CURVE

REPRESENTATION

Representation

Non-parametric form: $y = f(X)$

Explicit form: $y = mx + b$

Implicit form: $f(x, y) = 0$

Parametric form:
$x = x(t)$
$y = y(t)$

2^{nd} degree implicit representation:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

Any guess, why the factor 2 is used ?

This form of the expression, with the coefficients, provide a wide variety of 2D curve forms called:

CONIC SECTIONS

CONIC SECTIONS

PARABOLA

$y^2 = 4ax; a > 0$

$Focus: (a, 0);$

$Directrix = -a.$

$\text{eccentricity}, e = 1$

$x = at^2; y = \pm 2at.$

or

$x = \tan^2(\phi);$

$y = \pm 2\sqrt{a \tan(\phi)}.$

HYPERBOLA

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$

$b^2 = a^2(e^2 - 1);$

$e > 1; Foci: (\pm ae, 0).$

$Directrices: x = \pm a / e;$

$x = a \sec(t),$

$y = b \tan(t);$

$-\pi / 2 < t < \pi / 2.$

Rectangular

Hyperbola :

$\mathrm{e} = \sqrt{2}; x = ct; y = c / t.$

ELLIPSE

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$

$a \geq b > 0.$

$b^2 = a^2(1 - e^2);$

$0 \leq e \leq 1.$

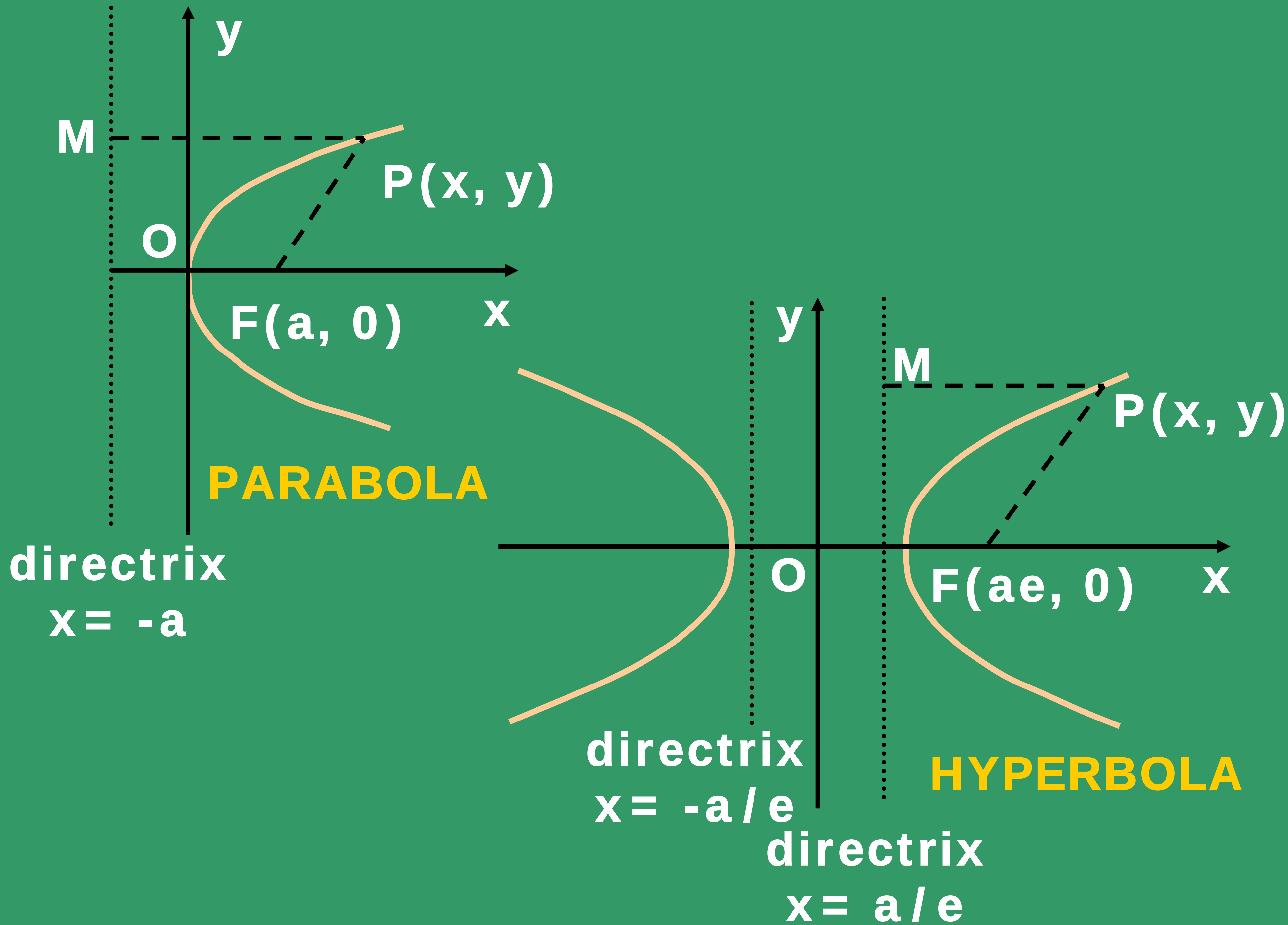
$Foci: (\pm ae, 0);$

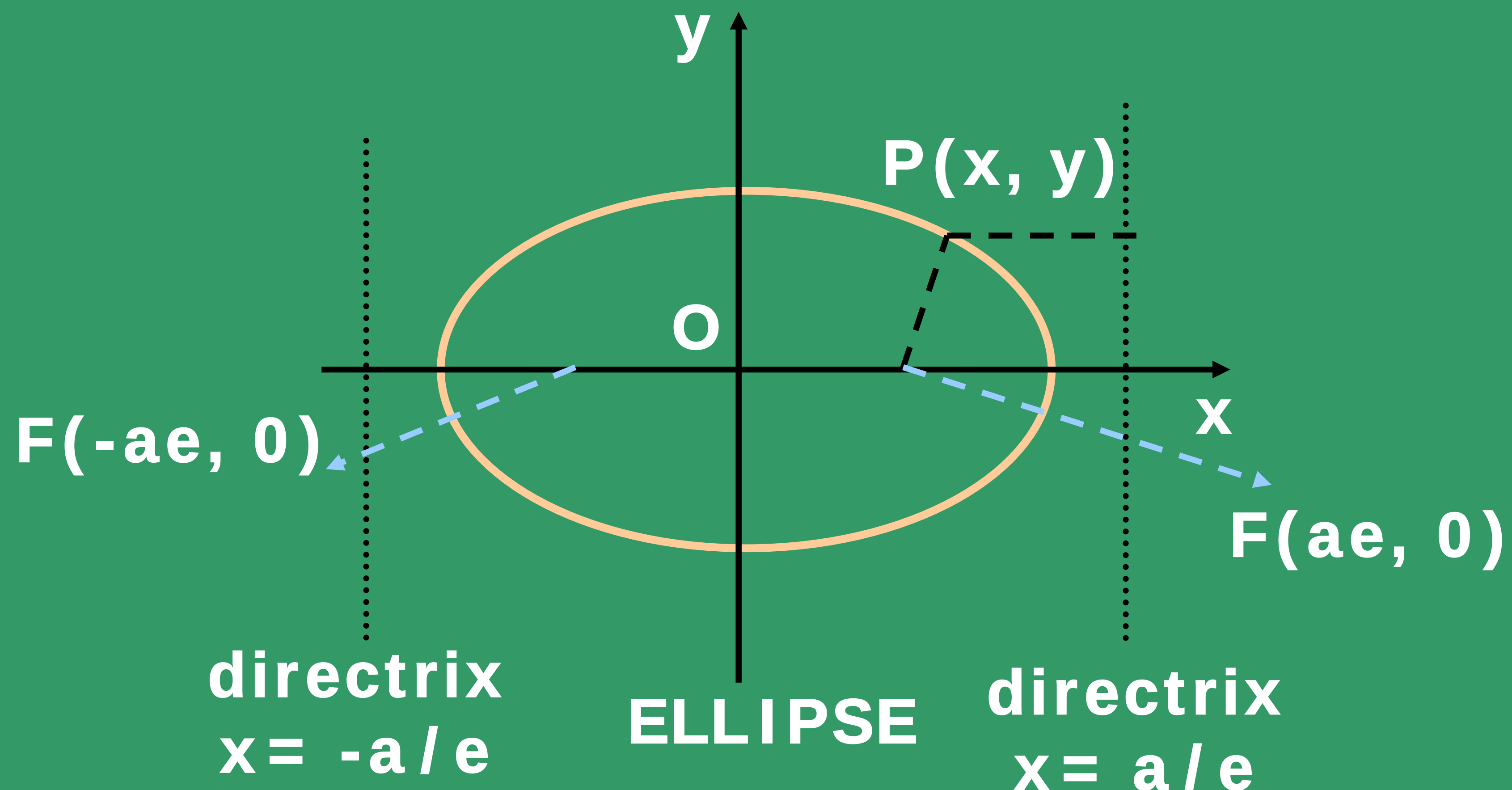
$Directrices: x = \pm a / e.$

$x = a \cos(t),$

$y = b \sin(t);$

$t \in [-\pi, \pi].$


y
M
P(x, y)
O
F(a, 0)
x
PARABOLA
directrix
x= -a
y
M
P(x, y)
O
F(ae, 0)
x
directrix
x= -a/e
HYPERBOLA
directrix
x= a/e


y
P(x, y)
O
x
F(-ae, 0)
F(ae, 0)
directrix
x= -a/e
ELLIPSE
directrix
x= a/e

Polar Equation of a conic (***home assignment***):

$$r = \frac{L}{1 + e\cos(\theta)}, \quad \text{where, } L = e.dist(F, d)$$

F – Focal Point; d – Directrix;

e – Eccentricity.

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

If the conic passes through the origin: f = 0.

Assuming, one of the parameters to be a constant, c = 1.0, f = 1.0

Remaining 5 Coeffs. may be obtained using 5 geometric conditions:

Say:

Boundary Conditions -

- **– two (2) end points**
- **- slope of the curves at two (2) end points. and**
- **- one (1) intermediate point**

Generalized CONIC

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

Re-organize:

as $\boldsymbol{XSX^T} = 0,$ **S is symmetric**

$$\Rightarrow \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or

$$\boldsymbol{XAX^T} + \boldsymbol{GX} + \boldsymbol{f} = 0$$

Special Conditions:

If $b^2 = ac$, the equation represents a PARABOLA;

If $b^2 < ac$, the equation represents an ELLIPSE;

If $b^2 > ac$, the equation represents a HYPERBOLA.

SPACE CURVE (3-D)

Explicit non-parametric representation:

x = x, y = f(x), z = g(x).

Non-parametric implicit representation:

f(x, y, z) = 0, g(x, y, z) = 0.

Intersection of the above two surfaces represents a curve.

Examples:

$$x = t^3, y = t^2, z = t.$$

A parametric space curve:

x = x(t), y = f(t), z = g(t).

Curve on the seam of a baseball:

$$x = \lambda[a.\cos(\theta + \pi/4) - b.\cos 3(\theta + \pi/4)],$$
$$y = \mu[a.\sin(\theta + \pi/4) - b.\sin 3(\theta + \pi/4)],$$
$$z = c.\sin(2\theta).$$

where,

$$\lambda = 1 + d.\sin(2\theta) = 1 + d(z/c),$$
$$\mu = 1 - d.\sin(2\theta) = 1 - d(z/c);$$
$$\theta = 2\pi t, 0 \le t \le 1.0.$$

HELIX:

$$x = r.\cos(t), y = r.\sin(t), z = bt;$$
$$b \ne 0, -\infty < t < \infty$$

PARAMETRIC CUBIC CURVES

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x,$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y,$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z.$$

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T.C,$$

$$\text{where, } T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

In general:

$$Q(t) = [x(t) \quad y(t) \quad z(t)] = T.M.G,$$

$$\text{where, } \mathrm{T} = [\mathrm{t}^3 \quad \mathrm{t}^2 \quad \mathrm{t} \quad 1],$$

$$\mathrm{M} = [m_{ij}]_{4x4} \text{ and } G = [g_1 \quad g_2 \quad g_3 \quad g_4]^T$$

M is a 4x4 ***basis matrix*** and G is a four element column vector of geometric constants, called the ***geometric vector***.

The curve is a weighted sum of the elements of the geometry matrix.

The weights are each cubic polynomials of t, and are called the ***blending functions***:

B = T.M.

CUBIC SPLINES

$$P(t) = \sum_{i=1}^{4} B_i t^{i-1}; t_i \leq t \leq t_2.$$

P(t) is the position vector of any point on the cubic spline segment.

P(t) = [x(t), y(t), z(t)]	***Cartesian***
or [r(t), θ(t), z(t)]	***Cylindrical***
or [r(t), θ(t), ϕ(t)]	***Spherical***

$$x(t) = \sum_{i=1}^{4} B_{ix} t^{i-1}$$

$$y(t) = \sum_{i=1}^{4} B_{iy} t^{i-1} \qquad t_1 \le t \le t_2.$$

$$z(t) = \sum_{i=1}^{4} B_{iz} t^{i-1}$$

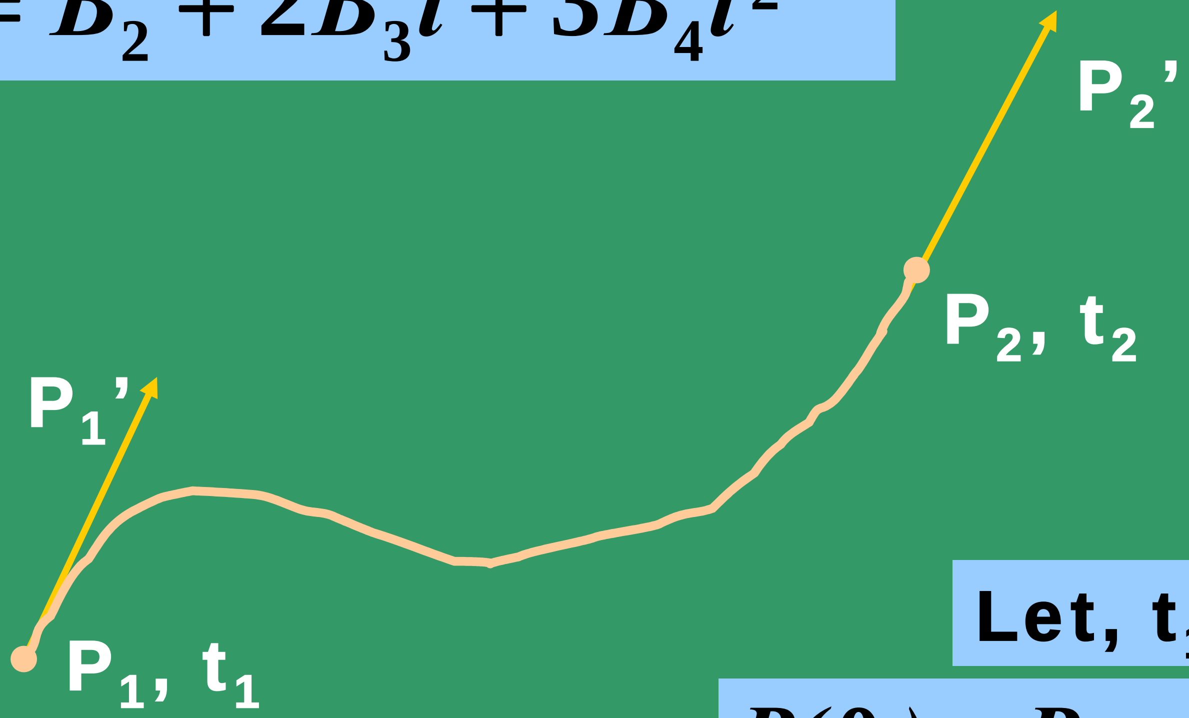
Use boundary conditions to evaluate the coeficients

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3,$$

$$t_1 \le t \le t_2$$

$$P'(t) = \sum_{i=1}^{4} (i-1) B_i t^{i-2}$$

$$= B_2 + 2B_3 t + 3B_4 t^2$$



Let, $t_1 = 0$:

$$P(0) = P_1; \quad P(t_2) = P_2.$$

$$P'(0) = P_1'; \quad P'(t_2) = P_1'.$$

Solutions:

$$B_1 = P_1;\ \ B_2 = P_1';$$

$$B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P(t_2);$$

$$B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P'(t_2);$$

$$B_3 =$$

$$B_4 =$$

Equation of a single cubic spline segment:

$$P(t) = P_1 + P_1' t + \left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right]t^2$$

$$+ \left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right]t^3;$$

For piece-wise continuity for complex curves, two or more curve segments are joined together.

In that case, use second derivative $P_2''(t)$ at end-points (joints).

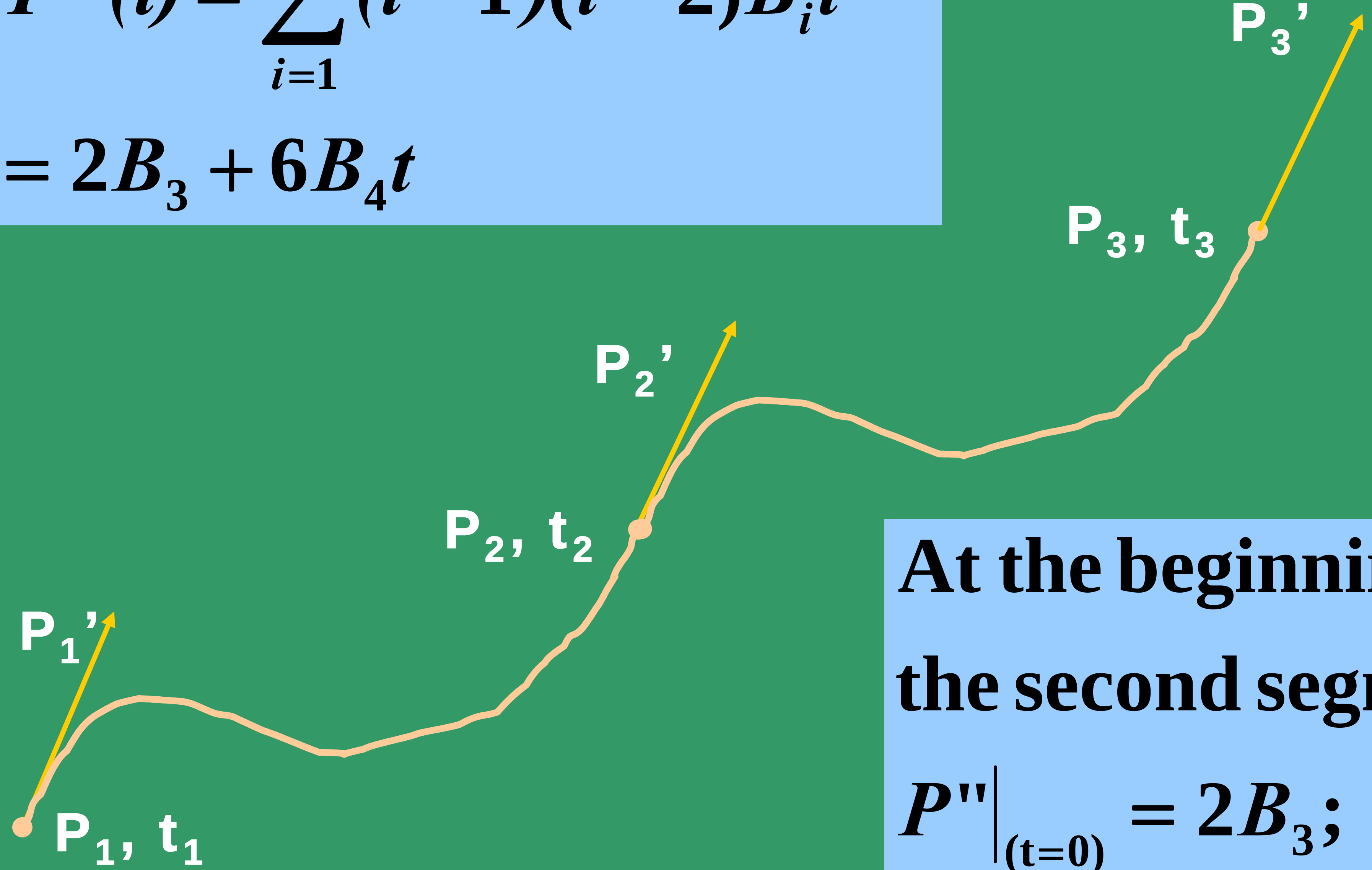
Various other approaches used are:

- Normalized Cubic splines
- Blending
- Weighting functions.

$$P''(t) = \sum_{i=1}^{4} (i-1)(i-2) B_i t^{i-3}$$

$$= 2B_3 + 6B_4 t$$

P_1' and P_3' known,
But what about P_2' ?



At the beginning of the second segment :

$$P''\Big|_{(t=0)} = 2B_3;$$

$$P''(t_2) = 2B_3 + 6B_4 t_2 = P''(0) = 2\bar{B}_3$$

$$\boldsymbol{B_3 = \frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2};}$$

$$\boldsymbol{B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2};}$$

$$6t_2\left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right] + 2\left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right] = 2\left[\frac{3(P_3 - P_2)}{t_2^2} - \frac{2P_2'}{t_2} - \frac{P_3'}{t_2}\right]$$

Multiplying both sides by $t_2 t_3$

$$t_3 P_1' + 2(t_3 + t_2)\boxed{P_2'} + t_2 P_3' = \frac{3}{t_2 t_3}\left[t_2^2(P_3 - P_2) + t_3^2(P_2 - P_1)\right]$$

Generalized equation for any two adjacent cubic spline segments, $P_k(t)$ and $P_{k+1}(t)$:

For first segment:

$$P_k(t) = P_k + P_k' t + \left[\frac{3(P_{k+1} - P_k)}{t_{k+1}^2} - \frac{2P_k'}{t_{k+1}} - \frac{P_{k+1}'}{t_{k+1}}\right] t^2 + \left[\frac{2(P_k - P_{k+1})}{t_{k+1}^3} + \frac{P_k'}{t_{k+1}^2} + \frac{P_{k+1}'}{t_{k+1}^2}\right] t^3;$$

For second segment:

$$P_{k+1}(t) = P_{k+1} + P_{k+1}' t + \left[\frac{3(P_{k+2} - P_{k+1})}{t_{k+2}^2} - \frac{2P_{k+1}'}{t_{k+2}} - \frac{P_{k+2}'}{t_{k+2}}\right] t^2 + \left[\frac{2(P_{k+1} - P_{k+2})}{t_{k+2}^3} + \frac{P_{k+1}'}{t_{k+2}^2} + \frac{P_{k+2}'}{t_{k+2}^2}\right] t^3;$$

Curvature Continuity ensured as:

$$t_{k+2} P_k' + 2(t_{k+1} + t_{k+2}) P_{k+1}' + t_{k+1} P_{k+2}' = \frac{3}{t_{k+1} t_{k+2}} \left[t_{k+1}^2 (P_{k+2} - P_{k+1}) + t_{k+2}^2 (P_{k+1} - P_k) \right]$$

Equation of a normalized cubic spline segment:

$$F = T.N;$$

$$P(t) = T.N.G =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}^T$$

Use, $t_2 = 1$;

For curvature Continuity:

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

For curvature Continuity:

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

For three control points (knots) this works as:

$$P_1' = [3(P_2 - P_0) - P_0' - P_2'] / 4;$$

$$t_{k+2}P_k' + 2(t_{k+1} + t_{k+2})P_{k+1}' + t_{k+1}P_{k+2}' = \frac{3}{t_{k+1}t_{k+2}}\left[t_{k+1}^2(P_{k+2} - P_{k+1}) + t_{k+2}^2(P_{k+1} - P_k)\right]$$

For N points ??

For 3 points – 1 Eqn. (+ unknown)

For 4 points – 2 eqns. (+ unknowns)

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For N points – (N-2) eqns. (+ unknowns)

$$\begin{bmatrix} t_3 & 2(t_2+t_3) & t_2 & 0 & . & . & . & . \\ 0 & t_4 & 2(t_3+t_4) & t_3 & 0 & & & \\ 0 & 0 & t_5 & 2(t_4+t_5) & t_4 & 0 & & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & t_n & 2(t_n+t_{n-1}) & t_{n-1} \end{bmatrix} \begin{bmatrix} P'_1 \\ P'_2 \\ P'_3 \\ . \\ . \\ P'_n \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{t_2 t_3}\left[t_2^2(P_3-P_2)+t_3^2(P_2-P_1)\right] \\ \frac{3}{t_3 t_4}\left[t_3^2(P_4-P_3)+t_4^2(P_3-P_2)\right] \\ . \\ . \\ . \\ \frac{3}{t_{n-1} t_n}\left[t_{n-1}^2(P_n-P_{n-1})+t_n^2(P_{n-1}-P_{n-2})\right] \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & & & & & & \\ 0 & t_3 & 2(t_2+t_3) & t_2 & . & . & . & . \\ 0 & 0 & t_4 & 2(t_3+t_4) & t_3 & & & \\ 0 & 0 & 0 & t_5 & 2(t_4+t_5) & t_4 & & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & t_n & 2(t_n+t_{n-1}) & t_{n-1} \\ . & . & . & . & & 0 & & 1 \end{bmatrix} \begin{bmatrix} P'_1 \\ P'_2 \\ P'_3 \\ . \\ . \\ P'_n \end{bmatrix}$$

$$= \begin{bmatrix} P'_1 \\ \frac{3}{t_2 t_3}\left[t_2^2(P_3-P_2)+t_3^2(P_2-P_1)\right] \\ \frac{3}{t_3 t_4}\left[t_3^2(P_4-P_3)+t_4^2(P_3-P_2)\right] \\ . \\ . \\ . \\ \frac{3}{t_{n-1} t_n}\left[t_{n-1}^2(P_n-P_{n-1})+t_n^2(P_{n-1}-P_{n-2})\right] \\ P'_n \end{bmatrix}$$

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

Lets solve for N = 4;

$$P_1' + 4P_2' + P_3' = 3[P_3 - P_1];$$

$$P_2' + 4P_3' + P_4' = 3[P_4 - P_2]$$

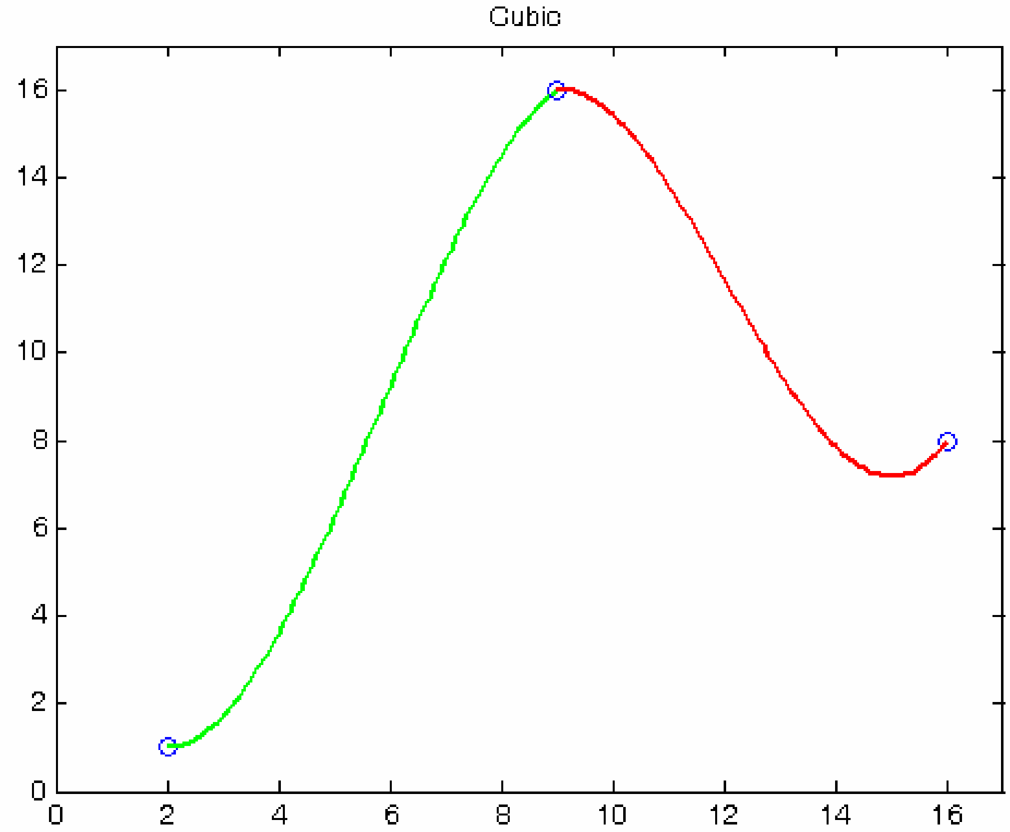
Re-arrange to get:

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} P_2' \\ P_3' \end{bmatrix} = \begin{bmatrix} 3(P_3 - P_1) - P_1' \\ 3(P_4 - P_2) - P_4' \end{bmatrix};$$

$$\begin{bmatrix} P_2' \\ P_3' \end{bmatrix} = (1/15) \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3(P_3 - P_1) - P_1' \\ 3(P_4 - P_2) - P_4' \end{bmatrix}$$

Problem: The position vectors of a normalized cubic spline are given as (0 0), (1 1), (2 -1) and (3 0). The tangent vectors at the ends are both (1 1).

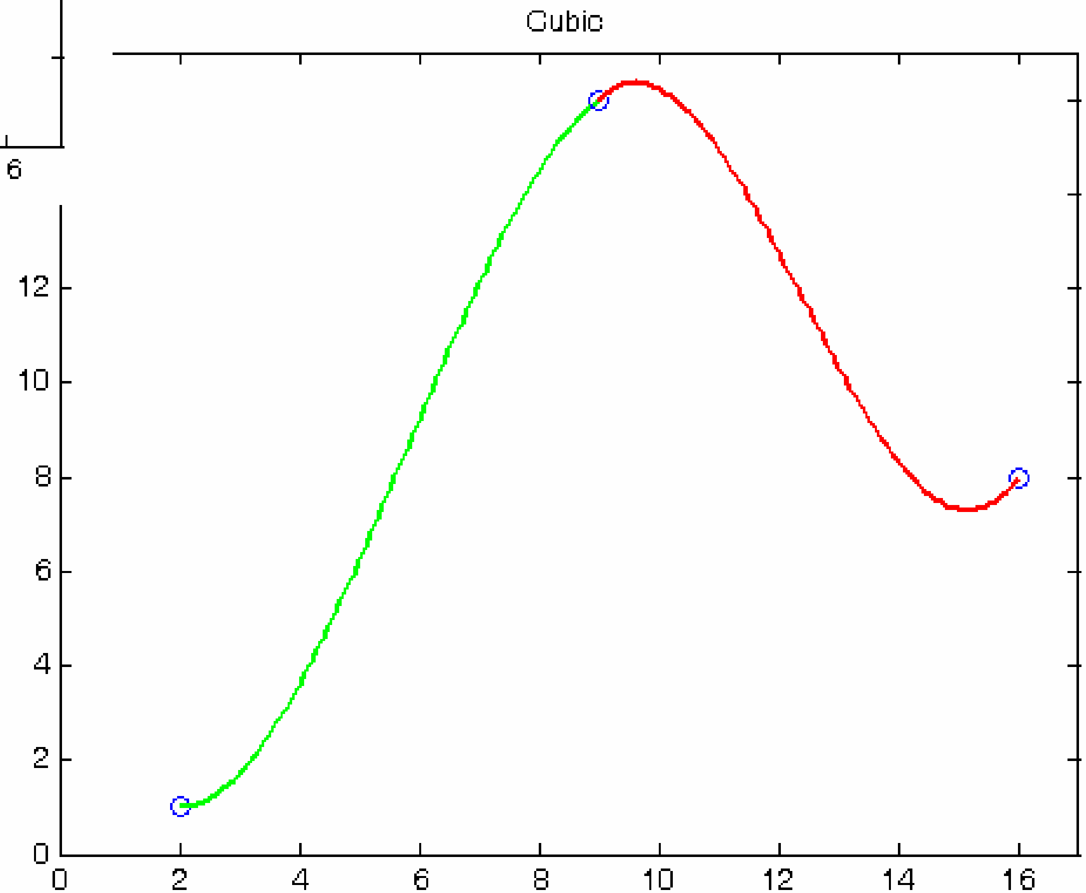
Soln: The 2 internal tangent vectors are calculated, and both are equal to (1 –0.8).

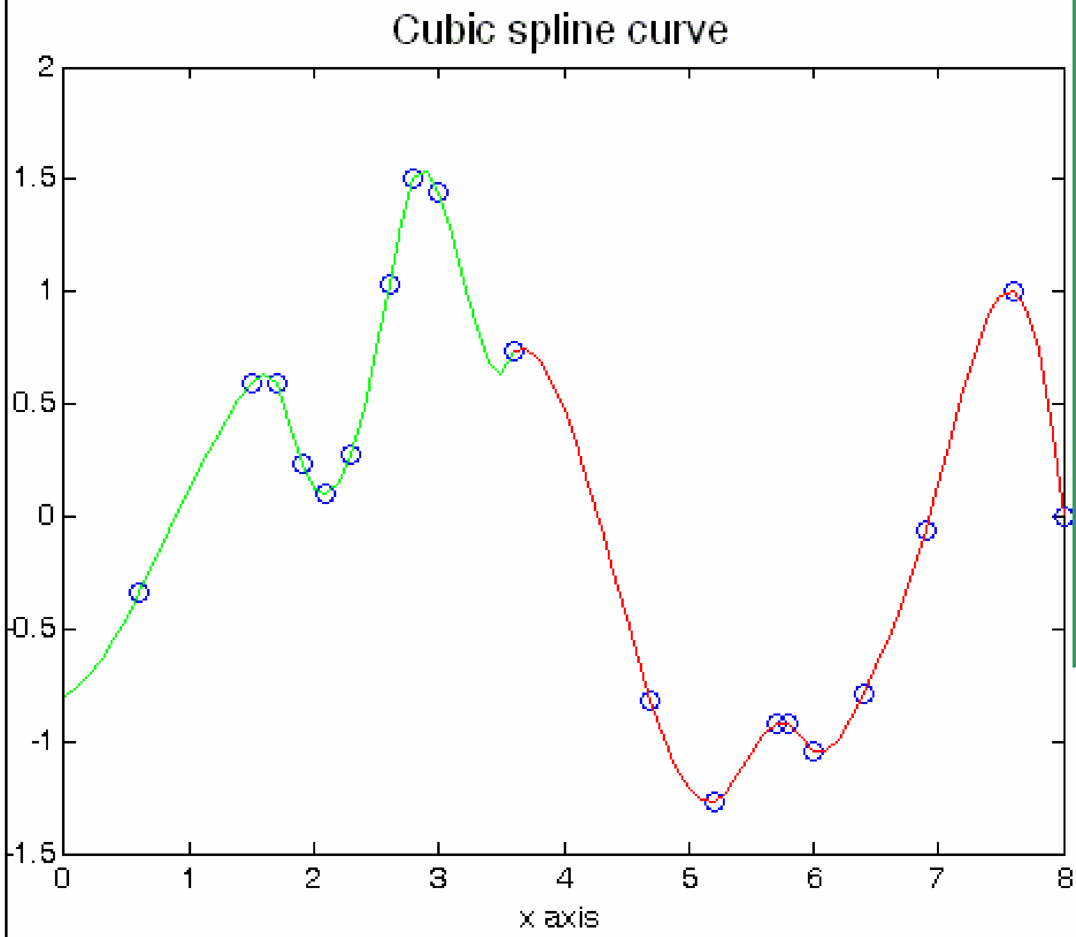


No use of 2^{nd} derivative smoothing

Two piecewise cubic spline segments

Using 2^{nd} derivative smoothing

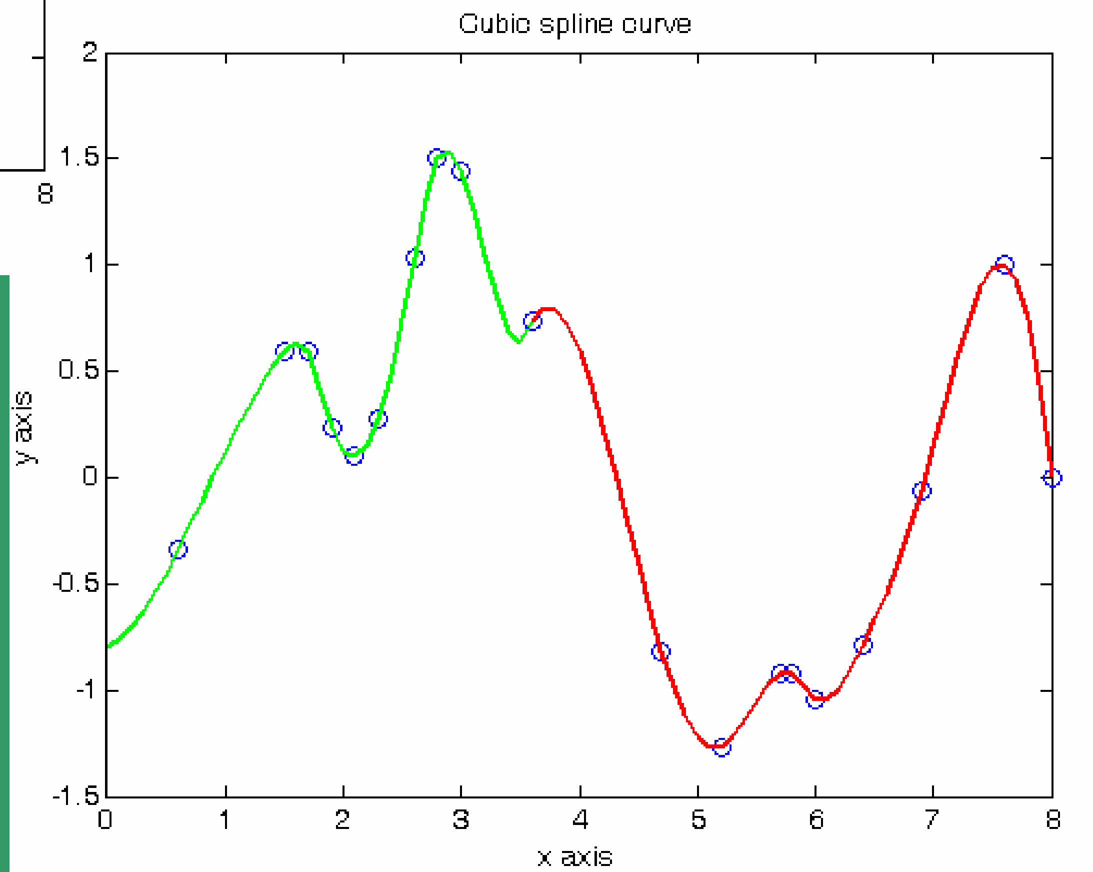




No use of 2nd derivative smoothing

Examples of spline interpolation

Using 2nd derivative smoothing

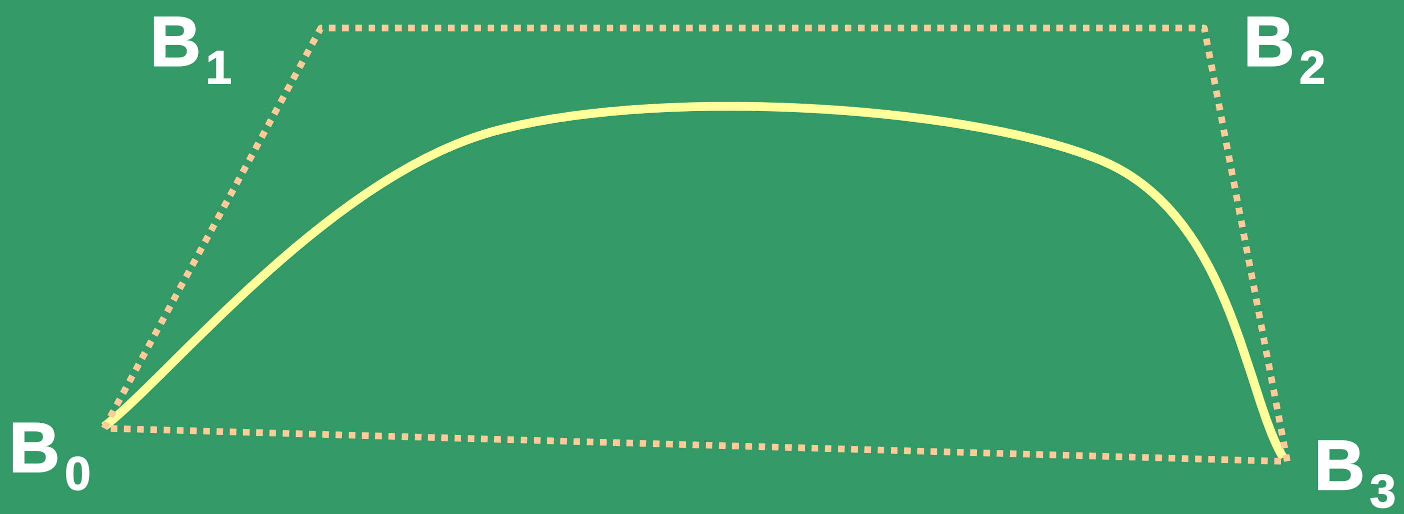


BEZIER CURVES

- Basis functions are real
- Degree of polynomial is one less than the number of points
- Curve generally follows the shape of the defining polygon
- First and last points on the curve are coincident with the first and last points of the polygon
- Tangent vectors at the ends of the curve have the same directions as the respective spans
- The curve is contained within the convex hull of the defining polygon
- Curve is invariant under any affine transformation.

A few typical examples of cubic polynomials for Bezier

BEZIER CURVES



Equation of a parametric Bezier curve:

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t); \ 0 \le t \le 1$$

where the **Bezier or Bernstein basis** or **blending function** is:

Binomial Coefficients:
(*i*th, *n*th-order **Bernstein basis** function)

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

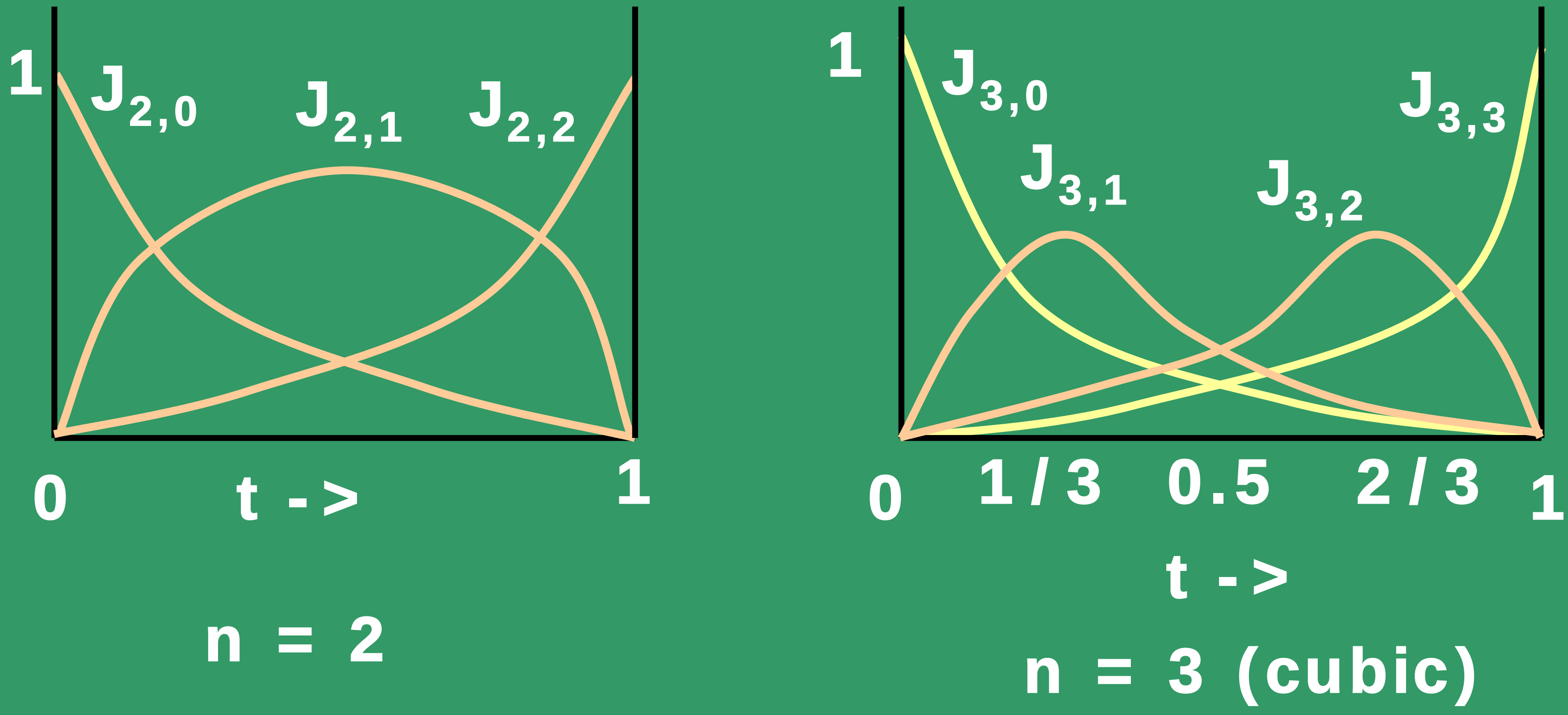
$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

B_i's are called the ***control points***.

$J_{n,i}(t)$ is the *i*th, *n*th order Bernstein basis function. n is the degree of the defining Bernstein basis function (polynomial curve segment).

This is one less than the number of points used in defining Bezier polygons.

Below are some examples of BBF (Bezier /Bernstein blending functions:



$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t); \quad 0 \le t \le 1$$

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Limits for $i = 0:$

$0^0 = 1; \quad 0! = 1$

$$J_{n,0}(0) = \frac{n!}{0!n!} 0^0 (1-0)^{n-0} = 1;$$

For $i \neq 0:$ $J_{n,i}(0) = \frac{n!}{i!(n-i)!} 0^i (1-0)^{n-i} = 0;$

Also:

$$J_{n,n}(1) = 1, i = n;$$
$$J_{n,i}(1) = 0, i \neq n.$$

Thus:

$$P(0) = B_0 J_{n,0}(0) = B_0.$$
$$P(1) = B_n J_{n,n}(1) = B_n.$$

For any t:

$$\sum_{i=0}^{n} J_{n,i}(t) = 1$$

Also Verify:

$$J_{n,i}(t) = (1-t).J_{(n-1),i}(t) + t.J_{(n-1),(i-1)}(t); \; n > i \geq 1$$

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}; \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Take n = 3:

$$\binom{n}{i} = \binom{3}{i} = \frac{6}{i!(3-i)!}$$

$$J_{3,0}(t) = 1.t^0(1-t)^3 = (1-t)^3;$$

$$J_{3,1}(t) = 3.t.(1-t)^2;$$

$$J_{3,2}(t) = 3.t^2.(1-t);$$

$$J_{3,3}(t) = t^3.$$

Thus, for Bezier:

$$P(t) = (1-t)^3 B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t) B_2 + t^3 B_3$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}; n = 3.$$

For B-splines:

$$P(t) = T.N.G =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}^T$$

For n = 4:

$$P(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

$$= T.N.G = F.G;$$

where:

$$F = [J_{n,o}(t) \;\; J_{n,1}(t) \;\; .... \; J_{n,n}(t)]$$

$$N = \lfloor \lambda_{ij} \rfloor_{nxn}$$

where:

$$\lambda_{ij} = \begin{cases} \binom{n}{j}\binom{n-j}{n-i-j}(-1)^{n-i-j} & 0 \le (\mathrm{i}+\mathrm{j}) \le \mathrm{n} \\ 0 & \text{otherwise} \end{cases}$$

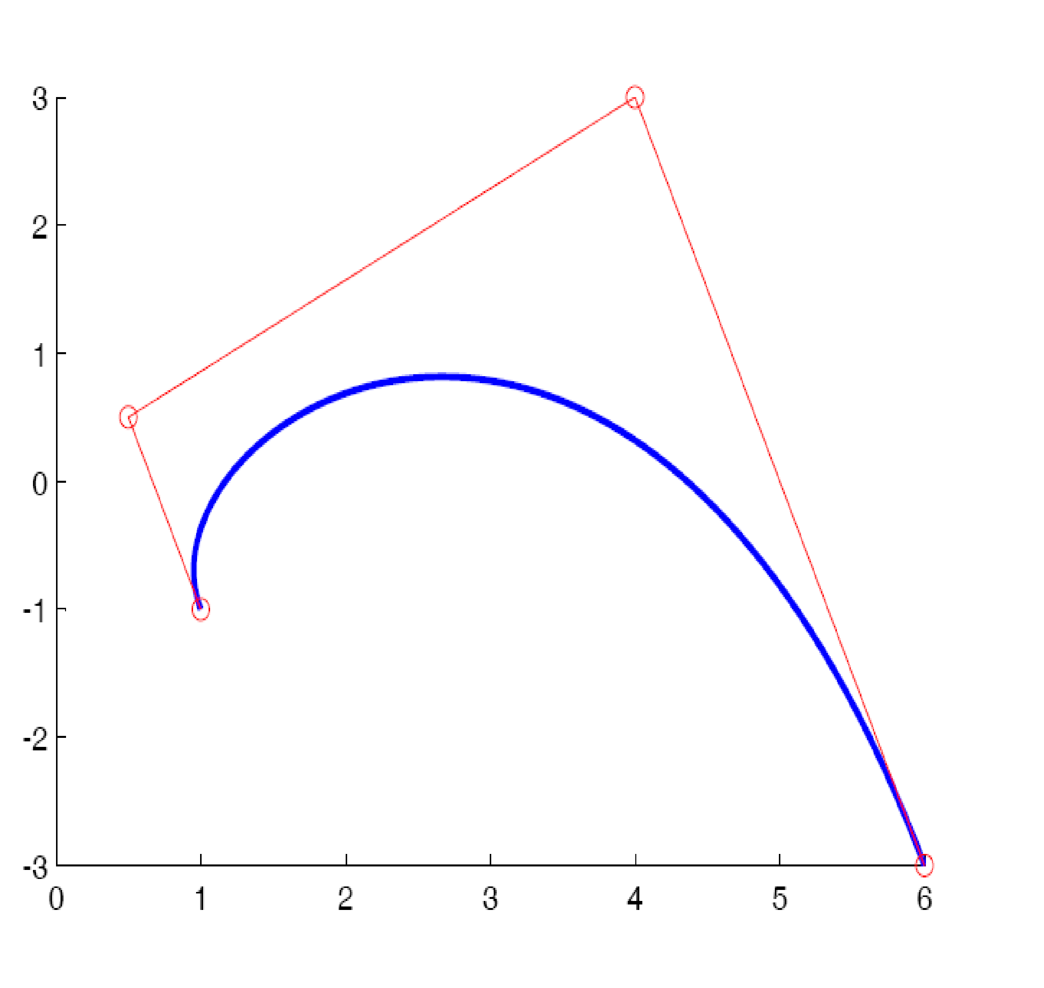
Computation of successive binomial coefficients:

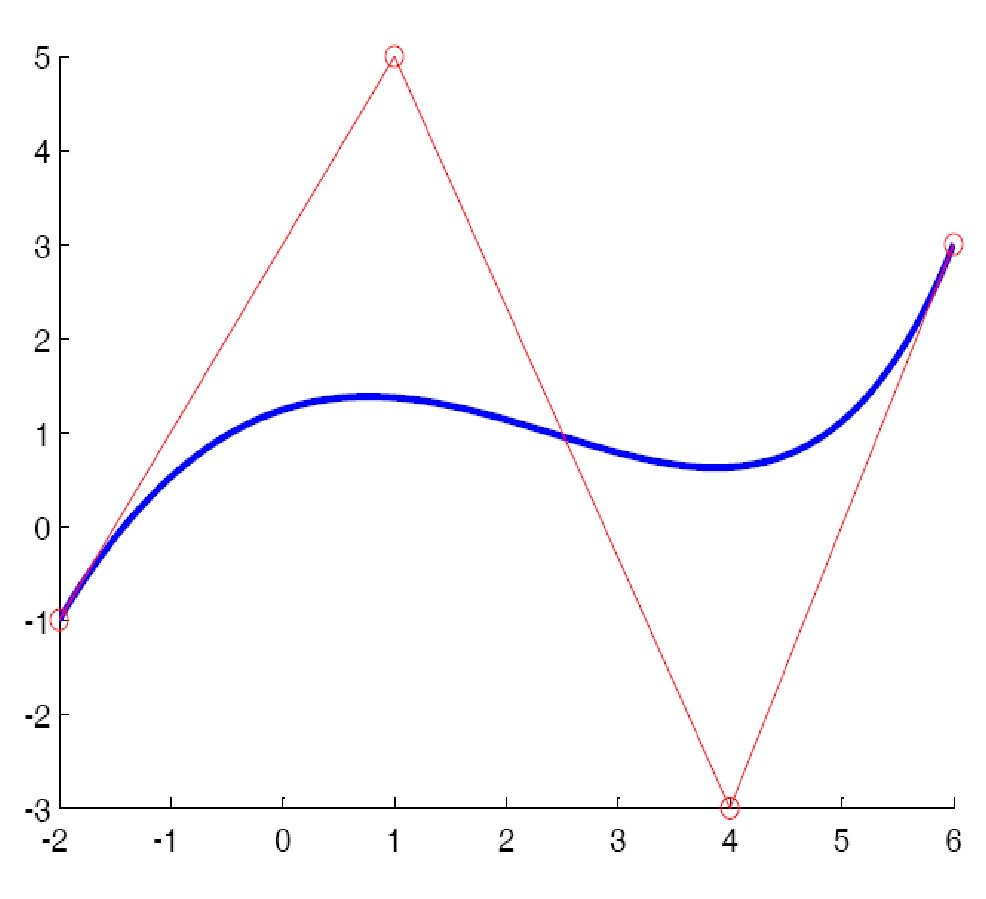
$$\binom{n}{i} = \left(\frac{n-i+1}{i}\right)\binom{n}{i-1}$$

Home Assignment:

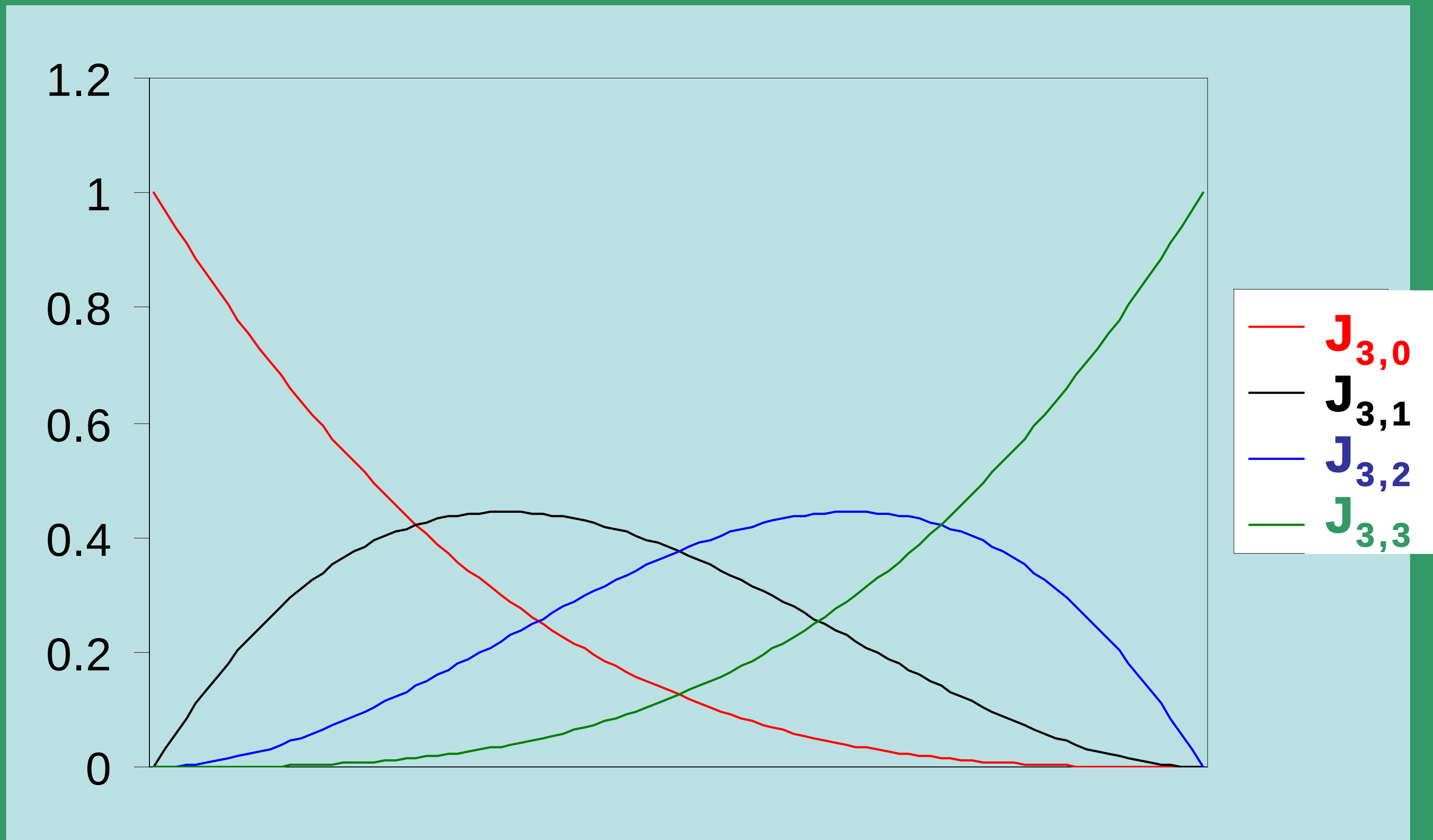
Get the expressions of $J_{2,i}$ and $J_{4,i}$

Bezier Curve Examples

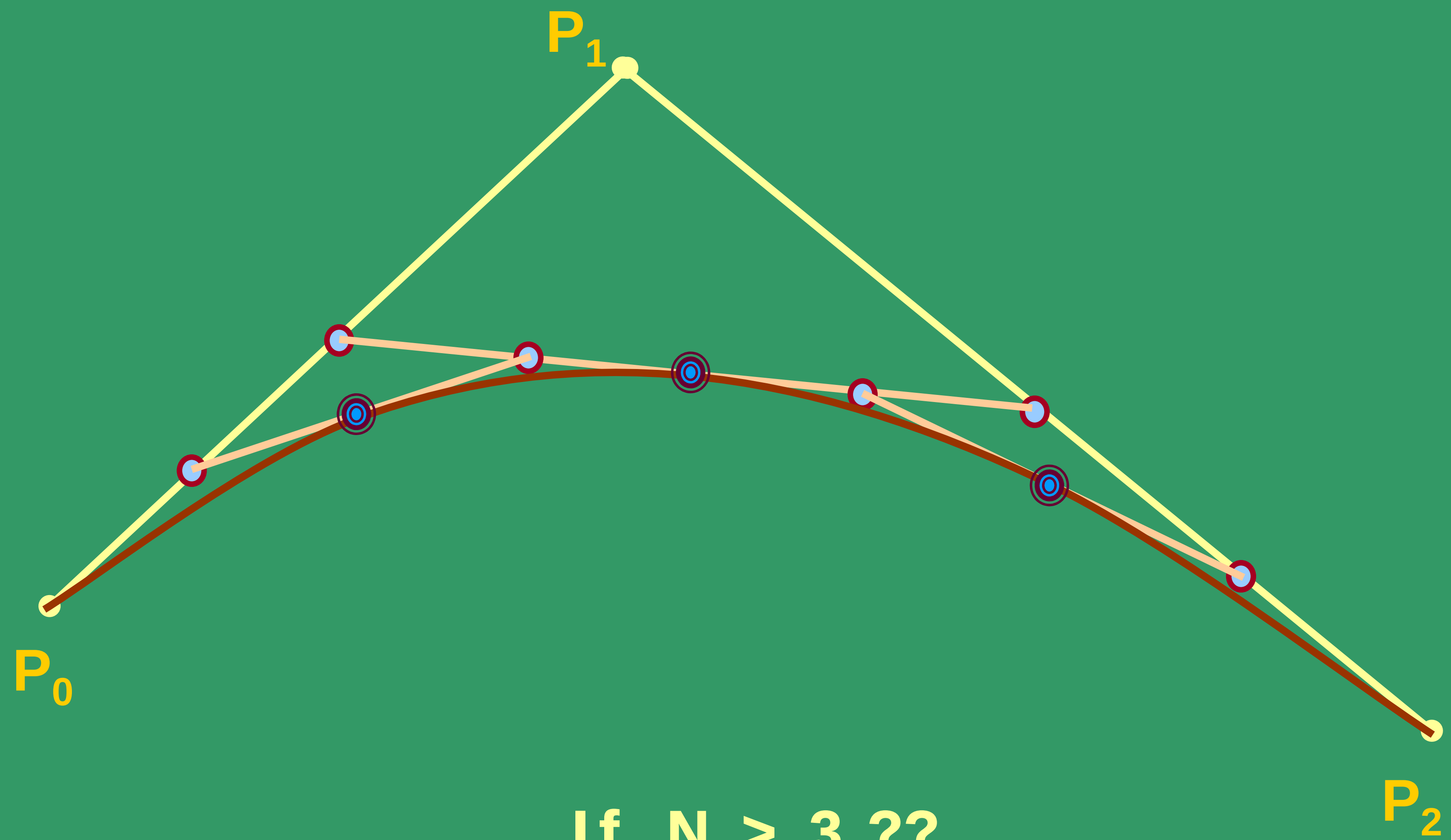




Bezier Basis Functions

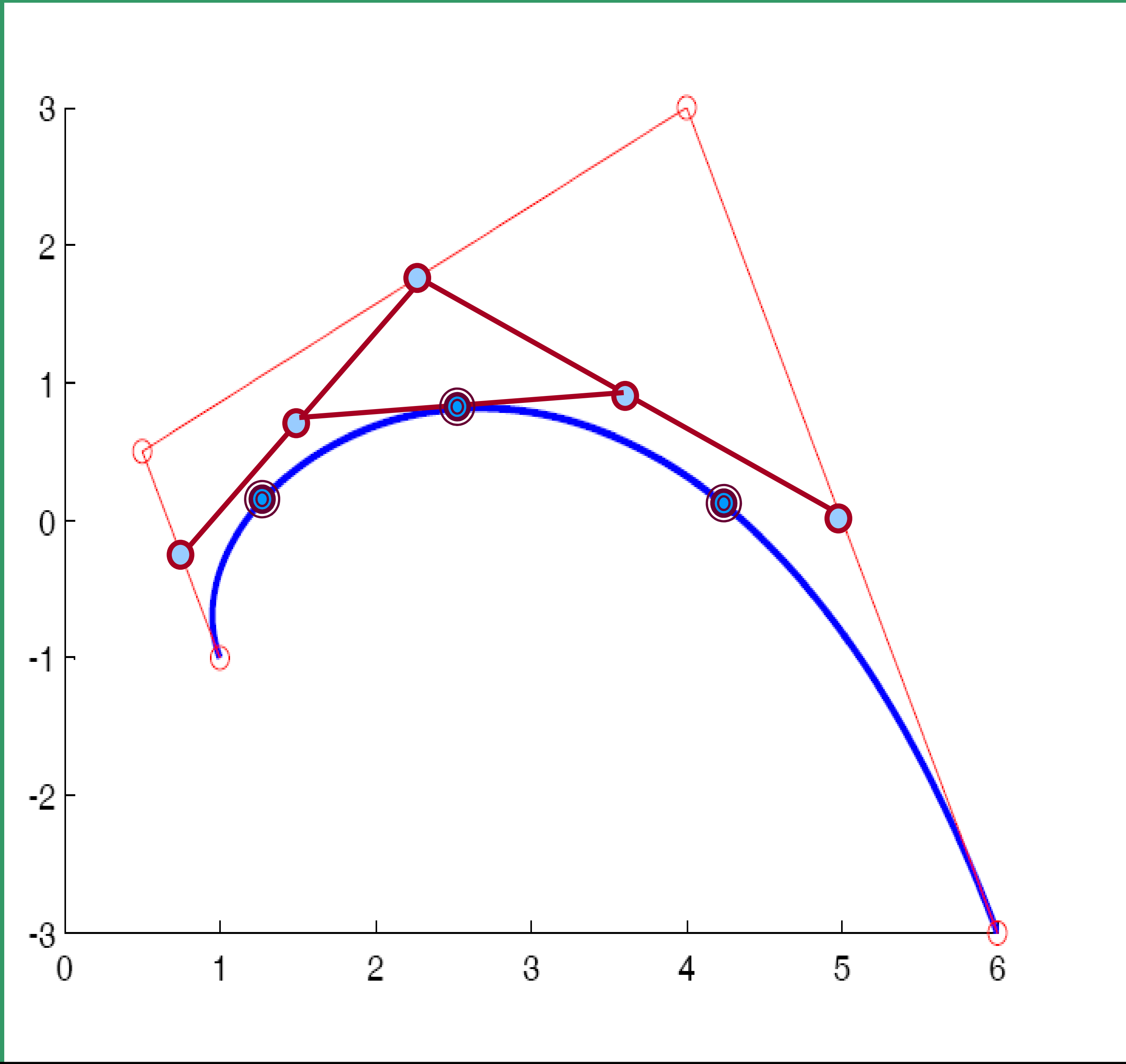


Recursive geometric definition of BEZIER CURVES



If, N > 3 ??

Recursive Bezier Curve Example



Read about:

- **B-splines represented as blending functions**
- **Conversion between one format to another.**
- **Knots and control points.**
- **When B-spline becomes a Bezier?**

QUADRICS – 3-D analogue of conics:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Jz + K = 0$$

QUADRIC SURFACES

Some trivial examples:

SPHERE

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2;$$

$$x=r.\cos\phi.\cos\theta,\ -\pi/2\le\phi\le\pi/2$$

$$y=r.\cos\phi.\sin\theta,\ \ -\pi\le\phi\le\pi$$

$$z=r.\sin\phi.$$

ELLIPSOID

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1;$$

$$x=a.\cos\phi.\cos\theta,\ -\pi/2\le\phi\le\pi/2$$

$$y=b.\cos\phi.\sin\theta,\quad -\pi\le\phi\le\pi$$

$$z=c.\sin\phi.$$

TORUS

$$\left[r-\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2}\right]^2+\left(\frac{z}{c}\right)^2=1;$$

$$x=a.(r+\cos\phi).\cos\theta,\ -\pi\le\phi\le\pi$$

$$y=b.(r+\cos\phi).\sin\theta,\quad -\pi\le\phi\le\pi$$

$$z=c.\sin\phi.$$

SUPERELLIPSOID

$$\left[\left(\frac{x}{a}\right)^{2/s_2} + \left(\frac{y}{b}\right)^{2/s_2}\right]^{s_2/s_1} + \left(\frac{z}{c}\right)^{2/s_1} = 1;$$

$$x = a.\cos^{s_1}\phi.\cos^{s_2}\theta, \ -\pi/2 \le \phi \le \pi/2$$

$$y = b.\cos^{s_1}\phi.\sin^{s_1}\theta, \ \ -\pi \le \phi \le \pi$$

$$z = c.\sin s_1\phi.$$

SUPERQUADRICS:

$$(\alpha x)^n + (\beta y)^n + (\gamma z)^n = k$$

General expression of a Quadric Surface

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz$$
$$+ Gx + Hy + Jz + K = 0.$$

The above is a generalization of the general conic equation in 3-D. In matrix form, it is:

$$XSX^T = 0,$$

$$\Rightarrow \begin{bmatrix} x & y & z & 1 \end{bmatrix} (1/2) \begin{bmatrix} 2A & D & F & G \\ D & 2B & E & H \\ F & E & 2C & J \\ G & H & J & 2K \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

Parametric forms of the quadric surfaces, are often used in computer graphics

Ellipsoid :

$$x = a\cos(\theta).\sin(\phi);\ \ 0 \le \theta \le 2\pi;$$
$$y = b\sin(\theta).\sin(\phi);\ \ 0 \le \phi \le 2\pi;$$
$$z = c\cos(\phi);$$

Elliptic Cone :

$$x = a\phi\cos(\theta);\ \ 0 \le \theta \le 2\pi$$
$$y = b\phi\sin(\theta);\ \ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = c\phi$$

Hyperbolic Paraboloid :

$$x = a\phi\cosh(\theta);\ \ -\pi \le \theta \le \pi$$
$$y = b\phi\sinh(\theta);\ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = \phi^2$$

Elliptic Paraboloid :

$$x = a\phi\cos(\theta);\ \ 0 \le \theta \le 2\pi$$
$$y = b\phi\sin(\theta);\ \ 0 \le \phi \le \phi_{\max}$$
$$z = \phi^2$$

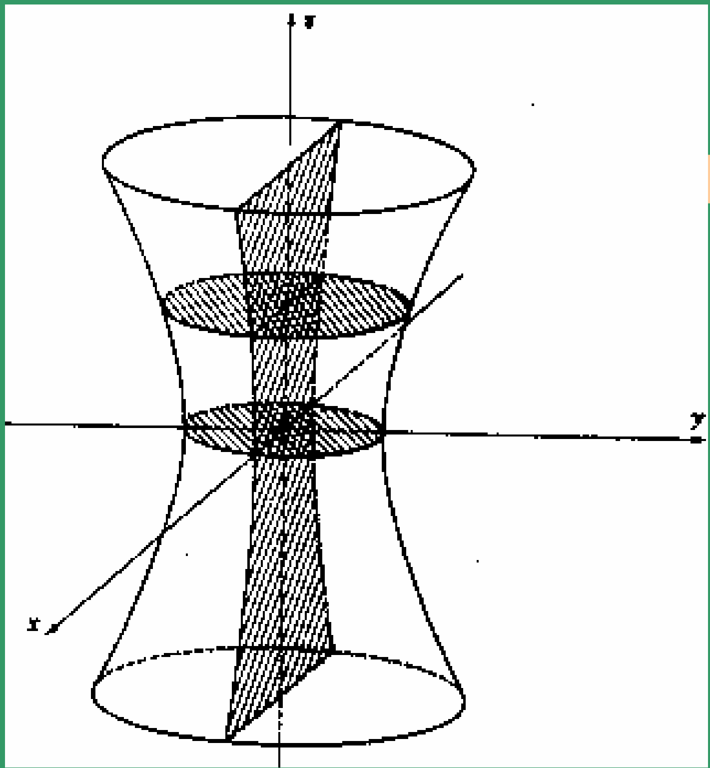
Hyperboloid:

$$x = a\cos(\theta)\cosh(\phi);\ \ 0 \le \theta \le 2\pi$$
$$y = b\sin(\theta)\sinh(\phi);\ \ -\pi \le \phi \le \pi$$
$$z = \sinh(\phi)$$

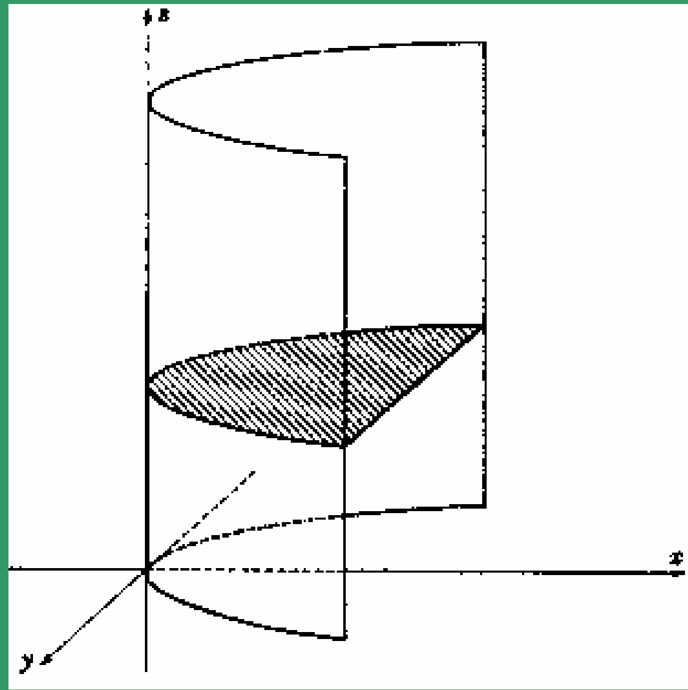
Parabolic Cylinder :

$$x = a\theta^2;\ \ 0 \le \theta \le \theta_{\max}$$
$$y = 2a\theta;\ \ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = \phi$$

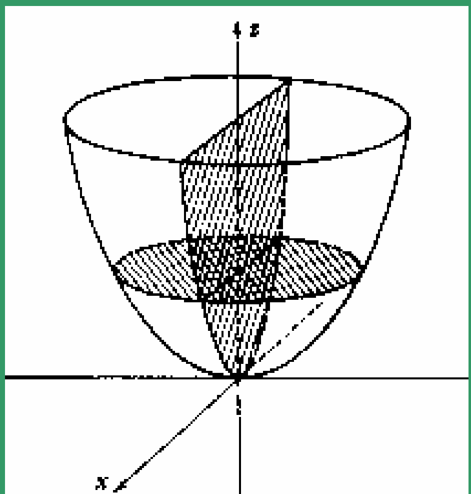
Some examples of Quadric Surfaces



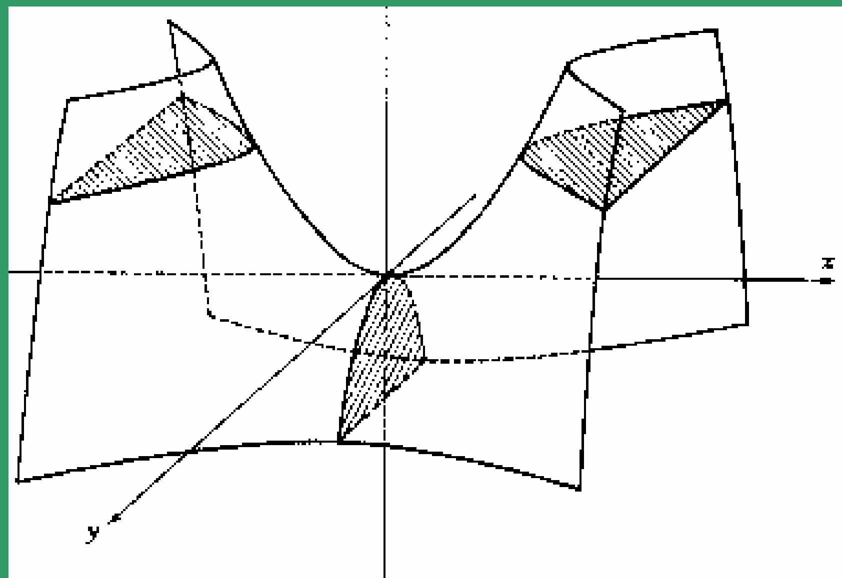
Hyperboloid



Parabolic Cylinder



Elliptic Paraboloid



Hyperbolic Paraboloid

BEZIER Surfaces

- Degree of the surface in each parametric direction is one less than the number of defining polygon vertices in that direction
- Surface generally follows the shape of the defining polygon net
- Continuity of the surface in each parametric direction is two less than the number of defining polygon net
- Only the corner points of the defining polygon net and the surface are coincident
- The surface is contained within the convex hull of the defining polygon
- Surface is invariant under any affine transformation.

Equation of a parametric Bezier surface:

$$J_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i};$$

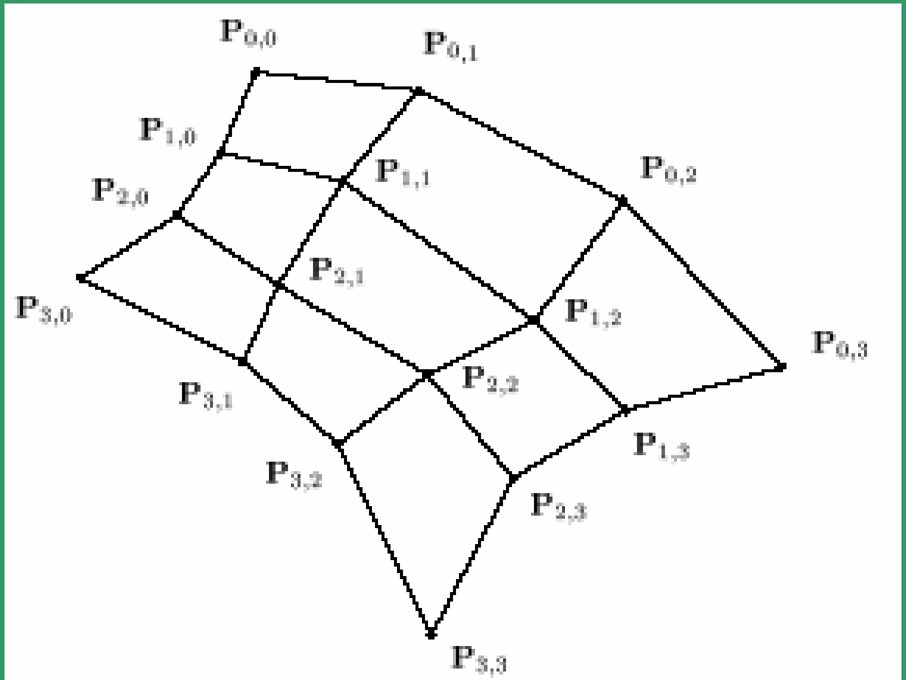
$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

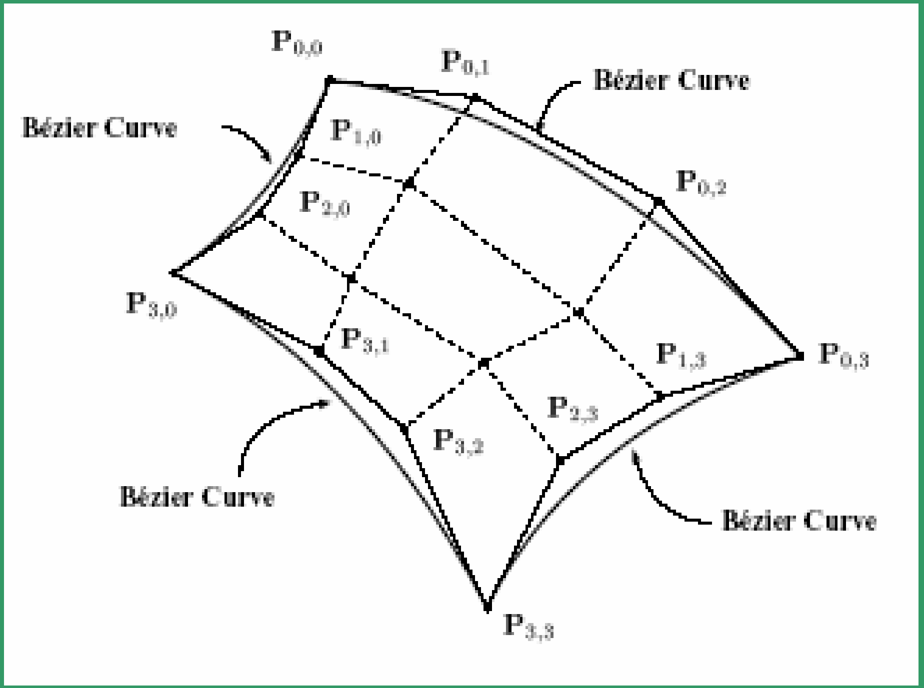
$$Q(u,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} J_{n,i}(u) K_{m,j}(w);$$

$$K_{m,j}(w) = \binom{m}{j} w^j (1-w)^{m-j};$$

$$\binom{m}{j} = \frac{m!}{j!(m-j)!}$$

BEZIER Surfaces





$$Q(u,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} J_{n,i}(u) K_{m,j}(w)$$

$$= \sum_{i=0}^{n} [\sum_{j=0}^{m} P_{i,j} J_{n,i}(u)] \, K_{m,j}(w);$$

BEZIER Surface in matrix form:

$$Q(u,w) = U.N.B.M^T W;$$

where,

$$U = [u^n \quad u^{n-1} \quad . \quad . \quad 1],$$

$$W = [w^m \quad w^{m-1} \quad . \quad . \quad 1]^T,$$

$$B = \begin{bmatrix} B_{0,0} & . & . & B_{0,m} \\ . & . & . & . \\ . & . & . & . \\ B_{n,0} & . & . & B_{n,m} \end{bmatrix}$$

4x4 bicubic BEZIER Surface in matrix form:

$$Q(u,w)=$$

$$\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3} \\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,2} \\ B_{2,0} & B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,0} & B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix}$$

$$X \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix};$$

Non-square 4x4 bicubic BEZIER Surface in matrix form:

$$Q(u,w) =$$

$$\begin{bmatrix} u^4 & u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & -12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} \\ B_{1,0} & B_{1,1} & B_{1,2} \\ B_{2,0} & B_{2,1} & B_{2,2} \\ B_{3,0} & B_{3,1} & B_{3,2} \\ B_{4,0} & B_{4,1} & B_{4,2} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w^2 \\ w \\ 1 \end{bmatrix};$$

End of Lectures on

CURVES
and SURFACE
REPRESENTATION