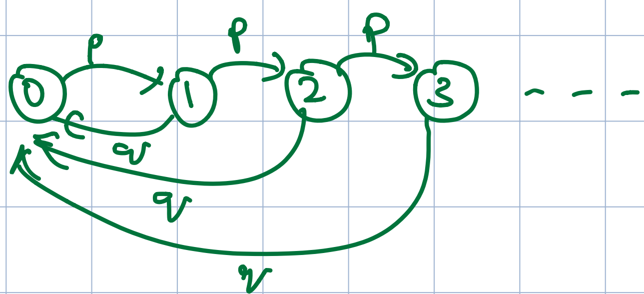
LECTURE-13

EXAMPLE 1> Toss a coin repeatedly until "k" consecutive heads are seen

$$P_{i,0} = q, \quad P_{i,i+1} = p, \quad i = 0, 1, 2, \dots$$

$$T = \min\{n \geq 0 \mid X_n = k\}, \quad m_i = E(T \mid X_0 = i)$$

$$(*) \begin{cases} m_i = 1 + q\, m_0 + p\, m_{i+1}, & 0 \leq i \leq k-1 \\ m_k = 0 \end{cases}$$



Solving $(*)$:

$$m_{k-1} = 1 + q m_0, \quad m_{k-2} = 1 + q m_0 + p(1 + q m_0)$$

$$m_{k-2} = (1 + q m_0)(1 + p)$$

$$m_{k-3} = (1 + q m_0)(1 + p + p^2)$$

$$m_i = (1 + q m_0)(1 + p + \dots + p^{k-i-1})$$

$$m_i = \frac{(1 + q m_0)(1 - p^{k-i})}{q}$$

For $i=0$, $$m_0 = \left(\frac{1}{q} + m_0\right)(1-p^k)$$

$$m_0 p^k = \frac{1}{q}(1-p^k) \quad \text{or} \quad m_0 = \frac{1}{q}\left(\frac{1}{p^k} - 1\right)$$

EXAMPLE 2) General simple random walk

$$P_{i,i+1} = p_i, \quad P_{i,i-1} = q_i, \quad P_{0,0} = 1$$

$$\alpha_i = \frac{q_1 \cdots q_i}{p_1 \cdots p_i}$$

$\forall i$, $v_i = 0$ if $\sum_j \alpha_j = \infty$

$\Leftrightarrow$ $\forall i$, $u_i = 1$ —— " ——

Assume $\sum_j \alpha_j = \infty$

Finding "m":
$$m_i = 1 + q_i m_{i-1} + p_i m_{i+1}$$
$$m_0 = 0$$

$$x_i = m_i - m_{i-1}$$

$$(p_i + q_i) m_i = 1 + q_i m_{i-1} + p_i m_{i+1}$$

$$q_i x_i = 1 + p_i x_{i+1}$$

$$x_{i+1} = \frac{q_i}{p_i} x_i - \frac{1}{p_i}$$

$$= \frac{v_i v_{i-1}}{p_i p_{i-1}} x_{i-1} - \frac{v_i}{p_i} \frac{1}{p_{i-1}} - \frac{1}{p_i}$$

$$= - - - -$$

$$\vdots$$

$$x_{i+1} = -\alpha_i b_i + \alpha_i m_1,$$

where $b_i = \sum_{j=1}^{i} \frac{1}{p_j \alpha_j}$

Using $x_i = m_i - m_{i-1}$ & $m_0 = 0$,

$$x_{i+1} + x_i + \cdots + x_1 = m_{i+1}$$

So, $m_{i+1} = -\sum_{j=1}^{i} \alpha_j b_j + \left(\sum_{j=0}^{i} \alpha_j\right) m_1$

Using "m" is a non-negative solution to $m = e + Bm$,

$$m_{i+1} \geq 0$$

$$\Leftrightarrow \quad m_1 \left(\sum_{j=0}^{i} \alpha_j\right) \geq \sum_{j=1}^{i} \alpha_j b_j$$

$$m_1 \geq \frac{\sum_{j=1}^{i} \alpha_j b_j}{\sum_{j=0}^{i} \alpha_j} \qquad \text{— } (**)$$

Proof by special case : $i=3$

$$\text{RHS in } (**) = \frac{\alpha_3 b_3 + \alpha_2 b_2 + \alpha_1 b_1}{\alpha_0+\alpha_1+\alpha_2+\alpha_3}$$

$$= \frac{\alpha_3\left(\frac{1}{P_3\alpha_3}+\frac{1}{P_2\alpha_2}+\frac{1}{P_1\alpha_1}\right)+\alpha_2\left(\frac{1}{P_2\alpha_2}+\frac{1}{P_1\alpha_1}\right)+\alpha_1\frac{1}{P_1\alpha_1}}{\alpha_0+\alpha_1+\alpha_2+\alpha_3}$$

$$= \frac{1}{\alpha_0+\alpha_1+\alpha_2+\alpha_3}\left(\frac{1}{P_1\alpha_1}(\alpha_1+\alpha_2+\alpha_3)+\frac{1}{P_2\alpha_2}(\alpha_2+\alpha_3)+\frac{1}{P_3\alpha_3}\alpha_3\right)$$

So, RHS of $(**)$ $(=)$

$$m_1 \geq \sum_{j=1}^{i}\frac{1}{P_j\alpha_j}\,\frac{\left(\sum_{r=j}^{i}\alpha_r\right)}{\sum_{r=0}^{i}\alpha_r} \rightarrow \text{Call this } f(i)$$

increasing function in "i"

Proof by special case with $i=2$

$$f(i) \leq f(i+1)$$

Put $i=2$

$$\frac{1}{\alpha_0+\alpha_1+\alpha_2}\left(\frac{1}{P_2\alpha_2}\alpha_2+\frac{1}{P_1\alpha_1}(\alpha_1+\alpha_2)\right) \leq \frac{1}{(\alpha_0+\alpha_1+\alpha_2+\alpha_3)}\left[\frac{\alpha_3}{P_3\alpha_3}+\frac{\alpha_2+\alpha_3}{P_2\alpha_2}+\frac{\alpha_1+\alpha_2+\alpha_3}{P_1\alpha_1}\right]$$

$$\left(1+\frac{\alpha_3}{\alpha_0+\alpha_1+\alpha_2}\right)\left(\frac{1}{P_1\alpha_1}(\alpha_1+\alpha_2)+\frac{1}{P_2\alpha_2}\alpha_2\right) < \left(\frac{\alpha_3}{P_3\alpha_3}+\frac{\alpha_2+\alpha_3}{P_2\alpha_2}+\frac{\alpha_1+\alpha_2+\alpha_3}{P_1\alpha_1}\right)$$

$$\frac{\alpha_3}{\alpha_0+\alpha_1+\alpha_2}\left(\frac{\alpha_1+\alpha_2}{p_1\alpha_1}+\frac{\alpha_2}{p_2\alpha_2}\right) \le \frac{\alpha_3}{p_1\alpha_1}+\frac{\alpha_3}{p_2\alpha_2}+\frac{\alpha_3}{p_3\alpha_3}$$

$$\left(\frac{\alpha_1+\alpha_2}{\alpha_0+\alpha_1+\alpha_2}\right)\frac{1}{p_1\alpha_1}+\left(\frac{\alpha_2}{\alpha_0+\alpha_1+\alpha_2}\right)\frac{1}{p_2\alpha_2} \le \frac{1}{p_1\alpha_1}+\frac{1}{p_2\alpha_2}+\frac{1}{p_3\alpha_3}$$

this holds

H.W. : Prove $f(i) \le f(i+1)$ for any $i \ge 1$

As $i \to \infty$, $\quad f(i) \to \sum_{j=1}^{\infty} \frac{1}{p_j\alpha_j}$ (Since $\sum \alpha_j = \infty$ by assumption)

$$m_1 \ge f(i) \quad \forall i$$

$$\Rightarrow \quad m_1 \ge \lim_{i\to\infty} f(i) = \sum_{j=1}^{\infty}\frac{1}{p_j\alpha_j}$$

→ not necessarily finite

So, $\quad m_1 = \sum_{j=1}^{\infty}\frac{1}{p_j\alpha_j}$ Since m_1 is the smallest "non-negative" solution

Using m_1, we have

$$m_i = \sum_{k=0}^{i-1}\alpha_k\left(\sum_{j=k+1}^{\infty}\frac{1}{p_j\alpha_j}\right)$$

< End of chapter >