CS 6846: Quantum Computing 15th August, 2024

Homework 1

Instructor: Shweta Agrawal *Due: 26th August at 5 PM*

Instructions:

1. Please type up your solutions using latex.
2. Please email TA and instructor.
3. You may collaborate with other students. Please mention the names of your collaborators or any other source that you use for the solution.
4. Please type up your solutions *individually* without any help.

Problem 1: Building Gates

1. (2 points) Construct a CNOT from two Hadamard gates and one controlled-Z (the controlled-Z gate maps $|11\rangle$ to $-|11\rangle$ and acts like the identity on the other basis states).
2. (3 points) A SWAP-gate interchanges two qubits: it maps basis state $|a, b\rangle$ to $|b, a\rangle$. Implement a SWAP gate using a few CNOTs (when using a CNOT, you're allowed to use either of the 2 bits as the control, but be explicit about this).

Problem 2: Understanding Unitary

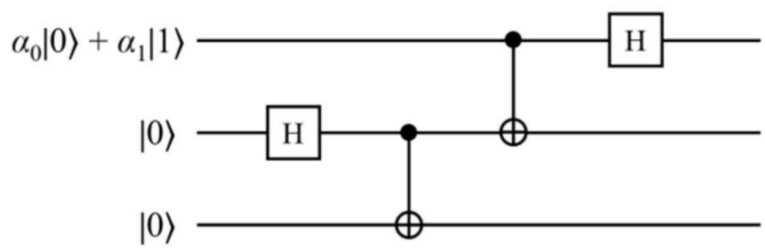
A matrix A is inner product-preserving if the inner product $\langle Av, Aw\rangle$ between Av and Aw equals the inner product $\langle v, w\rangle$, for all vectors v, w. A is norm-preserving if $||Av|| = ||v||$ for all vectors v, i.e., A preserves the Euclidean length of the vector. A is unitary if $A^\dagger A = AA^\dagger = I$. In the following, you may assume for simplicity that the entries of the vectors and matrices are real, not complex.

1. (2 points) Prove that A is norm-preserving if, and only if, A is inner product-preserving.
2. (2 points) Prove that A is inner product-preserving iff $A^\dagger A = AA^\dagger = I$.
3. (1 point) Conclude that A is norm-preserving iff A is unitary.

Bonus (3 points): prove the same for complex instead of real vector spaces.

Problem 3: Quantum Circuits

Consider the following quantum circuit:



- (3 points) Determine with proof the state of the three qubits at the end of the circuit's operation.
- (2 points) If we then measure the three qubits, give the outcomes and their probabilities that arise.

Problem 4: Playing with Quantum Gates

(a) (1 point) Show that a bit flip operation preceded and followed by Hadamard transforms equals a phase flip operation, i.e., $HXH = Z$ (Z gate maps $|1\rangle$ to $-|1\rangle$ and acts like the identity on the other basis states).

(b) (2 points) Show that surrounding a CNOT gate with Hadamard gates switches the role of the control-bit and target-bit of the CNOT: $(H \otimes H)\text{CNOT}(H \otimes H)$ is the 2-qubit gate where the second bit controls whether the first bit is negated (i.e., flipped).

Problem 5: Understanding Basics

(a) (1 point) Simplify the following: $((\langle 0| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)$.

(b) (2 points) Suppose we have the state $\frac{1}{\sqrt{2}}(|0\rangle|\phi\rangle + |1\rangle|\psi\rangle)$, where $|\phi\rangle$ and $|\psi\rangle$ are unknown normalized quantum states with the same number of qubits. Suppose we apply a Hadamard gate to the first qubit and then measure that first qubit in the computational basis. Give the probability of measurement outcome 1, as a function of the states $|\phi\rangle$ and $|\psi\rangle$.

(c) (2 points) Show that unitaries cannot "delete" information: there is no 1-qubit unitary U that maps $|\phi\rangle \mapsto |0\rangle$ for every 1-qubit state $|\phi\rangle$.

Problem 6: Mixed State

(a) (3 points) Consider the following **Experiment 1**. Sample a random bit b, and produce the state $|\phi\rangle = |b\rangle$.
Prove that the output of this experiment cannot be described by a pure state alone. In particular, suppose toward contradiction that there was a pure state $|\psi\rangle$ that described the output of the above experiment. Show that, in fact, $|\psi\rangle$ can be distinguished from $|\phi\rangle$. To do so, devise a unitary matrix U (based on $|\psi\rangle$) such that if you apply U to $|\psi\rangle$ and $|\phi\rangle$ and measure, the outcomes of measurement will have different probability distributions.
A state $|\phi\rangle$ sampled from a probability distribution like the procedure above is known as a mixed state.

(b) (4 points) Consider the following **Experiment 2**. Sample a random bit b, and produce the state $|\phi'\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{(-1)^b}{\sqrt{2}}|1\rangle$. Equivalently, sample $|\phi\rangle$ according to **Experiment 1**, and then apply the Hadamard transformation.
Show that there is no unitary transformation followed by a measurement that distinguishes $|\phi\rangle$ sampled from **Experiment 1** from $|\phi'\rangle$ sampled from **Experiment 2**. That is, show that for any unitary U, if you apply U to $|\phi\rangle$ and measure, or apply U to $|\phi'\rangle$ and measure, the probability distributions in the two cases are identical.

(c) The right way to describe a mixed state is by density matrix. Suppose that state $|\phi\rangle$ is sampled with probability p_i. Then the density matrix is the matrix

$$\rho = \sum_i p_i |\phi_i\rangle \langle\phi_i|.$$

It turns out that the density matrix captures all statistical information about the mixed state. That is, no sequence of unitary operations and measurements can distinguish two mixed states with the same density matrix, and for any two states with different density matrices, there is a unitary and measurement that distinguishes the two (with some nonzero probability).

1. (2 points) A pure state is a special case of a mixed state where the probability distribution has support on only a single state. Therefore, pure states also have density matrices. What is special about the density matrix for a pure state?
2. (2 points) What is the density matrix for the output of **Experiment 1?** Combined with part (1), Why does this show that the state can be distinguished from any pure state?
3. (2 points) What is the density matrix for the output of **Experiment 2?**
4. (2 points) Given an arbitrary mixed state, suppose you apply a unitary U to the state. Explain how to transform the corresponding density matrix?

5. (2 points) Given an arbitrary mixed state, suppose you measure the state. Let q_j be the probability the measurement gives j. What is q_j in terms of the density matrix for the state? **Hint:** start by analyzing pure states and then build up to a mixed state from there.

6. (3 points) Mixed states are useful for characterizing the state that remains after performing a measurement. What is the density matrix for the state that results from measuring a pure state $|\phi\rangle$, in terms of the entries in $|\phi\rangle$?

7. (3 points) The result of measuring a mixed state is another mixed state. How does measuring transform the density matrix?

Problem 7: Generalizing Measurement

Here, we will discuss how to generalize our notion of a measurement. Consider a quantum state over set B of size n. Fix an arbitrary orthonormal basis $C = \{|b_1\rangle, \ldots, |b_n\rangle\}$ for the space $\mathbb{C}^n$. That is, $|b_i\rangle$ are all orthogonal vectors.
The result of measuring $|\phi\rangle$ in basis C is the following. First, the measurement will output i with probability $|\langle b_i \mid \phi\rangle|^2$. Then, the state will collapse to $|b_i\rangle$.

(a) (1 point) Explain why the probability distribution over i is in fact a probability distribution (that is, the probabilities sum to 1).

(b) (3 points) Show that measuring in basis C is equivalent to (1) applying a unitary U, (2) measuring in the computational basis, and (3) applying a unitary U'. Thus, without loss of generality, we can usually consider measuring in the computational basis.