Polyhedral Compilation Foundations

Louis-Noël Pouchet

pouchet@cse.ohio-state.edu

Dept. of Computer Science and Engineering, the Ohio State University

Jan 25, 2010

888.11, Class #1

Motivating Example [1/2]

Example

```
for (i = 0; i < 3; ++i)
  for (j = 0; j < 3; ++j)
    A[i][j] = i * j;
```

Program execution:

```
1: A[0][0] = 0 * 0;
2: A[0][1] = 0 * 1;
3: A[0][2] = 0 * 2;
4: A[1][0] = 1 * 0;
5: A[1][1] = 1 * 1;
6: A[1][2] = 1 * 2;
7: A[2][0] = 2 * 0;
8: A[2][1] = 2 * 1;
9: A[2][2] = 2 * 2;
```

Motivating Example [2/2]

A few observations:

- Statement is executed 9 times
- There is a different values for i, j associated to these 9 instances
- There is an order on them (the execution order)

Objective:
find a representation where these 3 characteristics are modeled

Overview of the Solution

- Iteration domain: set of totally ordered n-dimensional vectors
  - **Iteration vector** $\vec{x}_S = (i,j)$
  - Iteration domain: the set of values of $\vec{x}_S$
- Convenient approach: **polytopes model sets** of totally ordered n-dimensional vectors
- **One condition: the set must be convex**

Convexity [1/2]

Convexity is the central concept of polyhedral optimization

Definition (Convex set)

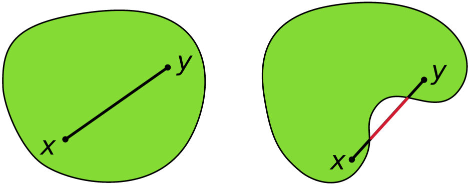
Given S a subset of $\mathbb{R}^n$. S is convex iff, $\forall \mu, \lambda \in S$ and given $c \in [0,1]$:

$$(1-c).\mu + c.\lambda \in S$$

With words: drawing a line segment between any two points of S, each point on this segment is also in S.

Warning: when $\mathbb{K} = \mathbb{Z}$, we use another definition

Convexity [2/2]



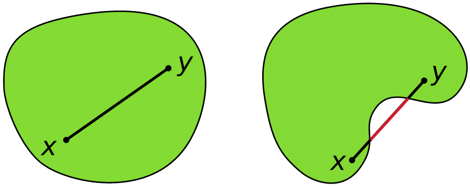
Definition (Convex combination)

Given S a convex set. For any family of vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r \in S$, and any nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = 1$, then:

$$\vec{v} = \sum_{i=1}^{r} u_i \lambda_i \in S$$

$\vec{v}$ is a convex combination of $\{\vec{u}_i\}$.

Convexity [2/2]



Definition (Convex combination)

Given S a convex set. For any family of vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r \in S$, and any nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = 1$, then:

$$\vec{v} = \sum_{i=1}^{r} u_i \lambda_i \in S$$

$\vec{v}$ is a convex combination of $\{\vec{u}_i\}$.

Exercise: Prove a statement surrounded by loops with unit-stride, no conditional and simple loop bounds has a convex iteration domain.

The Affine Qualifier

Definition (Affine function)

A function $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$ is affine if there exists a vector $\vec{b} \in \mathbb{K}^n$ and a matrix $A \in \mathbb{K}^{m \times n}$ such that:

$$\forall \vec{x} \in \mathbb{K}^m, \; f(\vec{x}) = A\vec{x} + \vec{b}$$

Definition (Affine half-space)

An affine half-space of $\mathbb{K}^m$ (affine constraint) is defined as the set of points:

$$\{\vec{x} \in \mathbb{K}^m \mid \vec{a}.\vec{x} \leq \vec{b}\}$$

Polyhedron (Implicit Representation)

Definition (Polyhedron)

A set $\mathcal{S} \in \mathbb{K}^m$ is a polyhedron if there exists a system of a finite number of inequalities $A\vec{x} \leq \vec{b}$ such that:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid A\vec{x} \leq \vec{b}\}$$

Equivalently, it is the intersection of finitely many half-spaces.

Definition (Polytope)

A polytope is a bounded polyhedron.

Integer Polyhedron

Definition ($\mathbb{Z}$-polyhedron)

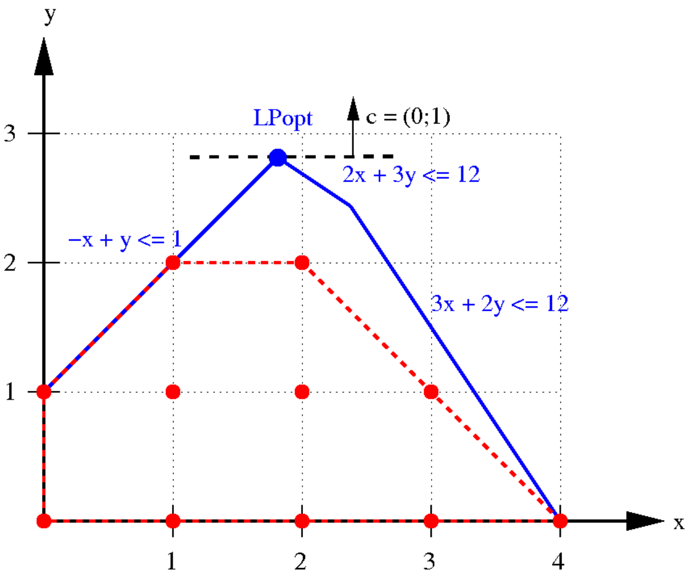
It is a polyhedron where all its extreme points are integer valued

Definition (Integer hull)

The integer hull of a rational polyhedron $\mathcal{P}$ is the largest set of integer points such that each of these points is in $\mathcal{P}$.

For the moment, we will "say" an integer polyhedron is a polyhedron of integer points (language abuse)

Rational and Integer Polytopes



Returning to the Example

Modeling the iteration domain:

- Polytope dimension: set by the number of surrounding loops
- Constraints: set by the loop bounds

$$\mathcal{D}_R : \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} . \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} . \begin{pmatrix} i \\ j \\ 1 \end{pmatrix} \geq \vec{0}$$

$$0 \leq i \leq 2, \quad 0 \leq j \leq 2$$

Relation with Operations on Polyhedra

Considering conjunctions:

Definition (Intersection)

The intersection of two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$ is a convex set $\mathcal{P}$:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid \vec{x} \in \mathcal{P}_1 \wedge \vec{x} \in \mathcal{P}_2\}$$

Considering disjunctions:

Definition (Union)

The union of two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$ is a set $\mathcal{P}$:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid \vec{x} \in \mathcal{P}_1 \vee \vec{x} \in \mathcal{P}_2\}$$

The union of two convex sets may not be a convex set

Lattices

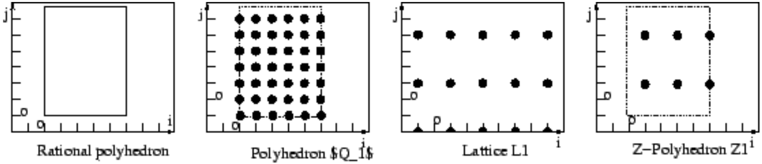
Definition (Lattice)

A subset L in $\mathbb{Q}^n$ is a lattice if is generated by integral combination of finitely many vectors: $a_1, a_2, \ldots, a_n$ ($a_i \in \mathbb{Q}^n$). If the a_i vectors have integral coordinates, L is an integer lattice.

Definition ($\mathcal{Z}$-polyhedron)

A $\mathcal{Z}$-polyhedron is the intersection of a polyhedron and an affine integral full dimensional lattice.

Pictured Example



Example of a $\mathcal{Z}$-polyhedron:

- $Q_1 = \{i, j \mid 0 \le i \le 5,\ 0 \le 3j \le 20\}$
- $L_1 = \{2i+1, 3j+5 \mid i, j \in \mathbb{Z}\}$
- $Z_1 = Q_1 \cap L_1$

Complex Example

Computing the set of cells of A accessed

Example

```
for (i = 0; i < N; ++i)
  for (j = i; j < N; ++j)
    A[2i + 3][4j] = i * j;
```

- $\mathcal{D}_S$: $\{i,j \mid 0 \le i < N,\ i \le j < N\}$
- Function: $f_A : \{2i+3, 4j \mid i,j \in \mathbb{Z}\}$
- Image($\mathcal{D}_S, f_A$) is the set of cells of A accessed (a $\mathcal{Z}$-polyhedron):
  - Polyhedron: $Q : \{i,j \mid 3 \le i < 2N+2,\ 0 \le j < 4N\}$
  - Lattice: $L : \{2i+3, 4j \mid i,j \in \mathbb{Z}\}$