# Online Estimation and Optimization of Utility-Based Shortfall Risk

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By

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## Declaration

I declare that the thesis titled "Online Estimation and Optimization of Utility-Based Shortfall Risk" submitted by me, Arvind Menon (EP17B017), to the Indian Institute of Technology, Madras for the award of the degree of Bachelor of Technology in Engineering Physics is a bona fide record of research work carried out by me under the supervision of Dr. L.A. Prashanth and Dr. Krishna Jagannathan. The contents of this thesis, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

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I certify that the statements above are true to the best of my knowledge.

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# Abstract

Utility-Based Shortfall Risk (UBSR) is a risk metric that is increasingly popular in financial applications, owing to certain desirable properties that it enjoys. We consider the problem of estimating UBSR in a recursive setting, where samples from the underlying loss distribution are available one-at-a-time. We cast the UBSR estimation problem as a root finding problem, and propose stochastic approximation-based estimations schemes. We derive non-asymptotic bounds on the estimation error in the number of samples. We also consider the problem of UBSR optimization within a parametrized class of random variables. We propose a stochastic gradient descent based algorithm for UBSR optimization, and derive non-asymptotic bounds on its convergence.

# Contents

# 1 Introduction

In several financial applications, it is necessary to understand risk sensitivity while maximizing the returns. Several risk measures have been studied in the literature, e.g., mean-variance, Value at Risk (VaR), Conditional Value at Risk (CVaR), distorted risk measure, and prospect theory. In [2], the authors consider four properties as desirable for a risk measure, namely positive homogeneity, translation invariance, sub-additivity, and monotonicity. They define a risk measure as being *coherent* if it possesses the aforementioned properties. In a related development, in[15], the authors chose to relax the sub-additivity and positive homogeneity requirements of a coherent risk measure, and instead impose a convexity condition on the underlying risk measure. Such a relaxation is justified in practical contexts where the risk is a non-linear function of the underlying random variable (e.g., a financial position).

CVaR is a popular risk measure that come under the umbrella of coherent risk measures. Utility-based shortfall risk (UBSR) [15] is a risk measure that is closely related to CVaR, and one that belongs to the class of convex risk measures. UBSR as a risk measure is preferable over CVaR for two reasons: (i) Unlike CVaR, UBSR is invariant under randomization; and (ii) UBSR involves a utility function that can be chosen to encode the risk associated with each value the r.v. $X$ takes, while CVaR is concerned primarily with values of $X$ beyond a certain quantile.

In real-world scenarios, the distribution of the underlying r.v. is seldom available in a closed form. Instead, one can obtain samples, which are used to estimate the chosen risk measure. Risk estimation has received a lot of attention in the recent past, cf. [19, 12, 24, 33, 13, 6, 28, 27, 35, 9, 22, 20], with CVaR being the dominant choice for the risk measure.

In this thesis, we focus on recursive estimation of UBSR, in a setting where data arrives in an online fashion. Stochastic approximation [30, 7] is a procedure that is well-suited for the purpose of online estimation. In the context of UBSR estimation, our main contribution is the *non-asymptotic* analysis of a stochastic approximation-based estimation scheme. We cast the estimation of UBSR as a stochastic root finding problem, and derive 'finite-sample' bounds for this scheme. Our analysis assumes that the underlying objective satisfies a monotonicity condition. If the monotonicity parameter is known and is used in setting the step-size, the algorithm results in an $O(1/n)$ rate of error decay. We also develop another variant that employs a universal step-size, and results in a $O(1/n^{\alpha})$ rate, where $0 < \alpha < 1$. These non-asymptotic results are obtained under similar technical assumptions as in [13, 18] — specifically, a finite second moment condition on the loss distribution. If the loss distribution is sub-Gaussian, we also obtain a 'high probability' result for the concentration of the approximation error.

Moving beyond UBSR estimation, we also consider the problem of optimizing UBSR within a parameterized class of random variables. The motivation for this problem lies in understanding the risk sensitivity in a portfolio management application [31, 18]. Specifically, an investor could choose to distribute his/her capital among different assets, and the decision parameter governing the capital distribution is to be optimized to decide the best allocation. The utility function that goes into the definition of UBSR would encode the investor's risk preference, and the goal is to find the best decision parameter to minimize risk, as quantified by UBSR.

For the problem of UBSR optimization, we propose a stochastic gradient algorithm, and derive non-asymptotic bounds on its performance. Stochastic gradient (SG) methods have a long history, and non-asymptotic analysis of such schemes has garnered a lot of attention over the last decade, see [8] for a survey.

Unlike in a classic SG setting, the UBSR optimization problem involves *biased* function measurements, which presents some technical challenges. Specifically, the UBSR estimation scheme is biased, in the sense that the estimation error does not have zero expectation. This is unlike in the classical SG settings, where the estimation error is assumed to be zero mean. In our setting, even though the estimation error is not zero-mean, the error can be reduced by increasing the batch size used for estimation. For the purpose of gradient estimation, we leverage the UBSR sensitivity formula derived in [18], and use a natural estimator of this quantity based on i.i.d. samples. By controlling the batch size, we are able to derive a $O(1/n)$ rate for the SG algorithm to optimize the UBSR.

*Related work.* Stochastic approximation has been explored in the context of CVaR estimation in [4, 5]. Recursive estimation of quantiles, variances and medians has been considered earlier in [11, 10, 17]. UBSR was introduced in [15], and non-recursive estimation schemes for UBSR were proposed in [18]. A paper closely related to our work from UBSR estimation viewpoint is [13], which uses a recursive estimation technique. The authors establish asymptotic convergence of their algorithm, and a 'central limit theorem' showing the asymptotic Gaussianity of the scaled estimation error. In contrast, we establish *non-asymptotic*, i.e., finite-sample bounds for the performance of our recursive estimation method, under similar technical assumptions as [13, 18]. In a recent paper [26], the authors use the estimation scheme from [18] to establish concentration inequalities for UBSR estimation.

The rest of the thesis is organized as follows: In Section 2, we define the notion of UBSR for a general random variable, and formulate the estimation as well as optimization problems under a UBSR objective. In Section 3, we describe the stochastic approximation-based scheme for estimating the UBSR of a random variable, and present concentration bounds for this estimation scheme. In Section 4, we present a stochastic gradient algorithm for optimizing the UBSR in a parameterized class of random variables, and present a non-asymptotic bound that quantifies the convergence rate of this algorithm. We provide proofs of convergence for all the proposed algorithms in the supplementary material. Finally, in Section 6, we provide our concluding remarks.

# 2 Problem Formulation

Let $X$ be a random variable, and $\ell(\cdot)$ be a convex utility function. Let $\lambda$ be a pre-specified "risk-level" parameter that lies in the interior of the range of $\ell$. We first define an acceptance set as follows:

$$\mathcal{A} := \{X \in L^\infty : E[\ell(-X)] \leq \lambda\}, \tag{1}$$

where $L^\infty$ represents the set of bounded random variables, and the expectation is taken w.r.t. the distribution of the random variable $X$.

Using the acceptance set, the utility-based shortfall risk (UBSR) $SR_\lambda(X)$ is defined by

$$SR_\lambda(X) := \inf\{t \in \mathcal{R} : t + X \in \mathcal{A}\}. \tag{2}$$

For notational convenience, we have made the dependence of UBSR $SR_\lambda(X)$ on the utility function $l$ implicit. Intuitively, if $X$ represents a financial position, then $SR_\lambda(X)$ denotes the minimum cash that has to be added to $X$ so that it falls into the acceptable set $\mathcal{A}$.

UBSR is a particular example of a convex risk measure [15], which is a generalization of a coherent risk measure [2]. In particular, a coherent risk measure satisfies sub-additivity and positive-homogeneity, and these two properties readily imply convexity.

As a risk measure, UBSR is preferable over the popular Value-at-Risk (VaR), owing to the fact that UBSR is convex. Another closely related risk measure is CVaR (Conditional Value at Risk), which is a coherent risk measure. UBSR has a few advantages over CVaR, namely (i) Unlike CVaR, UBSR is invariant under randomization; and (ii) UBSR involves an utility function that can be chosen to encode the risk associated with each value the r.v. $X$ takes, while CVaR is concerned with values of $X$ beyond VaR at a pre-specified level $\alpha$. For a loss r.v. $X$ in a financial application, it makes sense to associate more risk with larger losses, and this can be encoded using, for example, an exponential utility function. On the other hand, CVaR considers all losses beyond a certain threshold equally.

In this paper, we focus on two problems concerning shortfall risk, namely (i) UBSR estimation, and (ii) UBSR optimization within a parametrized family of distributions. We define these two problems below.

Define the function

$$g(t) := E[\ell(-X - t)] - \lambda. \tag{3}$$

We make the following assumption on the function $g$ defined above.

**(A1).** *There exists $t_l, t_u$ s.t. $g(t_l) > 0$ and $g(t_u) < 0$.*

Under the above assumption, the problem of UBSR estimation, i.e, estimating $SR_{\ell,\lambda}(X)$ of a r.v. $X$, can be cast as a root finding problem. Indeed, $SR_{\ell,\lambda}(X)$ is the unique root of the function $g$, i.e., the solution $t^*$ that satisfies $g(t^*) = 0$ coincides with $SR_{\ell,\lambda}(X)$. We consider a setting where the expectation in the definition of $g(\cdot)$ cannot be explicitly evaluated, Instead, we have access to samples from the distribution of $X$, and we use a stochastic root-finding scheme for the UBSR estimation.

Next, we define the the problem of UBSR optimization. Suppose that $X$ belongs to a parametrized family of distributions $\{X(\theta) : \theta \in \Theta\}$, where $\Theta$ is a compact and convex subset of $\mathbb{R}$. The SR optimization problem for this prametrized class is given as

$$\text{Find } \theta^* \in \arg\min_{\theta\in\Theta} SR_\lambda(X(\theta)). \tag{4}$$

For the sake of simplicity, we focus on the case of a scalar parameter $\theta$. Extending to a vector parameter is straightforward. Again, assuming that we have access to samples from the distribution of $X$, we use a stochastic gradient descent technique for SR optimization.

# 3 UBSR estimation

We consider a setting where the expectation in the definition of the function $g$ cannot be explicitly evaluated. Instead, we assume that have access to samples from the distribution of $X$ in an online fashion, and the goal is to have a recursive estimation scheme for UBSR.

Stochastic approximation [7] is a class of algorithms for solving stochastic root-finding problems. UBSR estimation is a root-finding problem since one has to find a $t^*$ satisfying $g(t^*) = 0$, or $E[\ell(-X - t^*)] = \lambda$. For this problem, [13] proposed a stochastic approximation scheme, and performed an asymptotic convergence analysis. In this paper, we focus on UBSR estimation from a non-asymptotic viewpoint.

We propose a method to incrementally estimate UBSR using each additional sample. Specifically, we use the following update iteration:

$$t_n = \Gamma(t_{n-1} + a_n (\hat{g}(t_{n-1}))), \tag{5}$$

where $\hat{g}(t) = \ell(\xi_n - t_{n-1}) - \lambda$ is an estimate of $g(t)$ using an i.i.d. sequence $\{\xi_i\}$ from the distribution of $-X$, and $\Gamma$ is a projection operator defined by $\Gamma(x) = \min(\max(t_l, x), t_u)$. Such a projection operator has been used in the context of UBSR estimation earlier, cf. [13].

## Main results

In addition to (A1), we make the following assumptions for the bounds on UBSR estimation.

**(A2).** *There exists $\mu_1 > 0$ such that $g'(t) \leq -\mu_1$, for all $t \in [t_l, t_u]$.*

**(A3).** *Let $\varepsilon_n = \hat{g}(t_n) - g(t_n)$. Then, there exists a $\sigma > 0$ such that $E[\varepsilon_n^2] \leq \sigma^2$ for all $n \geq 1$.*

Previous works on UBSR estimation, cf. [13, 18], require the utility function to be increasing, which is equivalent to the assumption (A2) that we make above. Next, (A3) requires that the underlying noise variance is bounded: a natural assumption in the context of an estimation problem. A similar assumption appears in [13, 18].

The first result below is a non-asymptotic bound on the estimation error $E[(t_n - SR_\lambda(X))^2]$ for a stepsize choice that requires the knowledge of $\mu_1$ from (A2).

**Theorem 1.** *Assuming (A1)-(A3) to hold and setting the step size $a_k = \frac{c}{k}$ with $\frac{1}{2} < \mu_1 c < 1$, we have*

$$E[(t_n - SR_\lambda(X))^2] \leq \frac{(t_0 - SR_\lambda(X))^2}{n^{2\mu_1 c}} + \frac{\sigma^2 2^{2\mu_1 c} c^2}{(2\mu_1 c - 1)n}. \tag{6}$$

*Proof.* See Appendix B.1. □

**Remark 1.** *The first term on the RHS in the bound above concerns the initial error, i.e., the rate at which the algorithm 'forgets' the starting point $t_1$. The second term relates to the noise variance in UBSR estimation. From the bound above, together with the fact that $\frac{1}{2} < \mu_1 c$, it is apparent that the initial error is forgotten faster than the error due to the noise. On a different note, from the bound in* (6)*, it is apparent that $E[(t_n - SR_\lambda(X))]$ scales linearly with the reciprocal of the monotonicity parameter $\mu_1$, since $c\mu_1$ is a constant.*

**Remark 2.** *In [13], the authors establish that $n^{1/2}(t_n - SR_\lambda(X))$ is asymptotically normal, say $N(0, \zeta^2)$ for a step-size choice that requires the knowledge of $g'(t*)$. Under mild regularity conditions (cf. [16]), the asymptotic normality result implies $nE(t_n - SR_\lambda(X))^2$ converges to a constant that depends on $\zeta^2$. The result we derived in Theorem 1 holds for all $n$, and matches the $O(1/n)$ bound from the asymptotic normality result of [13].*

Next, we present a high probability bound for the SR estimation algorithm in (5), under the following additional assumptions:

**(A4).** *The utility function $l$ is Lipschitz with constant $L_1$.*

**(A5).** *The r.v. $X$ is $\nu^2$-sub-Gaussian, i.e., $E\left[\exp\left(\frac{X^2}{2\nu^2}\right)\right] \leq 2$.*

The sub-Gaussianity condition above is equivalent to the following tail bound [34]:

$$\mathbb{P}\left(|X| > \epsilon\right) \leq 2\exp\left(\frac{\epsilon^2}{2\nu^2}\right), \text{ for any } \epsilon > 0.$$

**Theorem 2.** *Assume (A1)-(A5). Setting the step size $a_k = \frac{c}{k}$ with $\frac{1}{2} < \mu_1 c < 1$ and $cL_1^2 < \mu_1$. Then, we have the following bound for any $\delta \in (0, 1)$:*

$$\mathbb{P}\left(|t_n - SR_\lambda(X)| \leq \sqrt{\frac{\log(1/\delta)}{C_1 n}} + \frac{E[|t_0 - SR_\lambda(X)|]}{n^{\mu_1 c}} + \frac{c\sigma 2^{2\mu_1 c}}{\sqrt{(2\mu_1 c - 1)n}}\right) \geq 1 - \delta, \quad (7)$$

*where $C_1 = \frac{(2\mu_1 c - 1)}{2^{4\mu_1 c + 6} c^2 L_1^2 \nu^2}$.*

*Proof.* See Appendix B.2. □

The two results presented above required the knowledge of the monotonicity parameter $\mu_1$, which is typically unknown in a risk-sensitive learning setting. We now present a bound on the UBSR estimation error under a universal stepsize, i.e., one which does not require the knowledge of $\mu_1$. For this requirement, we require the following additional assumption that bounds the rate of growth of the utility function:

**(A6).** *There exists a $\mathcal{B} > 0$ such that $|g'(t)| \leq \mathcal{B}$, for all $t \in [t_l, t_u]$.*

**Theorem 3.** *Assume (A1)-(A6). Choose an $n_0$ such that $\mu_1 a_{n_0} < 1$. Then, we have the following bounds for two different step sizes:*

***Case I:*** *Set $a_k = \frac{c}{k}$. Then, for any $n \geq n_0$,*

$$E[(t_n - SR_\lambda(X))^2] \leq C(n_0)\left(E[(t_0 - SR_\lambda(X))^2] + \sigma^2\frac{\pi^2}{6}\right)\frac{1}{n^{2\mu_1 c}} + \mathcal{K}_1(n),$$

*where $C(n_0) = (1 + c^2\mathcal{B}^2)^{n_0}(n_0 + 1)^{2\mu_1 c}$, and* $\mathcal{K}_1(n) = \begin{cases} O\left(1/n^{2\mu_1 c}\right) & \textit{if } \mu_1 c < 1/2, \\ O\left(\log n/n\right) & \textit{if } \mu_1 c = 1/2, \textit{ and} \\ O\left(1/n\right) & \textit{if } \mu_1 c > 1/2. \end{cases}$

***Case II:*** *Set $a_k = \frac{c}{k^\alpha}$ for some $\alpha \in (0, 1)$. Then, for any $n \geq n_0$,*

$$E[(t_n - SR_\lambda(X))^2] \leq C(n_0)\left(E[(t_0 - SR_\lambda(X))^2] + \sigma^2 c^2 n_0\right)\exp\left(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha}\right) + \frac{2\sigma^2 c^2 (2\mu_1 c)^{\frac{\alpha}{1-\alpha}}}{(1-\alpha)n^\alpha}.$$

*Proof.* The proof proceeds by dividing the analysis into two parts about $n_0$. For a detailed proof, see Appendix B.3. □

A few remarks are in order.

**Remark 3.** *For Case I, the estimation error can decay as $1/n$ if $c$ is chosen such that $2\mu_1 c > 1$. However, if $\mu_1$ is not known, such a choice may not be feasible. Indeed, the error can decay much slower if $c$ is such that $2\mu_1 c$ is much smaller than $1$. For Case II above, the estimation error decays as $1/n^\alpha$ where $\alpha$ can be chosen arbitrarily close to 1 when deciding the step size, and this choice does not depend on $\mu_1$. However, as $\alpha$ approaches 1, the first term grows in an unbounded manner. An advantage with the larger stepsize $c/k^\alpha$ in Case II is that the initial error is forgotten exponentially fast, the corresponding rate is $1/n^{\mu_1 c}$ for the stepsize $c/k$.*

**Remark 4.** *The step size in Case II above is typically used in conjunction with* iterate averaging *[25, 32]. We can also use iterate averaging in this setting, but we can show that it does not improve the error decay rate derived for Case II without employing iterate averaging. From a practical perspective, outputting the 'last iterate' is often preferable over iterate averaging, especially when the latter does not improve the convergence rate appreciably.*

**Remark 5.** *The authors in [13] analyze a iterate-averaged variant of the SR estimation algorithm* (5)*, while assuming the knowledge of $g'(SR_\lambda(X))$ for setting the step-size constant $c$. The rate they derive under this assumption is $O(1/n)$ asymptotically. In comparison, our analysis is for a universal step-size, and we obtain a non-asymptotic bound of $O(1/n^\alpha)$, for $\alpha \in (0,1)$. In practice, the knowledge of $g'(SR_\lambda(X))$ is seldom available, motivating the universal step-size choice. The rate we derive in this case is comparable to the one obtained in [14] for general stochastic approximation schemes.*

The final result on UBSR estimation is a high probability bound for a universal stepsize choice.

**Theorem 4.** *Assume (A1)-(A5). Set the step size $a_k = \frac{c}{k^\alpha}$ with $\alpha \in (0,1)$, and choose an $n_0$ such that $L_1^2 a_{n_0} < \mu_1$. Then, we have the following bound for any $\delta \in (0,1)$, and for any $n \geq n_0$:*

$$\mathbb{P}\left( |t_n - SR_\lambda(X)| \leq C_2 \exp\left(-\frac{\mu_1 c n^{1-\alpha}}{2(1-\alpha)}\right) + \frac{C_3}{n^{\alpha/2}} \right) \geq 1 - \delta, \tag{8}$$

*where $C_2 = 8L_1\nu\sqrt{\frac{\log(1/\delta)(1+c^2L_1^2)^{n_0+1}c^2)}{c^2L_1^2}} + \sqrt{C(n_0)\left(E[(t_0 - SR_\lambda(X))^2] + \sigma^2c^2n_0\right)}$, and $C_3 = \left(8L_1\nu\sqrt{\frac{\log(1/\delta)2(\mu_1 c)^{\frac{\alpha}{1-\alpha}}c^2}{(1-\alpha)}} + \sqrt{\frac{\sigma^2 2(2\mu_1 c)^{\frac{\alpha}{1-\alpha}}c^2}{(1-\alpha)}}\right)$. In the above, $\mu_1, \sigma^2, L_1$ and $\nu$ are specified in (A2), (A3), (A4), and (A5), respectively, while the constant $C(n_0)$ is as defined in Theorem 3.*

*Proof.* See Appendix B.4. □

In the result above, we have chosen the stepsize to be $c/k^\alpha$ as choosing $c/k$ does not guarantee a $O(1/n)$ rate (see Remark 3).

# 4 UBSR Optimization

Recall that the UBSR optimization problem:

$$\text{Find } \theta^* \in \arg\min_{\theta\in\Theta} SR_\lambda(X(\theta)). \tag{9}$$

In this section, we devise a stochastic gradient algorithm that aims to solve the problem (9) using a gradient descent scheme with the following update iteration:

$$\theta_{n+1} = \theta_n - a_n h'_n(\theta_n), \tag{10}$$

where $a_n$ is a step-size parameter that satisfies standard stochastic approximation conditions, and $h'_n(\theta_n)$ is an estimate of $\frac{dSR_\lambda(\theta)}{d\theta}$.

We operate in a risk-sensitive learning framework, i.e., we do not have direct access to UBSR $SR_\lambda(\theta)$ and its derivative $\frac{dSR_\lambda(\theta)}{d\theta}$, for any $\theta$. Instead, we can obtain samples of the underlying r.v. corresponding to any parameter $\theta$, and use these samples to form the estimate $h'_n(\cdot)$. In the section below, we describe the derivative estimation scheme, and subsequently present non-asymptotic bounds for the iterate governed by (10).

## 4.1 Estimation of UBSR derivative

We begin by presenting the expression for the derivative of $SR_\lambda(X(\theta))$ w.r.t. $\theta$, derived in [18] : Letting $\xi = -X$,

$$\frac{dSR_\lambda(\theta)}{d\theta} = \frac{A(\theta)}{B(\theta)}, \tag{11}$$

where $A(\theta) \triangleq E[l'(\xi(\theta) - SR_\lambda(\theta)))\xi'(\theta)]$, and $B(\theta) \triangleq E[l'(\xi(\theta) - SR_\lambda(\theta))]$. The expression above is derived by first interchanging the differentiation and integration operators in $\frac{dSR_\lambda(\theta)}{d\theta}$, and then invoking the implicit function theorem. The assumptions justifying these steps are given below.

We now present a scheme for estimating the UBSR derivative $\frac{dSR_\lambda(\theta)}{d\theta}$, for a given $\theta$. Suppose we are given samples $\{\xi_1, \ldots, \xi_m\}$ from the distribution of $-X(\theta)$ for a given parameter $\theta$. Using these samples, we form a biased estimator $h'_m(\theta)$ of UBSR derivative as follows:

$$h'_m(\theta) = \frac{A_m}{B_m}, \text{ where } A_m(\theta) = \frac{1}{m}\sum_{i=1}^{m} l'(\xi_i(\theta) - t_m(\theta))\xi'_i(\theta), B_m(\theta) = \frac{1}{m}\sum_{i=1}^{n} l'(\xi_i(\theta) - t_m(\theta)), \tag{12}$$

and $t_m(\theta)$ is estimate of $SR_\lambda(\theta)$, which is obtained by running (5) for $m$ iterations. Notice that the estimate defined above is a ratio of estimates for the quantities $A(\theta)$ and $B(\theta)$, which are used in the expression (11) for $\frac{dSR_\lambda(\theta)}{d\theta}$. Notice that $A_m(\theta)$ and $B_m(\theta)$ are not unbiased estimates of $A(\theta)$ and $B(\theta)$, since the UBSR estimate $t_m(\theta)$ is biased. Hence, it is apparent that $h'_m(\theta)$ is a biased estimate of the UBSR derivative. An interesting question is if the estimate $h'_m(\theta)$ is consistent, and we answer this in the affirmative in Lemma 5.

**Assumptions.** We make the following assumptions for analyzing the consistency property of the UBSR derivative estimate (12). Recall that $\xi = -X$.

**(A7).** $\sup_{\theta\in\Theta} E(\xi(\theta)^2) \leq M_1$.

**(A8).** *(A1) and (A2) hold for every* $\theta \in \Theta$.

**(A9).** *The partial derivatives* $\partial l(\xi(\theta - t(\theta))))/\partial\theta, \partial l(\xi(\theta) - t(\theta))/\partial t$ *exist, and there exists* $\beta_1, \beta_2 > 0$ *such that*

$$E\left[(l'(\xi(\theta) - SR_\lambda(\theta))\xi'(\theta))^2\right] \leq \beta_1 < \infty, \textit{ and } E\left[(l'(\xi(\theta) - SR_\lambda(\theta)))^2\right] \leq \beta_2 < \infty, \forall\theta \in \Theta.$$

**(A10).** *The utility function $l(\cdot)$ satisfies*

$$|l'(\xi(\theta)-t)| \leq L_1, |l''(\xi(\theta)-t)| \leq L_2, \textit{ for all } (\theta,t) \in \Theta \times [t_l, t_u].$$

**(A11).** *The loss function $l(\cdot)$ is twice differentiable, and for any $\theta \in \Theta$, $l'(\xi(\theta) - SR_\lambda(\theta)) > \eta$.*

**(A12).** $\sup_{\theta \in \Theta} |\xi'(\theta)| \leq M_2$*, and $\xi'$ is $L_3$-Lipschitz for all $\theta \in \Theta$.*

We now discuss the motivation behind the assumptions listed above. First, a higher moment bound is usually necessary for ensuring asymptotic consistency of a sample-based estimate, and the bounded second moment requirement in (A7) encompasses a large class of unbounded r.v.s, while ensuring an $O(\frac{1}{\sqrt{m}})$ bound on the estimation error of $h'_m(\cdot)$ even in the non-asymptotic regime, i.e., for all $m \geq 1$. Assumption (A8) ensures that the scheme in (5) can be invoked to form the UBSR estimate $t_m$ in the derivative estimate (12). The second moment bounds in Assumption (A9) are necessary for obtaining a convergence rate result for the estimator (12), and a similar assumption has been made in [18] in the context of an asymptotic normality result. The Lipschitz conditions in (A10) are necessary for the interchange of expectation and differentiation operators in arriving at the expression (11) for UBSR derivative, see also [18]. From the condition in (A11) and the definition of $B_m$, it is apparent that $B_m(\theta) > \eta$. Finally, the conditions in (A11) and(A12) in conjunction with (A10) ensure that the function $l'(\xi(\theta) - SR_\lambda(\theta)))\xi'(\theta)$ is Lipschitz, and this in turn enables the derivation of a convergence rate result for the estimate 12.

We now present a rate result for the UBSR derivative estimate (12).

**Lemma 5.** *Assume (A7)–(A12). Then, for all $m \geq 1$, the UBSR derivative estimator* (12) *satisfies*

$$E\left|h'_m(\theta) - \frac{dSR_\lambda(\theta)}{d\theta}\right| \leq \frac{C_4}{\sqrt{m}}, \quad \textit{and} \quad E\left(h'_m(\theta) - \frac{dSR_\lambda(\theta)}{d\theta}\right)^2 \leq C_5,$$

*where $C_4 = \frac{\sqrt{\beta_2}(L_1 L_3 + M_2 L_2)\varsigma M_1 + \sqrt{\beta_1} L_2 \varsigma M_1}{\mu_1 \eta}$, and $C_5 = \frac{2\beta_2(\theta)\beta_1 + 2\beta_1(\theta)\beta_2}{\mu^2\eta^2}$. Here the constants $\beta_1, \beta_2, L_1, L_2, L_3, M_1, M_2$ are as specified in assumptions (A7)–(A12) above.*

*Proof.* The proof uses a connection between empirical and true mean of a r.v. to the 1-Wasserstein distance between empirical and true distribution functions. Specifically, for a given $t \in [t_l, t_u]$, define

$$f_m(t) = \frac{1}{m}\sum_{i=1}^{m} l'\left(\xi_i(\theta) - t\right), \text{ and } f(t) = E[l'(\xi(\theta) - t].$$

Let $F$ denote the cumulative distribution function of $\xi$, and $F_n$ denote the empirical distribution function, i.e., $F_n(x) = \frac{1}{m}\sum_{i=1}^{m} \mathbb{I}\left\{\xi - t \leq x\right\}$, for all $x \in \mathbb{R}$ . Then, we have

$$f_m(t) = \int l' dF_m, \text{ and } f(t) = \int l' dF.$$

Using the fact that $l'$ is $L_2$ Lipschitz from (A10), we obtain

$$|f_m(t) - f(t)| \leq L_2 W_1(F_n, F), \tag{13}$$

where $W_1(F_1, F_2) = \sup |E(f(X) - E(f(Y))|$, where the sup is over 1-Lipschitz functions.

Applying Theorem 3.1 of [21] with $p = 1, q = 2, d = 1$ there and using (A7), we obtain

$$EW_1(F_n, F) \leq \frac{\varsigma M_1}{\sqrt{m}}, \text{ leading to } E\left|f_m(t) - f(t)\right| \leq \frac{L_2 \varsigma M_1}{\sqrt{m}}.$$

In the above, $\varsigma$ is a universal constant.

Along similar lines, we can infer

$$E\left|\tilde{f}_m(t) - \tilde{f}(t)\right| \leq \frac{(L_1 L_3 + M_2 L_2)\varsigma M_1}{\sqrt{m}}, \tag{14}$$

where $\tilde{f}_m(t) = \frac{1}{m}\sum_{i=1}^{m} l'\left(\xi_i(\theta) - t\right)\xi'(\theta)$, and $\tilde{f}(t) = E[l'(\xi(\theta) - t)\xi'(\theta)]$.

$$\begin{aligned}
&E\left|h_m'(\theta) - \frac{dSR_\lambda(\theta)}{d\theta}\right| = E\left|\frac{A_m(\theta)}{B_m(\theta)} - \frac{A(\theta)}{B(\theta)}\right| \\
&\leq \frac{|B(\theta)|E[|A_m(\theta) - A(\theta)|] + |A(\theta)|E[|B_m(\theta) - B(\theta)|]}{\mu_1 \eta} \\
&\leq \frac{|B(\theta)|\sup_{t\in[t_l,t_u]} E|\tilde{f}(t_m) - \tilde{f}(t)|] + |A(\theta)|\sup_{t\in[t_l,t_u]} E[|f(t_m) - f(t)|]}{\mu_1 \eta} \\
&\leq \frac{\sqrt{\beta_2}(L_1 L_3 + M_2 L_2)\varsigma M_1 + \sqrt{\beta_1} L_2 \varsigma M_1}{\mu_1 \eta \sqrt{m}},
\end{aligned}$$

where the final inequality used (A9), (A10), (A7) and (A12). This proves the first claim. For a proof of the second claim, the reader is referred to Appendix C.1. □

Under assumptions similar to those listed above, the authors in [18] establish an asymptotic consistency as well as normality results. In contrast, we establish a result in the non-asymptotic regime, with a $O(\frac{1}{\sqrt{m}})$ that matches the aforementioned asymptotic rate.

## 4.2 Non-asymptotic bounds for UBSR optimization

In addition to the assumptions used for analyzing the convergence rate of UBSR derivative estimate (12), we require the following assumption for the non-asymptotic analysis of the stochastic gradient algorithm (10) for UBSR optimization:

**(A13).** *For any $\theta \in \Theta$, the function $h(\theta) = SR_\lambda(\theta)$ satisfies $h''(\theta) > \mu_2$, for some $\mu_2 > 0$.*

The assumption above implies that the UBSR objective $SR_\lambda(\theta)$ is a strongly convex function.

Using the results from Lemma 5 in conjunction with (A13), we present a bound on the error $E[\|\theta_n - \theta^*\|^2]$ in the optimization parameter in the theorem below.

**Theorem 6.** *Assume (A7)-(A13). Let $\theta^*$ denote the minimum of $SR_\lambda(\cdot)$. Set $a_k = c/k$ in* (10)*, with $\mu_2 c > \frac{1}{2}$. Let $m_n$ denote the batch size used for computing the estimate* (12) *corresponding to the parameter $\theta_n$. Then, for all $n \geq 1$, we have*

$$E[\|\theta_n - \theta^*\|^2] \leq \frac{3\|\theta_0 - \theta^*\|^2}{n^{2\mu_2 c}} + \frac{C_6}{n} + \frac{C_7}{m_n}, \tag{15}$$

*where $C_6 = \frac{3C_5 2^{2\mu_2 c} c^2}{(2\mu_2 c - 1)}$, and $C_7 = \frac{3C_4^2 c^2 2^{4\mu_2 c}}{(\mu_2 c)^2}$, with $C_4$ and $C_5$ as defined in Lemma 5.*

*Proof.* See Appendix C.2. □

The first term in (15) represents the initial error, and it is forgotten at a rate faster than $O(1/n)$ since $\mu_2 c > 1/2$. The overall rate for the algorithm would depend on the choice of the batch size $m_n$, and it is apparent that the error $E[\|\theta_n - \theta^*\|^2]$ does not vanish with a constant batch size. As in the case of Theorem 1, we observe that the error $E[\|\theta_n - \theta^*\|]$ has an inverse dependence on the strong convexity parameter $\mu_2$.

We now present a straightforward corollary of the result in Theorem 6 with an increasing batch size that ensures the error in the parameter vanishes asymptotically.

**Corollary 1.** *Under conditions of Theorem 6, with $m_n = n^\rho$ for some $\rho \in (0, 1]$, we have*

$$E[\|\theta_n - \theta^*\|^2] \le \frac{3\|\theta_0 - \theta^*\|^2}{n^{2\mu_2 c}} + \frac{C_6}{n} + \frac{C_7}{n^\rho} = O\left(\frac{1}{n^\rho}\right),$$

A few remarks are in order.

**Remark 6.** *From the result in the corollary above, it is easy to see that the optimal choice of batch size is $m_n = \Theta(n)$, and this in turn ensures an $O\left(\frac{1}{n}\right)$ rate of convergence for the stochastic gradient algorithm 10. With a biased derivative estimation scheme in a slightly different context, the authors in [3] show that an increasing batch size is necessary for the error of gradient descent type algorithm to vanish. Finally, the $O(1/n)$ bound in Theorem 6, which is for a setting where gradient estimates are biased, matches the minimax complexity result for strongly convex optimization with a stochastic first order oracle, cf. [1].*

**Remark 7.** *In the result above, we have bounded the error $E[\|\theta_n - \theta^*\|^2]$ in the optimization parameter. Using (A13) and $m_n = \Theta(n)$, we can also bound the optimization error $E[SR_\lambda(\theta_n)] - SR_\lambda(\theta^*)]$ using Corollary 1 as follows:*

$$E[SR_\lambda(\theta_n)] - SR_\lambda(\theta^*) \le \frac{1}{\mu_2} E[\|\theta_n - \theta^*\|^2] = O\left(\frac{1}{n}\right).$$

**Remark 8.** *To understand the deviation from the non-asymptotic analysis of a regular stochastic gradient algorithm (cf. [23]), we provide a brief sketch of the proof of Theorem 6. Letting $M_k = \int_0^1 [h''(m\theta_k + (1-m)\theta^*)]dm$, and $z_n = \theta_n - \theta^*$, we have*

$$z_{n+1} = z_n(1 - a_n M_n) - a_n \varepsilon_n, \text{ where } \varepsilon_n = h'_m(\theta_n) - h'(\theta_n)).$$

*Unlike the setting of [23], the noise in derivative estimate $\varepsilon_n$ is biased, i.e., $E[\varepsilon_n] \neq 0$. Now, unrolling the recursion above and taking expectations, we obtain*

$$\begin{aligned} E[\|z_{n+1}\|^2] &\le 3E[\|z_1\|^2] \prod_{k=1}^{n} (1 - a_k M_k)^2 + 3E[\sum_{k=1}^{n} [a_k \varepsilon_k \prod_{j=k+1}^{n} (1 - a_j M_j)]^2 \\ &\le 3E[\|z_1\|^2] n^{-2\mu_2 c} + 3 \underbrace{\sum_{k=1}^{n} \frac{c^2}{k^2} E[\varepsilon_k^2] (\prod_{j=k+1}^{n} (1 - a_j M_j))^2}_{I} \\ &\quad + 3 \underbrace{\sum_{k \neq l}^{n} a_k a_l E[|\varepsilon_l|] E[|\varepsilon_k|] \prod_{j=k+1}^{n} (1 - a_j M_j) \prod_{i=l+1}^{n} (1 - a_i M_i)}_{II}, \end{aligned}$$

*where we used strong convexity to bound the first term above. Term (II) is extra when compared to the analysis in the unbiased case. The rest of proof uses the bounds obtained in Lemma 5 to bound terms (I) and (II) above.*

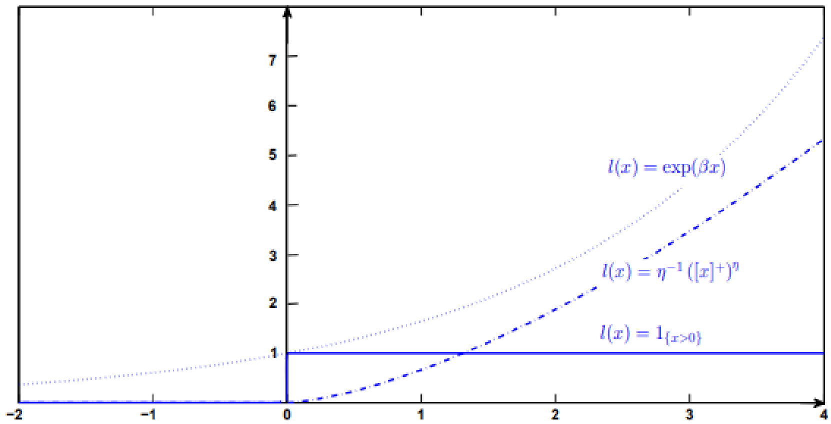
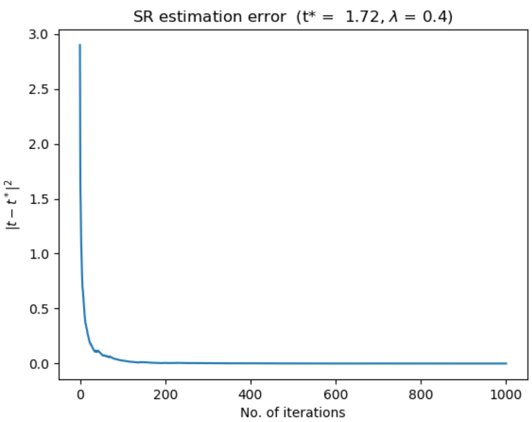


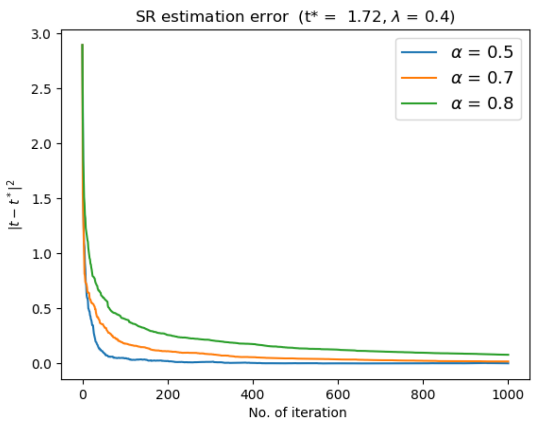

Figure 1: Loss Functions for VaR and SR

# 5 Experiments

Using simulations, the result in the above mentioned theorems for the bounds on estimation errors can be visually demonstrated. Loss function is chosen to be a piecewise polynomial function $l(x) = \eta^{-1}([x]^+)^\eta, \eta > 1$, since it follows the assumptions made regarding the loss function for a bounded domain. Modelling X as a gaussian random variable with $\mu = 0, \sigma^2 = 1$, for an acceptable risk level of $\lambda = 0.4$ and $\eta = 2$, we obtain the following trend for the squared estimation error:




Step size $a_k = 1/k$ and $t_0 = 0.02$




Step size $a_k = 0.1/k^\alpha$ and $t_0 = 0.02$

Figure 2: Squared estimation error for $t^*$

**Observations** :
Case I[$a_k = c/k$]: For high or low values of c [e.g. c = 10 and c = 0.02 ], the algorithm does not converge. This follows from the assumption $0.5 < \mu_1 c < 1$ made for Theorem (1). Figure 2 shows a decreasing trend depicting the convergence of the propose algorithm and supporting Theorem 1.

Case II [$a_k = c/k^\alpha$]: Unlike case I, this case works for any value of c. A lower value of c = 0.1 is chosen in order to have comparable graphs with same iterations, because case II has larger step sizes for the same value of c when compared to case I. Although Figure 2 shows that lower value of alpha leads to faster converges, this is simply a consequence of having larger step sizes for a simple model.

# 6 Concluding Remarks

We considered the problem of estimating Utility Based Shortfall Risk (UBSR) in an online setting, when samples from the underlying loss distribution are available one sample at a time. We cast the UBSR estimation problem as a stochastic approximation based root finding scheme. We derived non-asymptotic convergence guarantees for different step sizes, under a mild technical condition of a finite second moment. We also derived 'high probability' bounds for the concentration of the estimation error, when the loss distribution is sub-Gaussian.

Finally we considered the UBSR optimization problem, when the loss distribution belongs to a parametrized family. We proposed a stochastic gradient descent scheme, and derived non-asymptotic convergence guarantees under finite second moments. We faced the challenge of working with biased gradient estimates, which we addressed using batching. More broadly, the techniques developed in this work are applicable in a variety of settings, to characterize the finite sample performance of stochastic approximation and SGD algorithms.

As future work, it would be interesting to explore UBSR optimization in a risk-sensitive reinforcement learning setting. An orthogonal direction of future research is to extend the UBSR optimization algorithm to a vector parameter context, using a gradient estimation scheme based on finite differences, and the simultaneous perturbation method.

# A Appendix

# B Proofs for SR estimation

## B.1 Proof of Theorem 1

*Proof.* Move to $t_n$ and $t_0$ Starting with the definition of the variable $z_{n+1}$ and form of $t_{n+1}$ assumed in (A1),

$$
\begin{aligned}
z_n &= t_n - t^* \\
z_n &= \mathcal{T}(t_{n-1} + a_n\left(g(t_{n-1}) + \varepsilon_{n-1}\right)) - \mathcal{T}(t^*) \\
z_n &= \mathcal{T}(t_{n-1} + a_n\left(g(t_{n-1}) + \varepsilon_{n-1}\right)) - t^*.
\end{aligned} \tag{16}
$$

For any $k \geq 1$, define

$$
J_k = \int_0^1 g'(mt_k + (1-m)t^*)dy. \tag{17}
$$

Using (A2), we obtain $J_n \leq -\mu_1$. Using $J_n$ we can express $g(t_n)$ as,

$$
g(t_n) = \int_0^1 g'(mt_n + (1-m)t^*)dy(t_n - t^*) = J_n z_n.
$$

Squaring on both sides of (16), and using the fact that projection is non-expansive, we obtain

$$
\begin{aligned}
z_n^2 &\leq [z_{n-1} + a_n(g(t_{n-1}) + \varepsilon_{n-1})]^2 \\
&\leq [z_{n-1} + a_n(J_{n-1}z_{n-1} + \varepsilon_{n-1})]^2 \\
&\leq [z_{n-1}(1 + a_nJ_{n-1}) + a_n\varepsilon_{n-1}]^2 \\
&\leq z_{n-1}^2(1 + a_nJ_{n-1})^2 + a_n^2\varepsilon_{n-1}^2 + 2z_{n-1}(1 + a_nJ_{n-1})a_n\varepsilon_{n-1}
\end{aligned}
$$

Taking expectation $E[z_{n+1}|\mathcal{F}_n]$, where $\mathcal{F}_n$ is the sigma field generated by $t_k, k \leq n$, and using $E[\varepsilon_n] = 0$, we obtain

$$
\begin{aligned}
E[z_n^2] &\leq (1 + a_nJ_{n-1})^2E[z_{n-1}^2] + a_n^2E[\varepsilon_{n-1}^2] + 2 * z_{n-1}(1 + a_nJ_{n-1})a_nE[\varepsilon_{n-1}] \\
&\leq (1 + a_nJ_{n-1})^2E[z_{n-1}^2] + a_n^2E[\varepsilon_{n-1}^2].
\end{aligned}
$$

Using (A1) and (A2), we obtain

$$
\begin{aligned}
E[z_n^2] &\leq (1 + a_nJ_{n-1})^2E[z_{n-1}^2] + a_n^2\sigma^2 \\
&\leq E[z_0^2]\prod_{k=1}^{n}(1 + a_kJ_{k-1})^2 + \sigma^2\sum_{k=1}^{n}[a_k^2\prod_{j=k+1}^{n}(1 + a_jJ_{j-1})^2].
\end{aligned} \tag{18}
$$

Using $0.5 < \mu_1 c < 1$, we have $(1 + a_kJ_k)^2 \leq (1 - \mu_1 c)^2 \leq e^{-2\mu_1 c}$. Hence, we have

$$
E[z_n^2] \leq E[z_0^2]\prod_{k=1}^{n}(1 - a_k\mu_1)^2 + \sigma^2\sum_{k=1}^{n}[a_k^2\prod_{j=k+1}^{n}(1 - a_j\mu_1)^2]
$$

$$
\begin{aligned}
&\leq E[z_0^2]e^{-2\mu_1 \sum\limits_{k=1}^{n} a_k} + \sigma^2 \sum_{k=1}^{n}[a_k^2 e^{-2\mu_1 \sum\limits_{j=k+1}^{n} a_k}] \\
&\leq E[z_0^2]e^{-2\mu_1 c \log(n)} + \sigma^2 \sum_{k=1}^{n}[a_k^2 e^{-2\mu_1 c \log(\frac{n}{k+1})}] \\
&\leq \frac{E[z_0^2]}{n^{2\mu_1 c}} + \sigma^2 \sum_{k=1}^{n} a_k^2 (\frac{n}{k+1})^{-2\mu_1 c} \\
&\leq \frac{E[z_0^2]}{n^{2\mu_1 c}} + \sigma^2 n^{-2\mu_1 c} \sum_{k=1}^{n} \frac{c^2}{k^2}(k+1)^{2\mu_1 c} \\
&\leq \frac{E[z_0^2]}{n^{2\mu_1 c}} + \sigma^2 (\frac{2}{n})^{2\mu_1 c} \sum_{k=1}^{n} c^2 k^{2\mu_1 c - 2} \qquad (19) \\
&\leq \frac{E[z_0^2]}{n^{2\mu_1 c}} + \sigma^2 2^{4\mu_1 c} \frac{c^2}{(2\mu_1 c - 1)} \frac{1}{n}
\end{aligned}
$$

We have used the following inequality to upper bound the sum in (19):

$$
\frac{1}{n^{2\mu_1 c}} \sum_{k=1}^{n} k^{2\mu_1 c - 2} \leq \int\limits_{0}^{n+1} k^{2\mu_1 c - 2} dk \leq \frac{(n+1)^{2\mu_1 c - 1}}{n^{2\mu_1 c}(2\mu_1 c - 1)} \leq \frac{2^{2\mu_1 c}}{(2\mu_1 c - 1)} \frac{1}{n}. \qquad (20)
$$

Thus,

$$
E[z_n^2] \leq \frac{E[z_0^2]}{n^{2\mu_1 c}} + \sigma^2 2^{4\mu_1 c} \frac{c^2}{(2\mu_1 c - 1)} \frac{1}{n}.
$$

Hence proved. □

## B.2 Proof of Theorem 2

*Proof.* Move to $t_n$ and $t_0$ The centered form of the iterate $z_n = t_n - t*$, can be written as :

$$
|z_n| - E[|z_n|] = \sum_{k=1}^{n} g_k - g_{k-1} = \sum_{k=1}^{n} D_k
$$

where $g_k = E[|z_k||\mathcal{F}_k], D_k = g_k - g_{k-1}$ and $\mathcal{F}_k = \sigma(t_1, \dots, t_k)$.
Let $t_j^i(t)$ denote the iterate at time instant $j$, given that $t_i = t$. Using this notation, we have

$$
\begin{aligned}
&E[|t_{j+1}^i(t) - t_{j+1}^i(t')|^2] \leq E[|t_j^i(t) - t_j^i(t') + a_j(\hat{g}(t_{j+1}^i(t)) - \hat{g}(t_j^i(t'))|^2] \\
&\leq E[|t_j^i(t) - t_j^i(t')|^2] + 2a_j E[t_{j+1}^i(t) - t_{j+1}^i(t')]E[\hat{g}(t_{j+1}^i(t)) - \hat{g}(t_j^i(t'))] + a_j^2 E[|\hat{g}(t_{j+1}^i(t)) - \hat{g}(t_j^i(t'))|^2] \\
&\leq (1 - 2\mu_1 a_j + a_j^2 L_1^2) E[|t_j^i(t) - t_j^i(t')|^2].
\end{aligned}
$$

On unrolling the expression we obtain

$$
E[|t_n^i(t) - t_n^i(t')|^2] \leq |t - t'|^2 \prod_{j=1}^{n} (1 - 2\mu_1 a_j + a_j^2 L_1^2)
$$

Using the above inequality, we have

$$
\begin{aligned}
E[|t_n - t^*||t_i = t] - E[|t_n - t^*||t_i = t'] &\leq E[|t_n^i(t) - t_n^i(t')|] \\
&\leq |t - t'|(\prod_{j=1}^{n-1}(1 - 2\mu_1 a_j + a_j^2 L_1^2))^{1/2} \\
&\leq a_i|\hat{g} - \hat{g}'|(\prod_{j=1}^{n-1}(1 - 2\mu_1 a_j + a_j^2 L_1^2))^{1/2} \\
&\leq L_i|\hat{g} - \hat{g}'|
\end{aligned}
$$

where $L_i = a_i(\prod_{j=i}^{n-1}(1 - 2\mu_1 a_j + a_j^2 L_1^2))^{1/2}$, $t = t_{i-1} + a_i\hat{g}$, and $t' = t_{i-1} + a_i\hat{g}'$,

$$
\begin{aligned}
\mathbb{P}(|z_n| - E[|z_n|] > \varepsilon) &= P(\sum_{k=1}^{n} D_k > \varepsilon) \\
&\leq \exp(-\lambda\varepsilon)(E[\exp(\lambda\sum_{k=1}^{n} D_k)]) \\
&\leq \exp(-\lambda\varepsilon)E[\exp(\lambda\sum_{k=1}^{n-1} D_k)]E[\exp(\lambda D_n)|\mathcal{F}_{n-1}]. \qquad (21)
\end{aligned}
$$

It can be shown that an $L$-Lipschitz function $f$ of a $\nu^2$-sub-Gaussian r.v $Z$ is $4L^2\nu^2$-sub-Gaussian, i.e.,

$$
E[\exp(\lambda(f(Z))] \leq \exp\left(2\lambda^2 L^2 \nu^2\right).
$$

Using (A5), and the fact that $l$ is $L_1$ Lipschitz, we have $\hat{g}$ is $4L_1^2\nu^2$-sub-Gaussian. Next, $D_n$ is a $L_n$-Lipschitz function of $\hat{g}$, implying $D_n$ is $16L_n^2L_1^2\nu^2$-sub-Gaussian. Using the bound above in (21), we obtain

$$
E[\exp(\lambda D_n)|\mathcal{F}_{n-1}] \leq \exp\left(8\lambda^2 L_n^2 L_1^2 \nu^2\right).
$$

Plugging this bound into (21), followed by an optimization over $\lambda$, we obtain

$$
\mathbb{P}(|z_n| - E[|z_n|] > \varepsilon) \leq \exp(-\lambda\varepsilon)\exp(8\lambda^2 L_1^2\nu^2\sum_{k=1}^{n} L_k^2) \leq \exp\left(-\frac{\varepsilon^2}{64L_1^2\nu^2\sum\limits_{k=1}^{n} L_k^2}\right). \qquad (22)
$$

Computing $\sum\limits_{k=1}^{n} L_k^2$ gives the rate for the high probability bound. Substituting $a_k = c/k$ and using $0 < \mu_1 c < 1/2$, we obtain

$$
\begin{aligned}
\sum_{k=1}^{n} L_k^2 &= \sum_{k=1}^{n} a_k^2(\prod_{j=k}^{n-1}(1 - 2\mu_1 a_j + a_j^2 L_1^2)) \\
&\leq \sum_{k=1}^{n} a_k^2 \prod_{j=k}^{n}(1 - a_j(2\mu_1 - a_j L_1^2))
\end{aligned}
$$

$$\le \sum_{k=1}^{n} a_k^2 \exp(-\mu_1 \sum_{j=k}^{n} a_j)$$
$$\le \sum_{k=1}^{n} a_k^2 \exp\left(-\mu_1 c \log\left(\frac{n}{k+1}\right)\right)$$
$$\le \sum_{k=1}^{n} \frac{c^2}{k^2}\left(\frac{k+1}{n}\right)^{\mu_1 c}$$
$$\le \frac{2^{4\mu_1 c} c^2}{(2\mu_1 c - 1)} \frac{1}{n}.$$

Using the bound on $\sum_{k=1}^{n} L_k^2$ in (22), we obtain

$$\mathbb{P}(|z_n| - E[|z_n|] > \varepsilon) \le \exp\left(-\tilde{c} n \varepsilon^2\right), \tag{23}$$

where $\tilde{c} = \frac{(2\mu_1 c-1)}{2^{4\mu_1 c+6} c^2 L_1^2 \nu^2}$. Using the bound on $E[|z_n|$ from Theorem 1 in (33), we have

$$\mathbb{P}\left(|z_n| - E|z_n| \le \sqrt{\frac{\log(1/\delta)}{\tilde{c}n}} + \frac{E[|t_1 - t^*|]}{n^{\mu_1 c}} + \frac{c\sigma 2^{2\mu_1 c}}{\sqrt{(2\mu_1 c - 1)}\sqrt{n}}\right) \ge 1 - \delta, \tag{24}$$

□

## B.3 Proof of Theorem 3

*Proof.* Move to $t_n$ and $t_0$ The passage leading up to (18) holds for any choice of stepsize, and does not require $0.5 < \mu_1 c < 1$. Using (18) as the starting point, we have

$$E[z_n^2] \le E[z_0^2] \prod_{k=1}^{n} (1 - a_k |J_{k-1}|)^2 + \sigma^2 \sum_{k=1}^{n} [a_k^2 \prod_{j=k+1}^{n} (1 - a_j |J_{j-1}|)^2]. \tag{25}$$

We split the analysis into two regimes: $k < n_0$ and $k \ge n_0$. Using(A6), we have $|J_k| < \mathcal{B}$. We shall now simplify (25) under two different stepsize choices.

**Case I:** $a_k = \frac{c}{k}$

$$\prod_{k=1}^{n} (1 - a_k |J_{k-1}|)^2 = \prod_{k=1}^{n_0} (1 + a_k^2 |J_{k-1}|^2 - 2a_k J_{k-1}) \prod_{k=n_0+1}^{n} (1 - a_k |J_{k-1}|)^2$$
$$\le (1 + c^2 \mathcal{B}^2)^{n_0} e^{-2\mu_1 \sum_{n_0+1}^{n} a_k}$$
$$\le (1 + c^2 \mathcal{B}^2)^{n_0} e^{-2\mu_1 c \log(\frac{n}{k_0+1})}$$
$$\le (1 + c^2 \mathcal{B}^2)^{n_0} (\frac{n_0+1}{n})^{2\mu_1 c}$$
$$\le C(n_0) \frac{1}{n^{2\mu_1 c}}$$

Where $C(n_0) = (1 + c^2 \mathcal{B}^2)^{n_0} (n_0 + 1)^{2\mu_1 c}$
For the second term in the general expression, we have to divide the summation about $n_0$ to

obtain

$$\sum_{k=1}^{n}[a_k^2\prod_{j=k+1}^{n}(1-a_j|J_{j-1}|)^2]=\sum_{k=1}^{n_0-1}[a_k^2\prod_{j=k+1}^{n}(1-a_j|J_{j-1}|)^2]+\sum_{k=n_0}^{n}[a_k^2\prod_{j=k+1}^{n}(1-a_j|J_{j-1}|)^2]$$
$$\leq(1+c^2\mathcal{B}^2)^{n_0}(\frac{n_0+1}{n})^{2\mu_1 c}\sum_{k=1}^{n_0-1}a_k^2+\sum_{k=n_0}^{n}a_k^2(\frac{k+1}{n})^{2\mu_1 c}$$
$$\leq(1+c^2\mathcal{B}^2)^{n_0}(n_0+1)^{2\mu_1 c}\frac{\pi^2}{6}\frac{1}{n^{2\mu_1 c}}+\frac{c^2}{n^{2\mu_1 c}}\sum_{k=n_0}^{n}\frac{c^2}{k^2}(k+1)^{2\mu_1 c} \tag{26}$$

After observing that $\sum\limits_{k=1}^{n}a_k^2=\sum\limits_{k=1}^{n}\frac{c^2}{k^2}<c^2\frac{\pi^2}{6}$ to obtain (26), We now compute (26) for different cases of $\mu_1 c$ values to compute the rates,

**Case a:** $\mu_1 c>1/2$

Using the bound in (20), $\sum\limits_{k=n_0}^{n}\frac{c^2}{k^2}(\frac{k+1}{n})^{2\mu_1 c}\leq\frac{2^{4\mu_1 c}c^2}{(2\mu_1 c-1)}\frac{1}{n}$ and substituting in the general equation we have,

$$E[z_n^2]\leq C(n_0)\left(E[z_0^2]+\sigma^2\frac{\pi^2}{6}\right)\frac{1}{n^{2\mu_1 c}}+\frac{\sigma^2c^22^{4\mu_1 c}}{(2\mu_1 c-1)}\frac{1}{n} \tag{27}$$

**Case b:** $\mu_1 c=1/2$

For this condition the sum reduces to $\sum\limits_{k=n_0}^{n}\frac{c^2}{k^2}(\frac{k+1}{n})^{2\mu_1 c}\leq\frac{2}{n}\sum\limits_{k=n_0}^{n}\frac{c^2}{k}\leq 2c^2\frac{\log(n+1)}{n}$ and substituting in the general equation we have,

$$E[z_n^2]\leq C(n_0)\left(E[z_0^2]+\sigma^2\frac{\pi^2}{6}\right)\frac{1}{n^{2\mu_1 c}}+2\sigma^2c^2\frac{\log(n+1)}{n} \tag{28}$$

**Case c:** $\mu_1 c<1/2$

The sum in (26) reduces to $\frac{1}{n^{2\mu_1 c}}\sum\limits_{k=n_0}^{n}\frac{c^2}{k^2}(k+1)^{2\mu_1 c}\leq\frac{2^{2\mu_1 c}}{n^{2\mu_1 c}}\sum\limits_{k=n_0}^{n}\frac{c^2}{k^{(1+2(1/2-\mu_1 c))}}\leq\frac{2^{2\mu_1 c+1}c^2}{(1-2\mu_1 c)n^{2\mu_1 c}}$, the sum is bounded and substituting in the general equation we have,

$$E[z_n^2]\leq C(n_0)\left(E[z_0^2]+\sigma^2\frac{\pi^2}{6}\right)\frac{1}{n^{2\mu_1 c}}+\sigma^2\frac{2^{2\mu_1 c+1}c^2}{(1-2\mu_1 c)}\frac{1}{n^{2\mu_1 c}}. \tag{29}$$

We now turn to analyzing the case when the stepsize $a_k$ is larger than $c/k$.

**Case II:** $a_k=\frac{c}{k^\alpha}$ for $\alpha\in(0,1)$ First, we bound a factor in the first term of (25) as follows:

$$\prod_{k=1}^{n}(1-a_k|J_{k-1}|)^2=\prod_{k=1}^{n_0}(1+a_k^2|J_{k-1}|^2-2a_kJ_k)\prod_{k=n_0+1}^{n}(1-a_k|J_{k-1}|)^2$$
$$\leq(1+c^2\mathcal{B}^2)^{n_0}e^{-2\mu_1\sum\limits_{n_0+1}^{n}a_k}$$
$$\leq(1+c^2\mathcal{B}^2)^{n_0}\exp(-\frac{2\mu_1 c(n^{1-\alpha}-n_0^{1-\alpha})}{1-\alpha})$$
$$\leq(1+c^2\mathcal{B}^2)^{n_0}\exp(\frac{2\mu_1 cn_0^{1-\alpha}}{1-\alpha})\exp(-\frac{2\mu_1 cn^{1-\alpha}}{1-\alpha})$$

$$\leq C(n_0)\exp(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha}), \tag{30}$$

where $C(n_0) = (1+c^2\mathcal{B}^2)^{n_0}\exp(\frac{2\mu_1 c n_0^{1-\alpha}}{1-\alpha})$.

We now bound the second term in (25) by splitting the term around $n_0$ as follows:

$$\begin{aligned}
&\sum_{k=1}^{n}[a_k^2\prod_{j=k+1}^{n}(1-a_j|J_{j-1}|)^2] \\
&= \sum_{k=1}^{n_0-1}[a_k^2\prod_{j=k+1}^{n}(1-a_j|J_{j-1}|)^2] + \sum_{k=n_0}^{n}[a_k^2\prod_{j=k+1}^{n}(1-a_j|J_{j-1}|)^2] \\
&\leq C(n_0)\exp(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha})\sum_{k=1}^{n_0-1}a_k^2 + \sum_{k=n_0}^{n}a_k^2\exp(-\frac{2\mu_1 c(n^{1-\alpha}-k^{1-\alpha})}{1-\alpha}) \\
&\leq C(n_0)c^2 n_0\exp(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha}) + c^2\exp(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha})\sum_{k=n_0}^{n}k^{-2\alpha}\exp(\frac{2\mu_1 c k^{1-\alpha}}{1-\alpha}) \\
&\leq C(n_0)c^2 n_0\exp(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha}) + \frac{2(2\mu_1 c)^{\frac{\alpha}{1-\alpha}}c^2}{1-\alpha}\frac{1}{n^\alpha}.
\end{aligned} \tag{31}$$

The sum $c^2\exp(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha})\sum_{k=n_0}^{n}k^{-2\alpha}\exp(\frac{2\mu_1 c k^{1-\alpha}}{1-\alpha})$ is bounded using the proof in [29] eqn. (79), which uses Jensen's inequality and convexity of the function $f(x) = x^{-2\alpha}\exp(x^{1-\alpha})$.

Substituting the bounds in (30) and (31) in (25), we obtain

$$E[z_n^2] \leq C(n_0)\left(E[z_0^2]+\sigma^2 c^2 n_0\right)\exp(-\frac{2\mu_1 c n^{1-\alpha}}{1-\alpha}) + \frac{\sigma^2 2(2\mu_1 c)^{\frac{\alpha}{1-\alpha}}c^2}{(1-\alpha)n^\alpha}. \tag{32}$$

Hence proved. □

## B.4 Proof of Theorem 4

*Proof.* Move to $t_n$ and $t_0$ Recall that $n_0$ is chosen such that for all $n \geq n_0$, we have $\frac{c}{n^\alpha}L_1^2 < \mu_1$.

$$\begin{aligned}
\sum_{k=1}^{n_0-1}L_k^2 &= \sum_{k=1}^{n_0-1}a_k^2(\prod_{j=k}^{n-1}(1-2\mu_1 a_j + a_j^2 L_1^2)) \\
&= \sum_{k=1}^{n_0-1}a_k^2(\prod_{j=k}^{n_0-1}(1-2\mu_1 a_j + a_j^2 L_1^2))(\prod_{j=n_0}^{n}(1-2\mu_1 a_j + a_j^2 L_1^2)) \\
&\leq (1+c^2L_1^2)^{n_0}\sum_{k=1}^{n_0-1}a_k^2(1+c^2L_1^2)^{-k}\prod_{j=n_0}^{n}(1-a_j(2\mu_1 - a_j L_1^2)) \\
&\leq (1+c^2L_1^2)^{n_0}\sum_{k=1}^{n_0-1}a_k^2\exp(-\mu_1\sum_{j=n_0}^{n}a_j) \\
&\leq (1+c^2L_1^2)^{n_0}\exp\left(-\frac{\mu_1 c(n^{1-\alpha}-n_0^{1-\alpha})}{1-\alpha}\right)\sum_{k=1}^{n_0-1}a_k^2(1+c^2L_1^2)^{-k}
\end{aligned}$$

$$\leq \frac{(1+c^2L_1^2)^{n_0+1}c^2}{c^2L_1^2}\exp\left(-\frac{\mu_1 c(n^{1-\alpha}-n_0^{1-\alpha})}{1-\alpha}\right)$$

Computing the second part of the complete summation:

$$\begin{aligned}
\sum_{k=n_0}^{n} L_k^2 &= \sum_{k=n_0}^{n} a_k^2(\prod_{j=k}^{n-1}(1-2\mu_1 a_j + a_j^2 L_1^2)) \\
&\leq \sum_{k=n_0}^{n} a_k^2 \prod_{j=k}^{n}(1-a_j(2\mu_1 - a_j L_1^2)) \\
&\leq \sum_{k=n_0}^{n} a_k^2 \exp(-\mu_1 \sum_{j=k}^{n} a_j) \\
&\leq \sum_{k=n_0}^{n} a_k^2 \exp\left(-\frac{\mu_1 c(n^{1-\alpha}-k^{1-\alpha})}{1-\alpha}\right) \\
&\leq \exp\left(-\frac{\mu_1 c n^{1-\alpha}}{1-\alpha}\right)\sum_{k=n_0}^{n}\frac{c^2}{k^{2\alpha}}\exp\left(\frac{\mu_1 c k^{1-\alpha}}{1-\alpha}\right) \\
&\leq \frac{2(\mu_1 c)^{\frac{\alpha}{1-\alpha}}c^2}{1-\alpha}\frac{1}{n^\alpha}
\end{aligned}$$

Using the two summations we obtain the final summation given by,

$$\begin{aligned}
\sum_{k=1}^{n} L_k^2 &= \sum_{k=1}^{n_0-1} L_k^2 + \sum_{k=n_0}^{n} L_k^2 \\
&\leq \frac{(1+c^2L_1^2)^{n_0+1}c^2}{c^2L_1^2}\exp\left(-\frac{\mu_1 c(n^{1-\alpha}-n_0^{1-\alpha})}{1-\alpha}\right) + \frac{2(\mu_1 c)^{\frac{\alpha}{1-\alpha}}c^2}{1-\alpha}\frac{1}{n^\alpha}
\end{aligned}$$

Using the bound on $\sum_{k=1}^{n} L_k^2$ in (22), we obtain

$$\mathbb{P}(|z_n| - E[|z_n|] > \varepsilon) \leq \exp\left(-\tilde{c}n\varepsilon^2\right), \tag{33}$$

where $\tilde{c} = \frac{(1-\alpha)}{2(\mu_1 c)^{\frac{\alpha}{1-\alpha}}c^2}$. Using the bound on $E[|z_n|$ from Theorem 1 in (33), we have

$$\mathbb{P}\left(|t_n - t^*| \leq C_2\exp(-\frac{\mu_1 c n^{1-\alpha}}{2(1-\alpha)}) + C_3\frac{1}{n^{\alpha/2}}\right) \geq 1-\delta, \tag{34}$$

where $C_2$ and $C_3$ are as defined in the theorem statement. Hence proved. □

# C Proofs for SR optimization

## C.1 Proof of Lemma 5

*Proof.* We prove the second claim in the statement of the lemma, i.e., $E\left(h'_m(\theta) - \frac{dSR_\lambda(\theta)}{d\theta}\right)^2 \leq C_5$. Notice that

$$E[|h'_m(\theta) - \frac{dSR_\lambda(\theta)}{d\theta}|^2] = E[|\frac{A_m(\theta)}{B_m(\theta)} - \frac{A(\theta)}{B(\theta)}|^2]$$

$$
\begin{aligned}
&= E[|\frac{B(\theta)A_m(\theta) - A(\theta)B(\theta) + A(\theta)B(\theta) - A(\theta)B_m(\theta)}{B_m(\theta)B}|^2] \\
&= E[|\frac{B(\theta)(A_m(\theta) - A(\theta)) - A(\theta)(B_m(\theta) - B(\theta))}{B_m(\theta)B}|^2] \\
&\leq \frac{2B^2(\theta)E[|A_m(\theta) - A(\theta)|^2] + 2A^2(\theta)E[|B_m(\theta) - B(\theta)|^2]}{\mu_1^2\eta^2} \\
&\leq \frac{2B^2(\theta)E[A_m^2(\theta)] + 2A^2(\theta)E[B_m(\theta)^2]}{\mu_1^2\eta^2} \\
&\leq \frac{2\beta_2\beta_1 + 2\beta_1\beta_2}{\mu^2\eta^2} = C_5,
\end{aligned}
$$

where the final inequality used (A9).

□

## C.2 Proof of Theorem 6

*Proof.* Move to $z_n$ and $z_0$ Iteration over the variable $\theta$ are given by,

$$
\begin{aligned}
\theta_n &= \theta_{n-1} - a_n h'_m(\theta_{n-1}) \\
&= \theta_{n-1} - a_n \left(h'(\theta_{n-1}) + \varepsilon_{n-1}\right)
\end{aligned}
$$

where $\varepsilon_{n-1} = h'_m(\theta_{n-1}) - h'(\theta_{n-1}))$. Thus,

$$
\begin{aligned}
\theta_n - \theta^* &= \theta_{n-1} - \theta^* - a_n \left(h'(\theta_{n-1}) + \varepsilon_{n-1}\right) \\
z_n &= z_{n-1} - a_n \left(h'(\theta_{n-1}) + \varepsilon_{n-1}\right)
\end{aligned}
$$

Let $M_k = \int\limits_0^1 [h''(m\theta_k + (1-m)\theta^*)]dm$ . Letting $z_n = \theta_n - \theta^*$, we have

$$
\begin{aligned}
h'(\theta_n) &= \int_0^1 [h''(m\theta_k + (1-m)\theta^*)]dm(\theta_n - \theta^*) = M_n z_n, \text{ and} \\
z_n &= z_{n-1}(1 - a_n M_{n-1}) - a_n \varepsilon_{n-1}
\end{aligned}
$$

Unrolling the equation above, we obtain

$$
\begin{aligned}
z_n &= z_0 \prod_{k=1}^n (1 - a_k M_{k-1}) - \sum_{k=1}^n [a_k \varepsilon_{k-1} \prod_{j=k+1}^n (1 - a_j M_{j-1})] \\
E[\|z_n\|^2] &\leq 3E[\|z_0\|^2] \prod_{k=1}^n (1 - a_k M_{k-1})^2 + 3E[\sum_{k=1}^n [a_k \varepsilon_{k-1} \prod_{j=k+1}^n (1 - a_j M_{j-1})]^2 \\
&\leq 3E[\|z_0\|^2] \prod_{k=1}^n (1 - a_k M_{k-1})^2 + 3E[(\sum_{k=1}^n [a_k \varepsilon_{k-1} \prod_{j=k+1}^n (1 - a_j M_{j-1}))^2] \\
&\leq 3E[\|z_0\|^2](\mathcal{P}_{1:n})^2 + 3E[(\sum_{k=1}^n a_k \varepsilon_{k-1} \mathcal{P}_{k+1:n})^2]
\end{aligned}
$$

$$\leq 3E[\|z_0\|^2]n^{-2\mu_2 c} + 3E[(\sum_{l=1}^{n}\sum_{k=1}^{n}[a_k a_l \varepsilon_{l-1}\varepsilon_{k-1}\mathcal{P}_{k+1:n}\mathcal{P}_{l+1:n})]$$

$$\leq 3E[\|z_0\|^2]n^{-2\mu_2 c} + 3E[\sum_{k=1}^{n} a_k^2 \varepsilon_{k-1}^2 (\mathcal{P}_{k+1:n})^2 + \sum_{k\neq l}^{n} a_k a_l \varepsilon_{l-1}\varepsilon_{k-1}\mathcal{P}_{k+1:n}\mathcal{P}_{l+1:n}]$$

$$\leq 3E[\|z_0\|^2]n^{-2\mu_2 c} + 3\underbrace{\sum_{k=1}^{n}\frac{c^2}{k^2}E[\varepsilon_{k-1}^2](\mathcal{P}_{k+1:n})^2}_{\text{I}} + 3\underbrace{\sum_{k\neq l}^{n} a_k a_l E[|\varepsilon_{l-1}|]E[|\varepsilon_{k-1}|]\mathcal{P}_{k+1:n}\mathcal{P}_{l+1:n}}_{\text{II}}$$

where $\mathcal{P}_{i:j} = \prod_{k=i}^{j}(1 - a_k M_{k-1})^2$

Solving each expression separately using lemma 5, we obtain Fix the hard-coded refs here

$$\begin{aligned}
I &= \sum_{k=1}^{n}\frac{c^2}{k^2}E[\varepsilon_{k-1}^2](\mathcal{P}_{k+1:n})^2 \\
&\leq C_5 \sum_{k=1}^{n}\frac{c^2}{k^2}(\frac{k+1}{n})^{2\mu_2 c} \\
&\leq \frac{C_5 2^{2\mu_2 c}c^2}{(2\mu_2 c - 1)}\frac{1}{n} \\
II &= \sum_{k\neq l}^{n} a_k a_l E[|\varepsilon_{l-1}|]E[|\varepsilon_{k-1}|]\mathcal{P}_{k+1:n}\mathcal{P}_{l+1:n} \\
&\leq \left(\frac{C_4}{\sqrt{m}}\right)^2 \sum_{k\neq l}^{n} a_k a_l \mathcal{P}_{k+1:n}\mathcal{P}_{l+1:n} \\
&\leq \frac{C_4^2}{m}\sum_{k>l}\frac{c^2}{kl}\prod_{j=k+1}^{n}(1 - a_j M_{j-1})\prod_{j=l+1}^{n}(1 - a_j M_{j-1}) \\
&\leq \frac{C_4^2}{m}\sum_{k>l}\frac{c^2}{kl}\prod_{j=k+1}^{n}(1 - a_j M_{j-1})\prod_{j=l+1}^{n}(1 - a_j M_{j-1}) \\
&\leq \frac{C_4^2}{m}\sum_{l=1}^{n}\sum_{k=l+1}^{n}\frac{c^2}{kl}(\frac{l+1}{n})^{\mu_2 c}(\frac{k+1}{n})^{\mu_2 c} \\
&\leq \frac{C_4^2}{m}\sum_{l=1}^{n}\frac{c^2}{l}(\frac{l+1}{n})^{\mu_2 c}\sum_{k=l+1}^{n}\frac{1}{k}(\frac{k+1}{n})^{\mu_2 c} \\
&\leq \frac{C_4^2}{m}\frac{2^{2\mu_2 c}}{n^{2\mu_2 c}}\sum_{l=1}^{n}c^2 l^{\mu_2 c - 1}\sum_{k=l+1}^{n}k^{\mu_2 c - 1} \\
&\leq \frac{C_4^2}{m}\frac{2^{2\mu_2 c}}{n^{2\mu_2 c}}\sum_{l=1}^{n}c^2 l^{\mu_2 c - 1}\frac{(n+1)^{\mu_2 c} - (l+1)^{\mu_2 c}}{\mu_2 c} \\
&\leq \frac{C_4^2}{m}\frac{2^{3\mu_2 c}}{\mu_2 c}\frac{c^2}{n^{\mu_2 c}}\sum_{l=1}^{n}l^{\mu_2 c - 1} \\
&\leq \frac{C_4^2 c^2}{m}\frac{2^{3\mu_2 c}}{\mu_2 c}\frac{1}{n^{\mu_2 c}}\frac{(n+1)^{\mu_2 c}}{\mu_2 c}
\end{aligned}$$

$$\leq \frac{C_4^2 c^2}{m} \frac{2^{4\mu_2 c}}{(\mu_2 c)^2}.$$

$\square$

# References


1. A. Agarwal, M. J. Wainwright, P. Bartlett, and P. Ravikumar. Information-theoretic lower bounds on the oracle complexity of convex optimization. In Y. Bengio, D. Schuurmans, J. Lafferty, C. Williams, and A. Culotta, editors, *Advances in Neural Information Processing Systems*, volume 22. Curran Associates, Inc., 2009.

2. P. Artzner, F. Delbaen, J. Eber, and D. Heath. Coherent measures of risk. *Mathematical finance*, 9(3):203–228, 1999.

3. Y. F. Atchade, G. Fort, and E. Moulines. On stochastic proximal gradient algorithms. *arXiv preprint arXiv:1402.2365*, 23, 2014.

4. O. Bardou, N. Frikha, and G. Pagès. Computing VaR and CVaR using stochastic approximation and adaptive unconstrained importance sampling. *Monte Carlo Methods and Applications*, 15(3):173–210, 2009.

5. Bernard Bercu, Manon Costa, and Sébastien Gadat. Stochastic approximation algorithms for superquantiles estimation. *arXiv preprint arXiv:2007.14659*, 2020.

6. Sanjay P Bhat and L. A. Prashanth. Concentration of risk measures: A Wasserstein distance approach. *Advances in Neural Information Processing Systems*, 32:11762–11771, 2019.

7. V. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.

8. L. Bottou, F. E Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. *Siam Review*, 60(2):223–311, 2018.

9. D. B. Brown. Large deviations bounds for estimating conditional value-at-risk. *Operations Research Letters*, 35(6):722–730, 2007.

10. H. Cardot, P. Cénac, and P. Zitt. Efficient and fast estimation of the geometric median in Hilbert spaces with an averaged stochastic gradient algorithm, 2011.

11. H. Cardot, P. Cénac, and A. Godichon. Online estimation of the geometric median in Hilbert spaces : non asymptotic confidence balls, 2015.

12. A. Cassel, S. Mannor, and A. Zeevi. A general approach to multi-armed bandits under risk criteria. In *Proceedings of the 31st Conference On Learning Theory*, pages 1295–1306, 2018.

13. J. Dunkel and S. Weber. Stochastic root finding and efficient estimation of convex risk measures. *Operations Research*, 58(5):1505–1521, 2010.

14. M. Fathi and N. Frikha. Transport-entropy inequalities and deviation estimates for stochastic approximation schemes. *Electronic Journal of Probability*, 18, 2013.

15. H. Föllmer and A. Schied. Convex measures of risk and trading constraints. *Finance and stochastics*, 6(4):429–447, 2002.

**16**. L. Gerencsér. Convergence rate of moments in stochastic approximation with simultaneous perturbation gradient approximation and resetting. *IEEE Trans. Autom. Contr.*, 44(5): 894–905, 1999.

**17**. A. Godichon-Baggioni. Estimating the geometric median in hilbert spaces with stochastic gradient algorithms: Lp and almost sure rates of convergence. *Journal of Multivariate Analysis*, 146:209–222, 2016.

**18**. Z. Hu and Z. Dali. Convex risk measures: efficient computations via monte carlo. *Available at SSRN 2758713*, 2016.

**19**. A. Kagrecha, J. Nair, and K. Jagannathan. Distribution oblivious, risk-aware algorithms for multi-armed bandits with unbounded rewards. In *Advances in Neural Information Processing Systems*, pages 11269–11278, 2019.

**20**. J. Lee, S. Park, and J. Shin. Learning bounds for risk-sensitive learning. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 13867–13879. Curran Associates, Inc., 2020.

**21**. J. Lei. Convergence and concentration of empirical measures under Wasserstein distance in unbounded functional spaces. *Bernoulli*, 26(1):767–798, 2020.

**22**. Z. Mhammedi, B. Guedj, and R. C. Williamson. Pac-bayesian bound for the conditional value at risk. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 17919–17930. Curran Associates, Inc., 2020.

**23**. E. Moulines and F. Bach. Non-asymptotic analysis of stochastic approximation algorithms for machine learning. *Advances in neural information processing systems*, 24:451–459, 2011.

**24**. A. K. Pandey, L. A. Prashanth, and S. P. Bhat. Estimation of spectral risk measures. In *AAAI Conference on Artificial Intelligence*, 2021.

**25**. B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM Journal on Control and Optimization*, 30(4):838–855, 1992.

**26**. L. A. Prashanth and Sanjay P. Bhat. A Wasserstein distance approach for concentration of empirical risk estimates. *arXiv preprint arXiv:1902.10709*, 2020.

**27**. L. A. Prashanth, J. Cheng, M. C. Fu, S. I. Marcus, and C. Szepesvári. Cumulative prospect theory meets reinforcement learning: prediction and control. In *International Conference on Machine Learning*, pages 1406–1415, 2016.

**28**. L. A. Prashanth, K. Jagannathan, and R. K. Kolla. Concentration bounds for CVaR estimation: The cases of light-tailed and heavy-tailed distributions. In *International Conference on Machine Learning (Accepted)*, 2020.

**29**. L. A. Prashanth, N. Korda, and R. Munos. Concentration bounds for temporal difference learning with linear function approximation: the case of batch data and uniform sampling. *Machine Learning*, 110(3):559–618, 2021. doi: 10.1007/s10994-020-05912-5.

**30**. H. Robbins and S. Monro. A stochastic approximation method. *The annals of mathematical statistics*, pages 400–407, 1951.

**31**. R. T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *Journal of risk*, 2:21–42, 2000.

**32**. D. Ruppert. Stochastic approximation. *Handbook of Sequential Analysis*, pages 503–529, 1991.

**33**. P. Thomas and E. Learned-Miller. Concentration inequalities for conditional value at risk. In *International Conference on Machine Learning*, pages 6225–6233, 2019.

**34**. R. Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press, 2018.

**35**. Y. Wang and F. Gao. Deviation inequalities for an estimator of the conditional value-at-risk. *Operations Research Letters*, 38(3):236–239, 2010.

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