Multiple Facets of Algebraic Computation

C Ramya
IMSc, Chennai.

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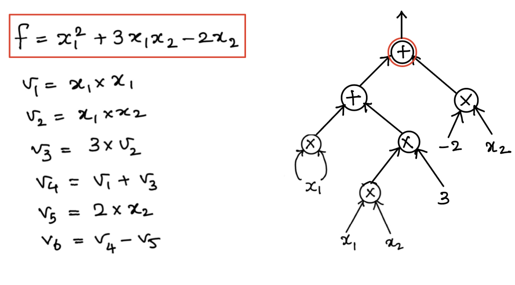

- **Model:** Arithmetic circuits: DAGs with leaves labelled by variables or constants(from $\mathbb{F}$) and internal gates labelled by $\{+, \times\}$.

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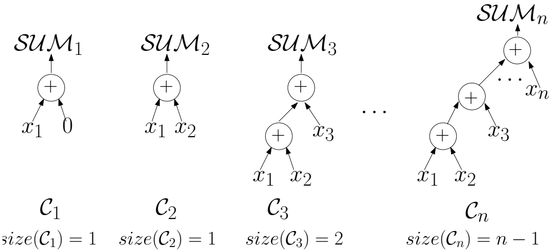

$SUM = (\mathcal{SUM}_n)_{n \geq 1}$ is a polynomial family.

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Valiant's Conjecture: Any circuit for perm_n requires size $n^{\omega(1)}$.

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> -Valiant(1975)

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Consider $P(z_1, z_2, z_3) = z_2^2 - 4z_1z_3$. Then,

- $f(x) \in \mathcal{C} \Rightarrow P(\overline{\text{coeff}}(f)) = 0$.
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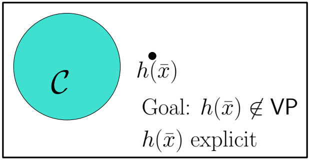
Proving Lower Bounds against $\mathcal{C}$: Find a property P that every polynomial in $\mathcal{C}$ satisfies and then find an explicit h that does not satisfy P.



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For more(infact most) sophisticated circuit classes $\mathcal{C}$:

Construct a measure $\mu : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{R}$:

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- $\mu(f)$ is $\text{rank}(M_f)$ for a matrix M_f associated with polynomial f.

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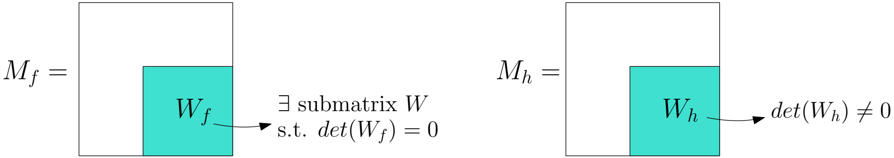

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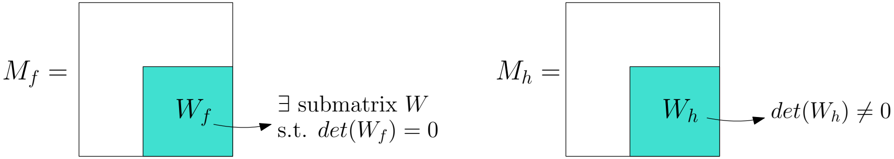

For $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree $d = \text{poly}(n)$: $M_f \in \mathbb{F}^{N \times N}, N = \binom{n+d}{n}$.

- $M_f[x^\alpha, x^\beta] =$ coefficient of x^α in $\frac{\partial f}{\partial x^\beta}$.
- Entries of M_f are linear in the coefficients of f.

Towards Algebraically Natural Proofs

Let $f \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d. Then, $f = \sum c_m \cdot m$.
Coefficient-vector: $\overline{\text{coeff}}(f) = (c_1, c_2, \ldots, c_N) \in \mathbb{F}^N$ where $N = \binom{n+d}{n}$.
"Natural" lower bound proof for $\mathcal{C} \subseteq \mathbb{F}[x_1, \ldots, x_n]^{\leq d}$:

$\mathcal{C}$ has a *natural proof* if there is a non-zero polynomial $P(z_1, \ldots, z_N)$:

1. **Usefulness**: $\forall f \in \mathcal{C}$, $P(\overline{\text{coeff}}(f)) = 0$.
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Example 2: $\mathcal{C} = \{\Sigma\Pi\Sigma,\ \Sigma\Pi\Sigma\Pi,\ \Sigma \wedge \Sigma\}$.
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Example 1: $\mathcal{C} = \{(\alpha x - \beta)^2 \mid \alpha, \beta \in \mathbb{C}\}$. Then, $P(z_1, z_2, z_3) = z_2^2 - 4z_1z_3$ such that $P(\overline{\text{coeff}}(f)) = 0$ for all $f \in \mathcal{C}$.

Example 2: $\mathcal{C} = \{\Sigma\Pi\Sigma,\ \Sigma\Pi\Sigma\Pi,\ \Sigma \wedge \Sigma\}$.
Then, $P(z_1, \ldots, z_N) = \det(W)$ such that $P(\overline{\text{coeff}}(f)) = 0$ for all $f \in \mathcal{C}$.

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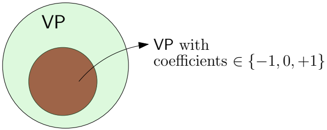
Theorem (Chatterjee, Kumar, R., Saptharishi, Tengse)

Answer: *Yes, for polynomials with small integer coefficients.*



On the Existence of Natural Proofs

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For n, d and $N = \binom{n+d}{n}$, there exists a non-zero $P(z_1, \ldots, z_N)$ such that

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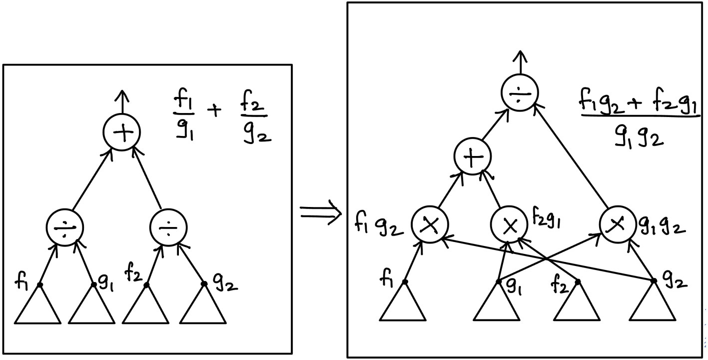
- What does this result suggest? An evidence for the power of *natural lower bound techniques* for proving super-polynomial lower bounds.

Strassen's Vermeidung von Divisionen



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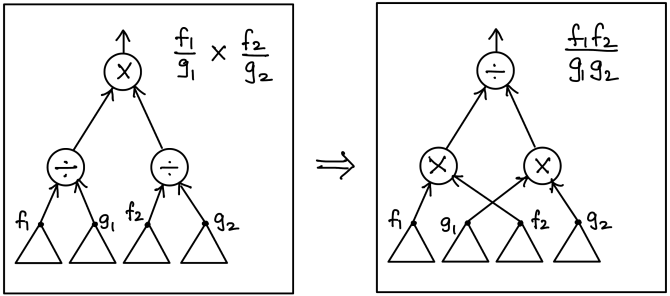

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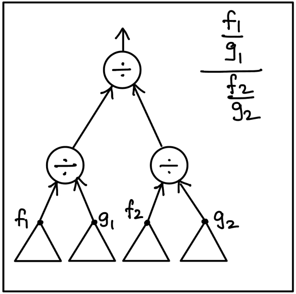

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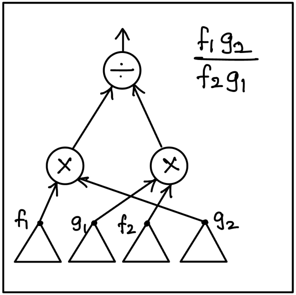

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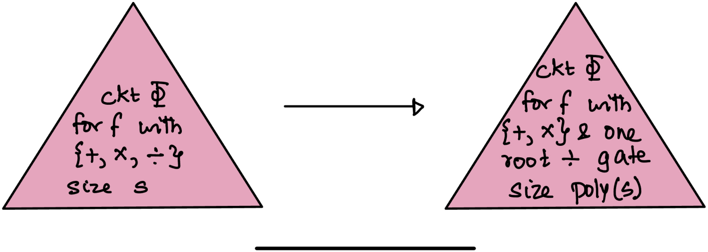

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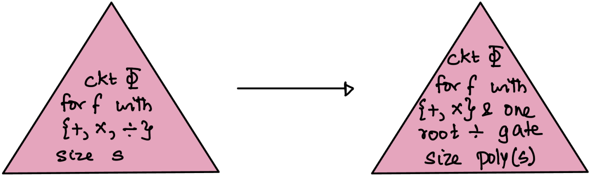

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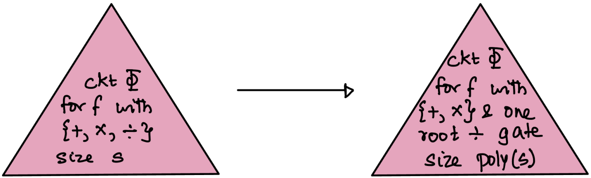

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- Division gates can be eliminated with polynomial blow up in size.

Polynomial Identity Testing

- A polynomial($f \equiv 0$) is *identically zero* if all its coefficients are zero.
- E.g.: $(x+y)^2 - x^2 - y^2 - 2xy \equiv 0$ and $(x+y)^2 - x^2 - y^2 + 2xy \not\equiv 0$.

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- Multivariate case: Can have infinitely many roots.
- Randomized polynomial time algorithm for multivariate PIT is known.
- **Open Question:** Derandomizing PIT.

Non-commutative PIT

- Set of non-commuting variables $\{x_1, \ldots, x_n\}$ i.e., $x_i x_j \neq x_j x_i \ \forall i \neq j$. E.g., $(x_1 + x_2)(x_1 - x_2) \neq x_1^2 - x_2^2$.
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- **Open:** Randomized polynomial time algorithm for ncPIT on circuits of polynomial size.

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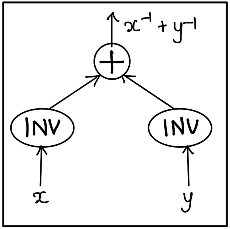

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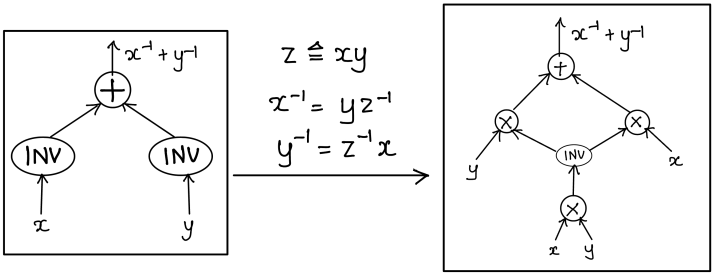

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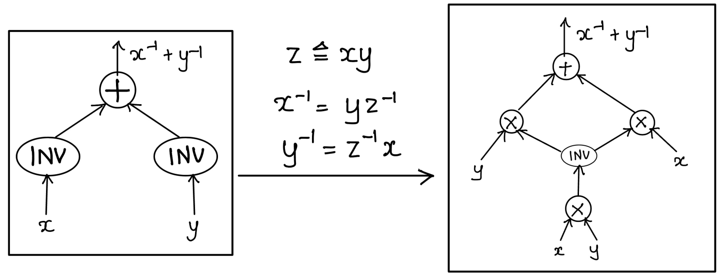

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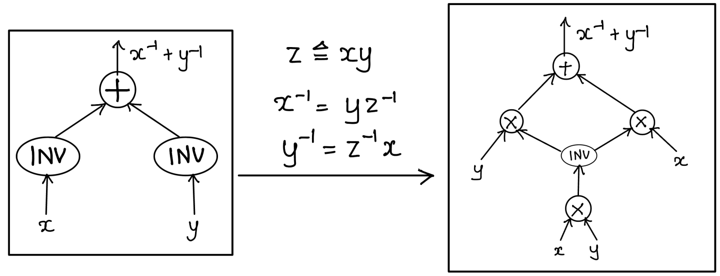

- Hua's identity: $(x + xy^{-1}x)^{-1} \equiv x^{-1} + (x + y)^{-1}$.
- Nested inversions cannot always be eliminated. e.g., $(u + xy^{-1}z)^{-1}$.
- Inversion height: number if nested inversions.

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- A rational expression $r(x_1, \ldots, x_n)$ computes the zero function[1] if
  - r has a nonempty domain of definition
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(Garg et al. '20, Ivanyos et al. '18) Deterministic polynomial time algorithm in the white-box model for non-commutative formula[2].

(Derksen, Makam '17) Randomized polynomial time in black-box model for non-commutative formula.

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(Hrubes, Wigderson) A rational formula of size s has inversion height $O(\log s)$.

Thank you