An Embarrassingly Simple 2^n-Time Algorithm for SVP—And How We Hope to Improve It

Divesh Aggarwal
Noah Stephens-Davidowitz

Game Plan

- Basics of sieving
- Embarrassingly simple algorithm: Sieving by averages
- Hope to simplify the proof of correctness
- Hope to make it faster

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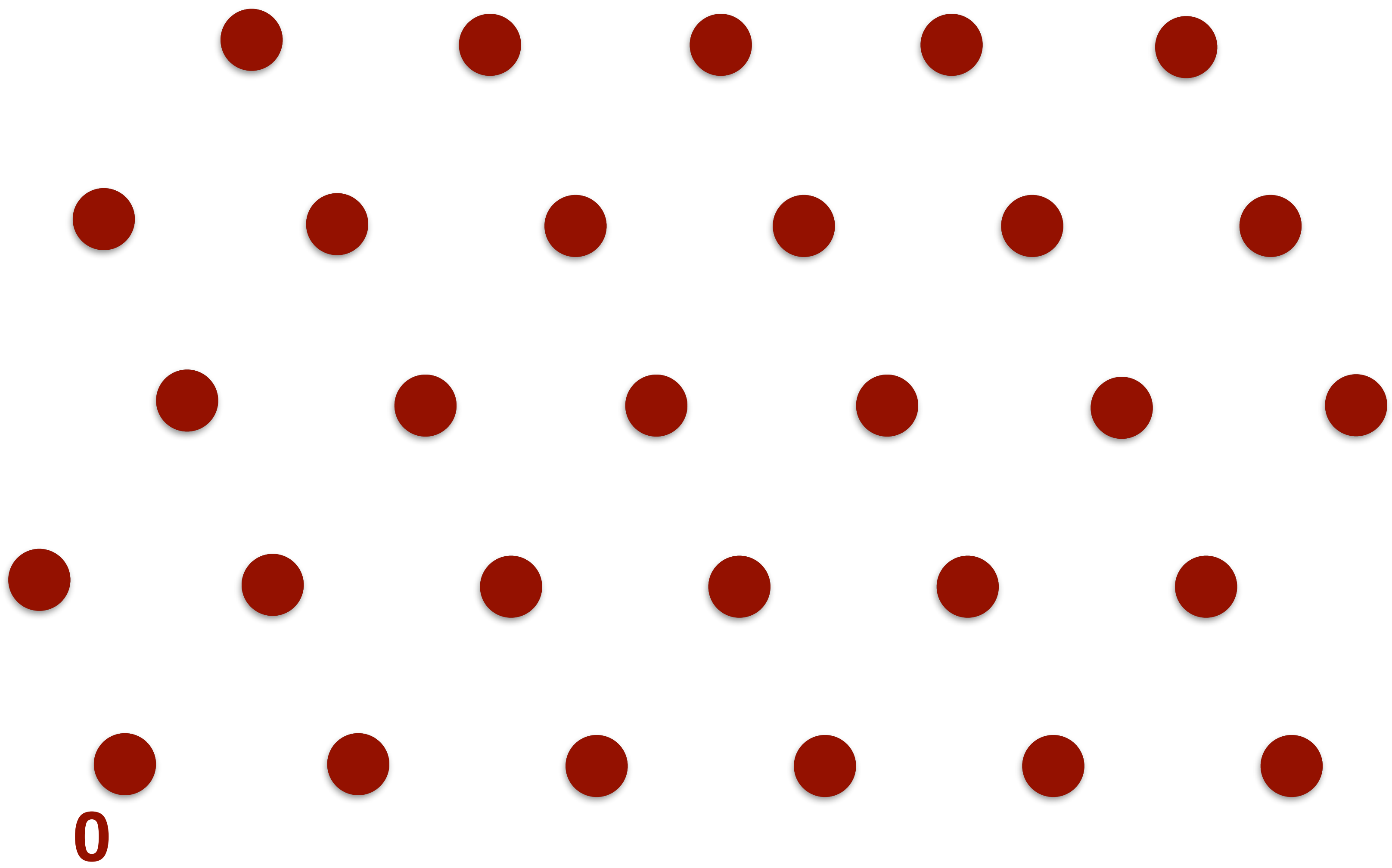

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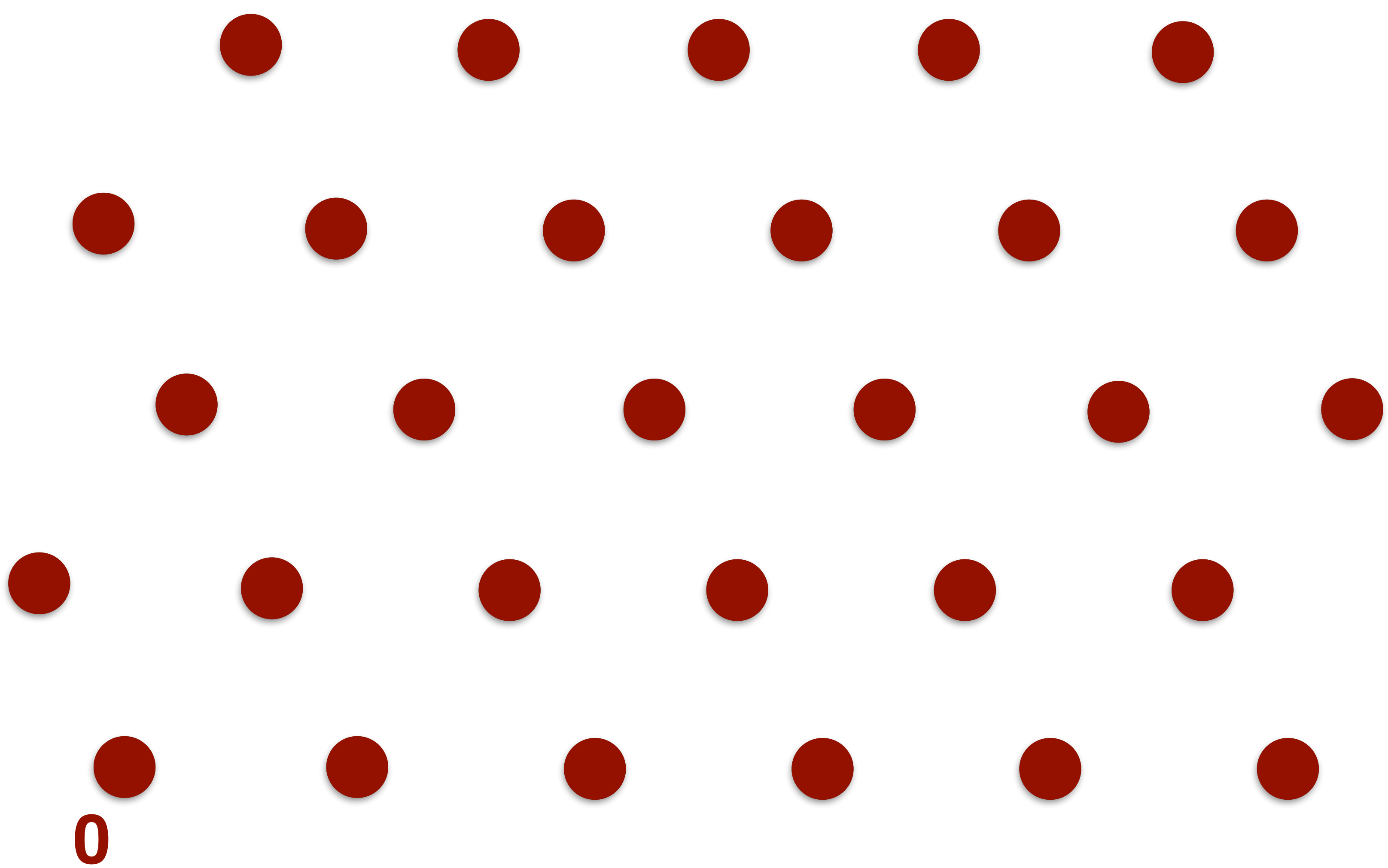

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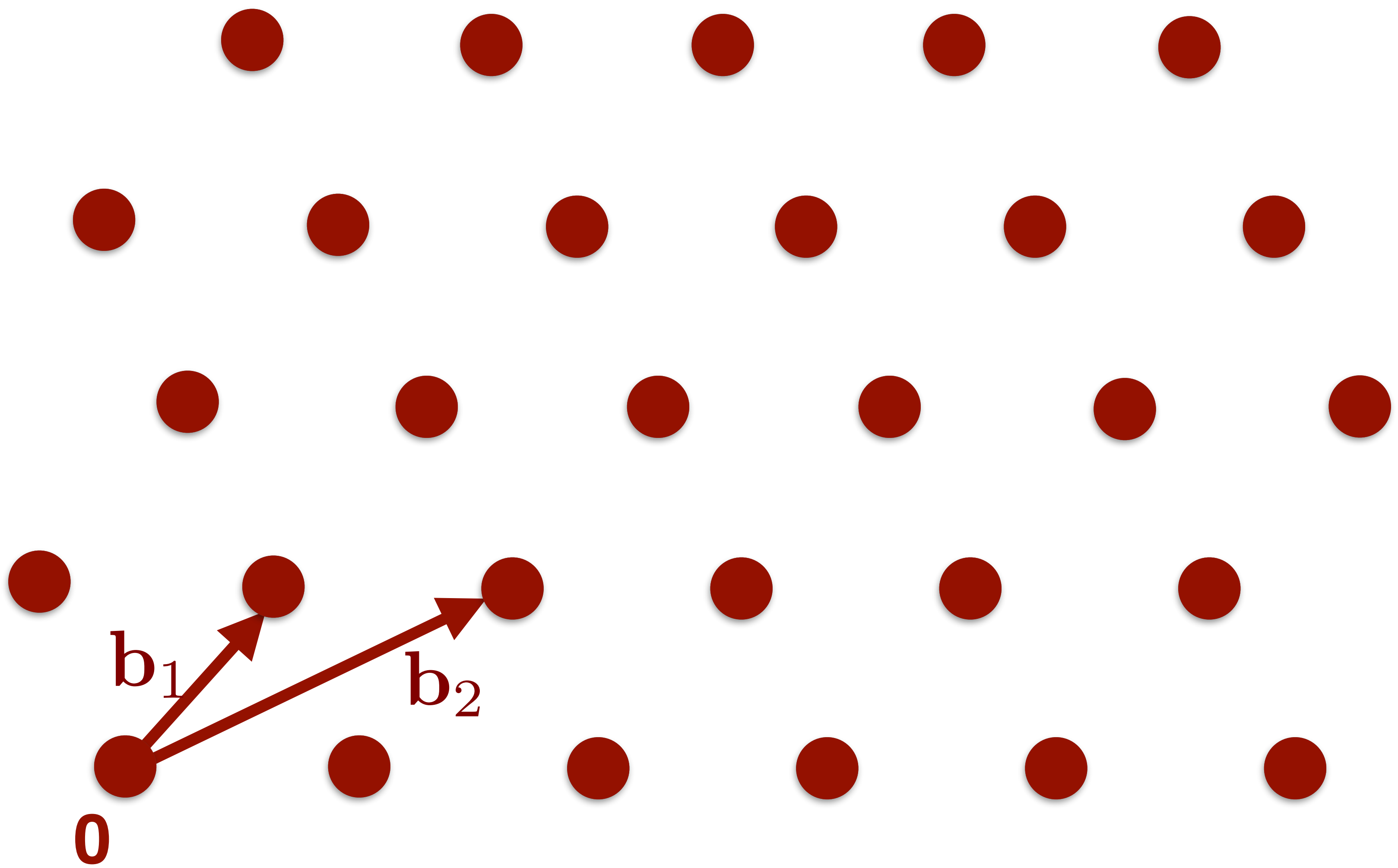

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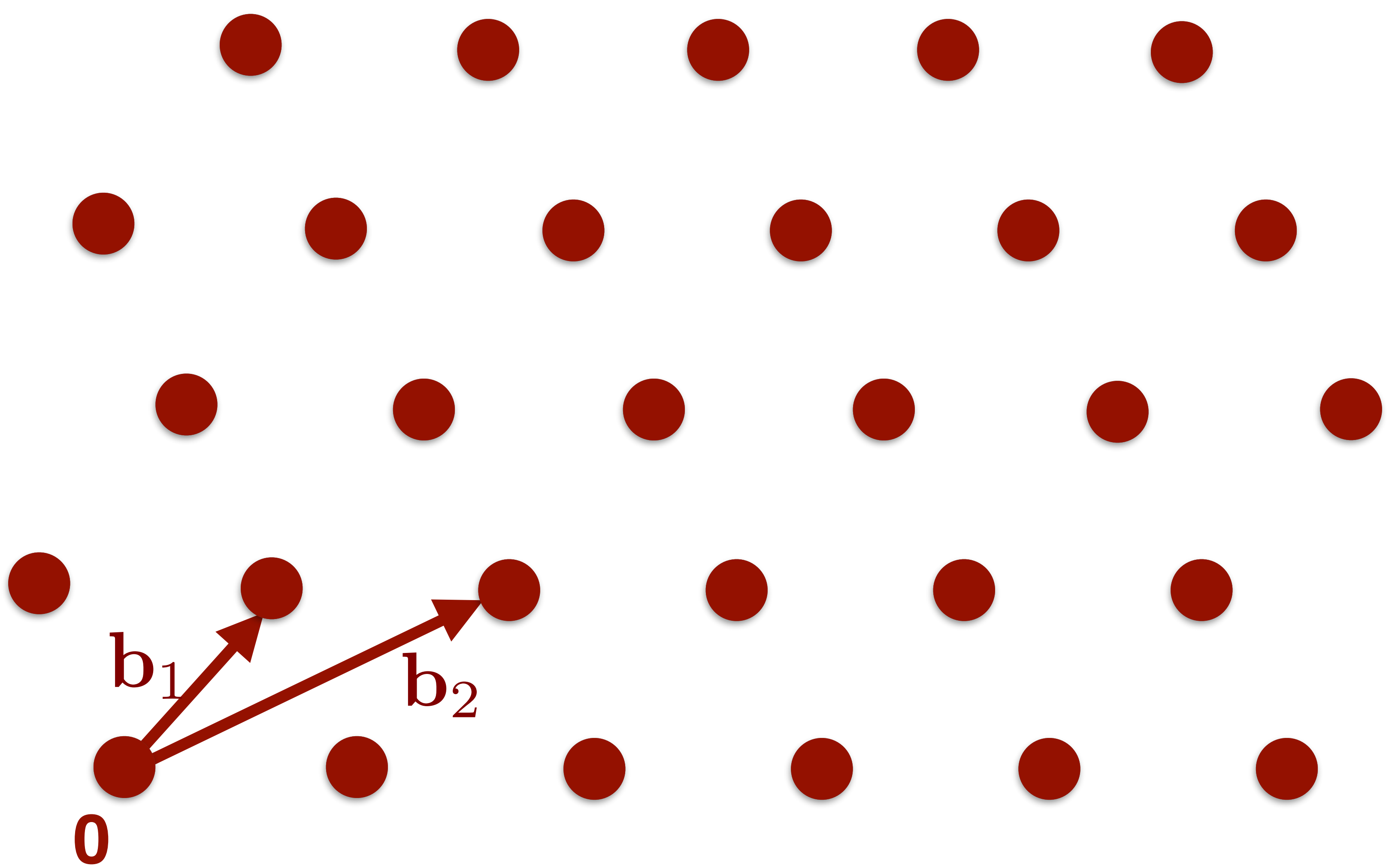
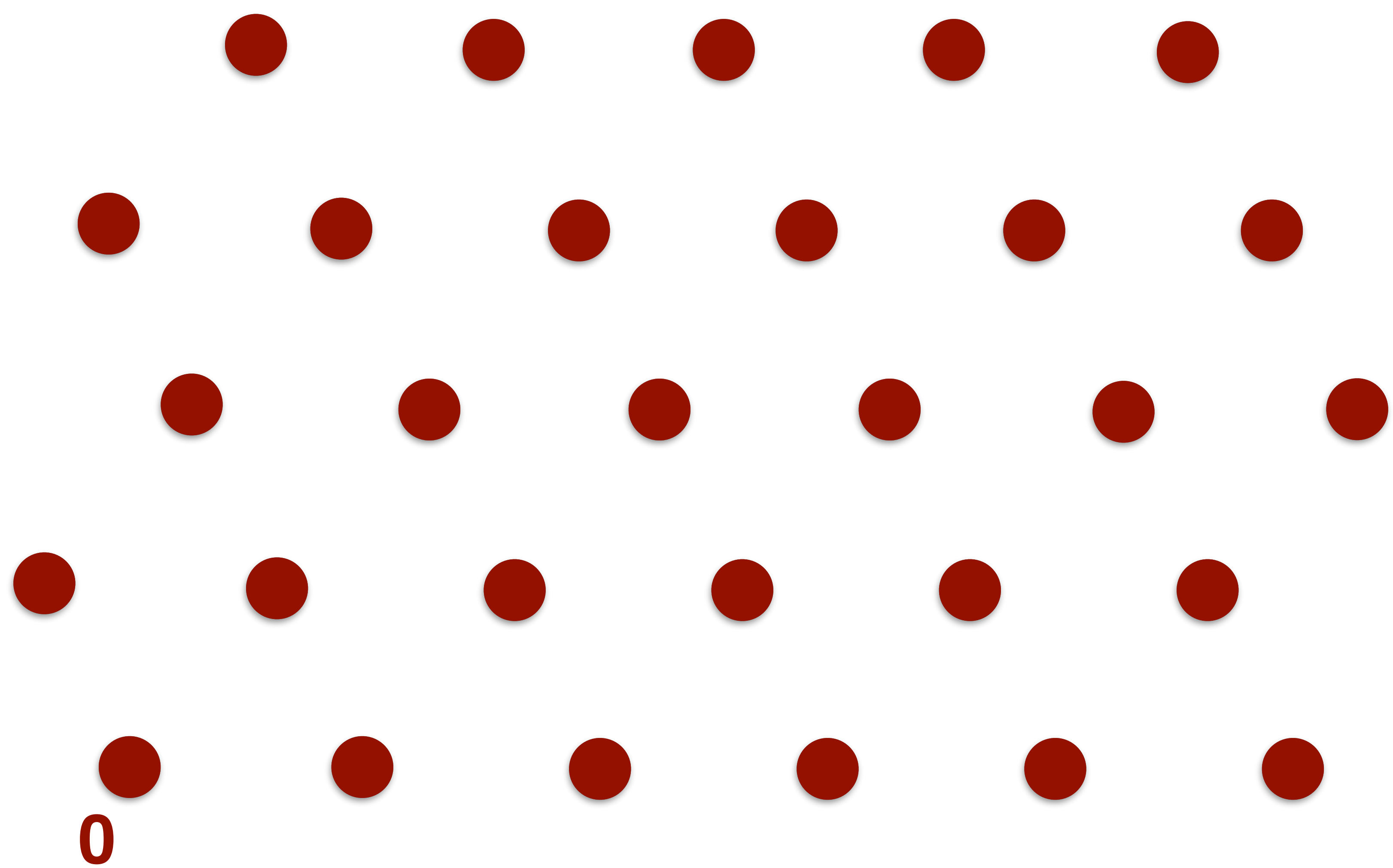


Shortest Vector Problem



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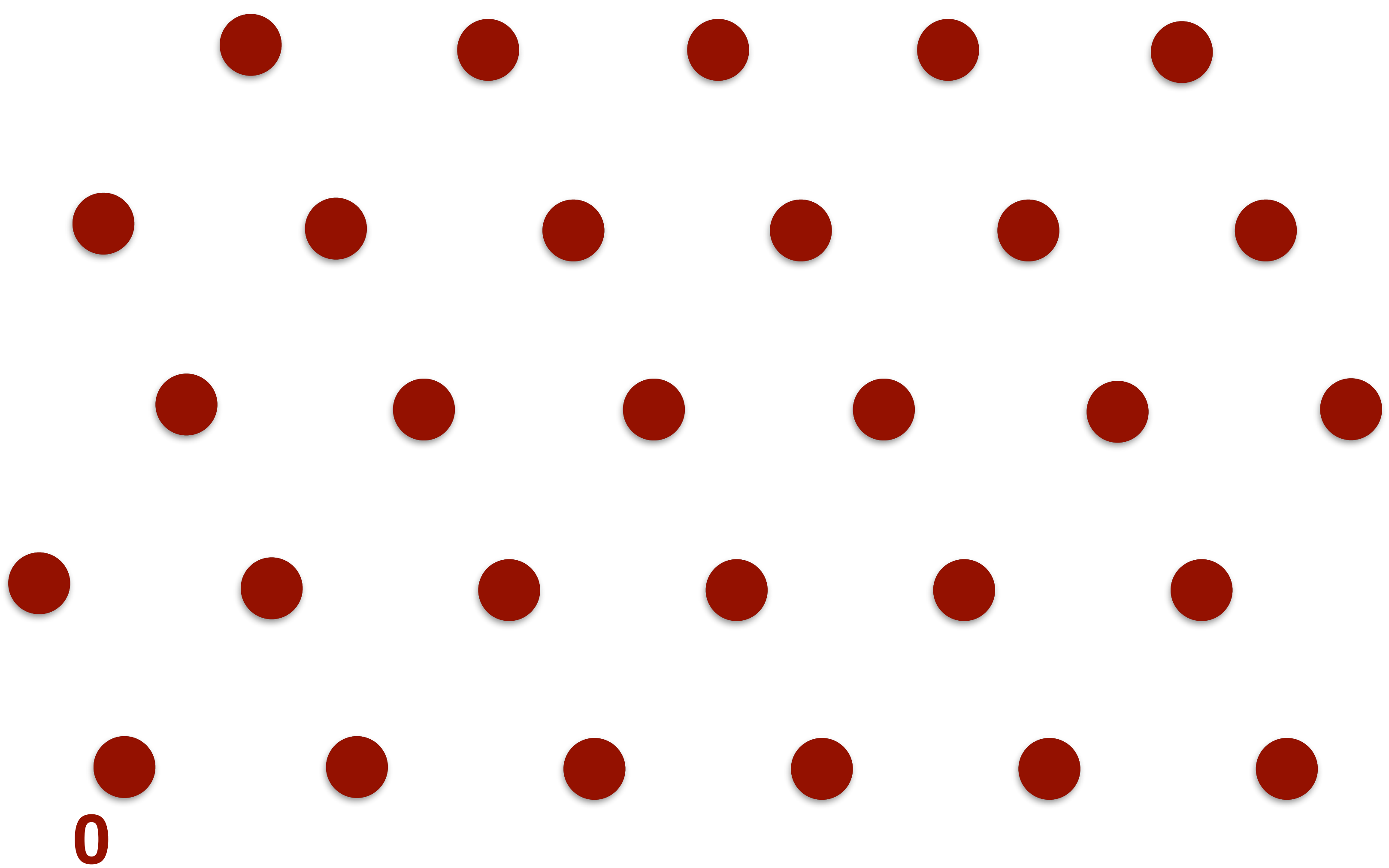

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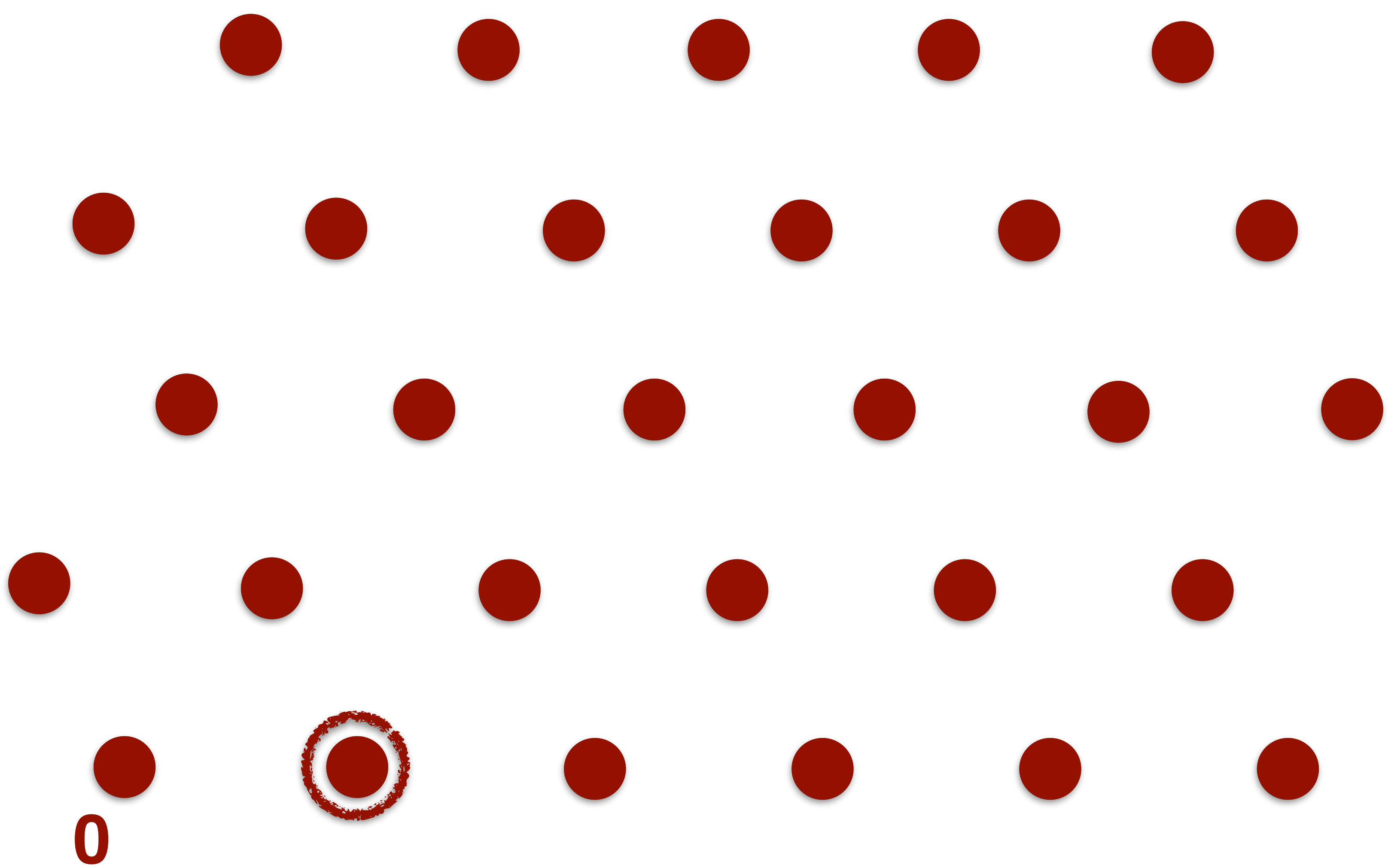

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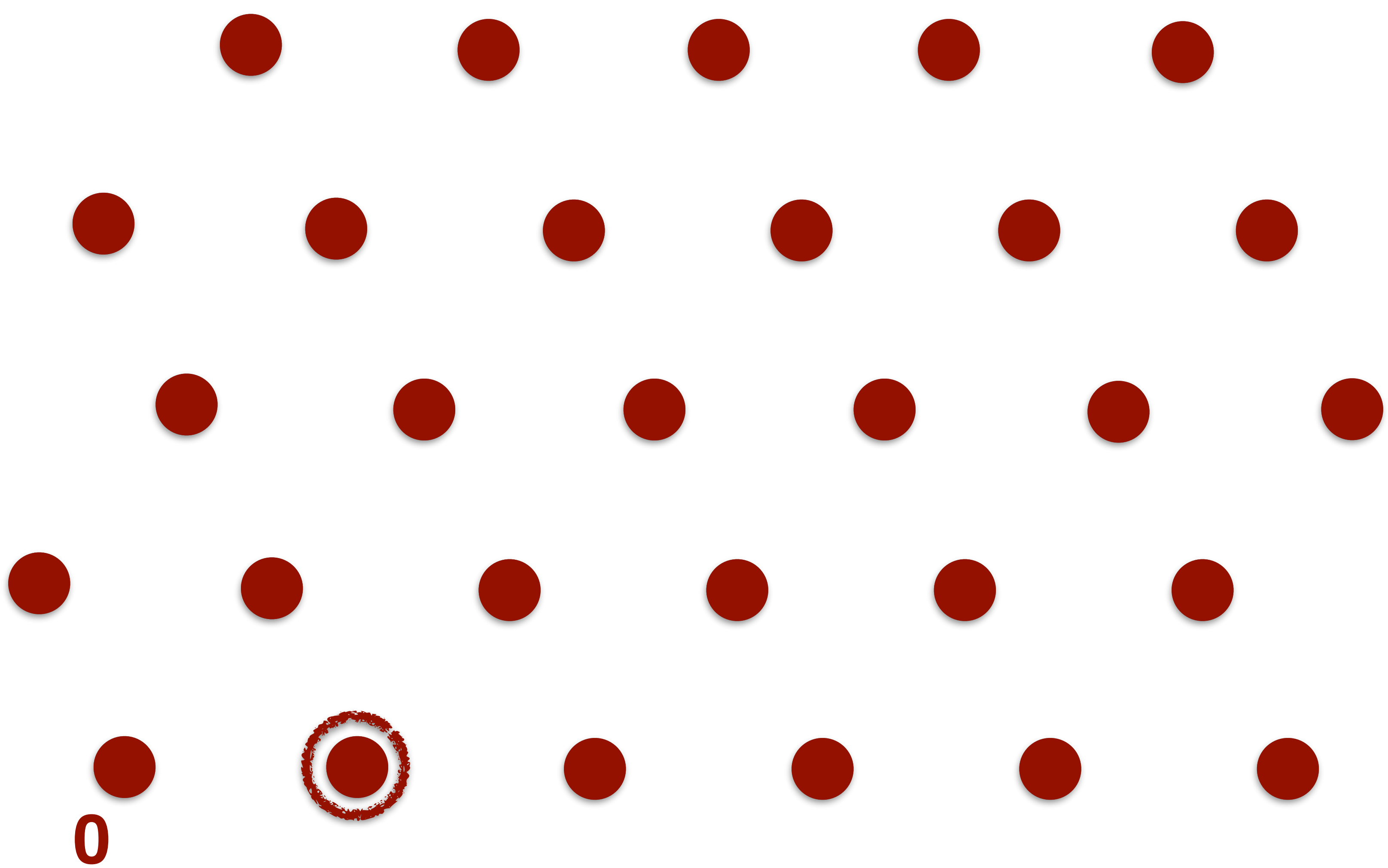

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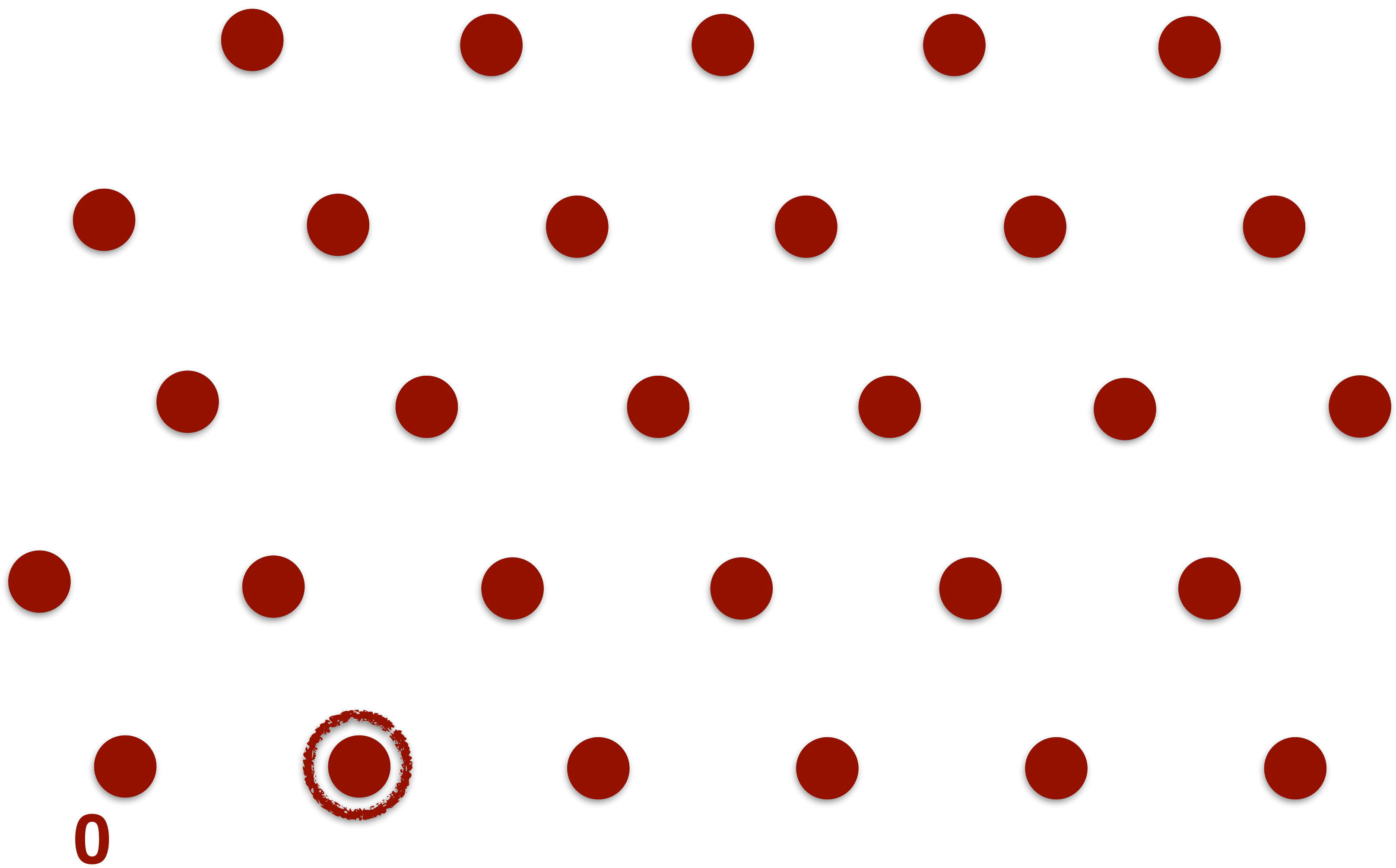

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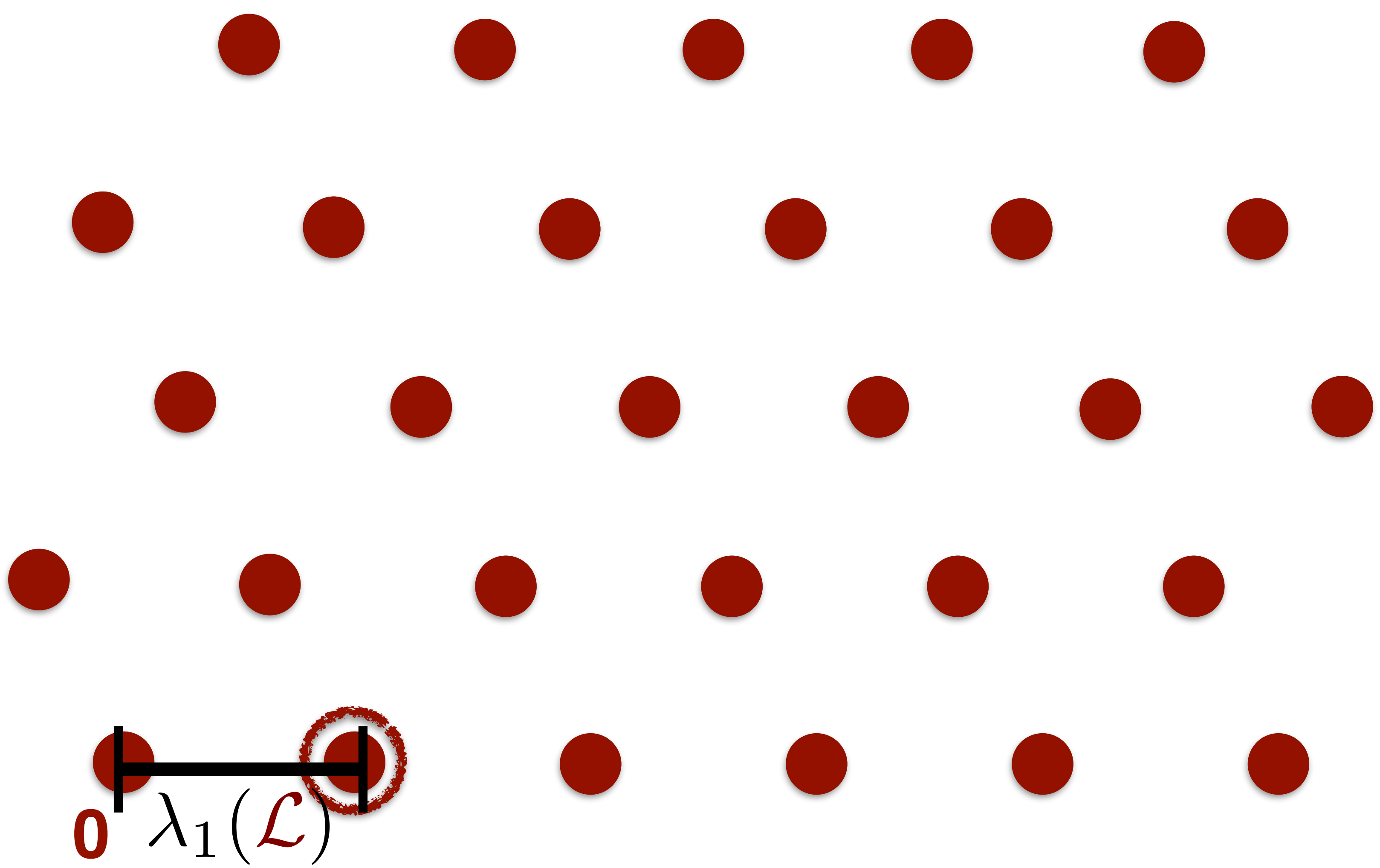

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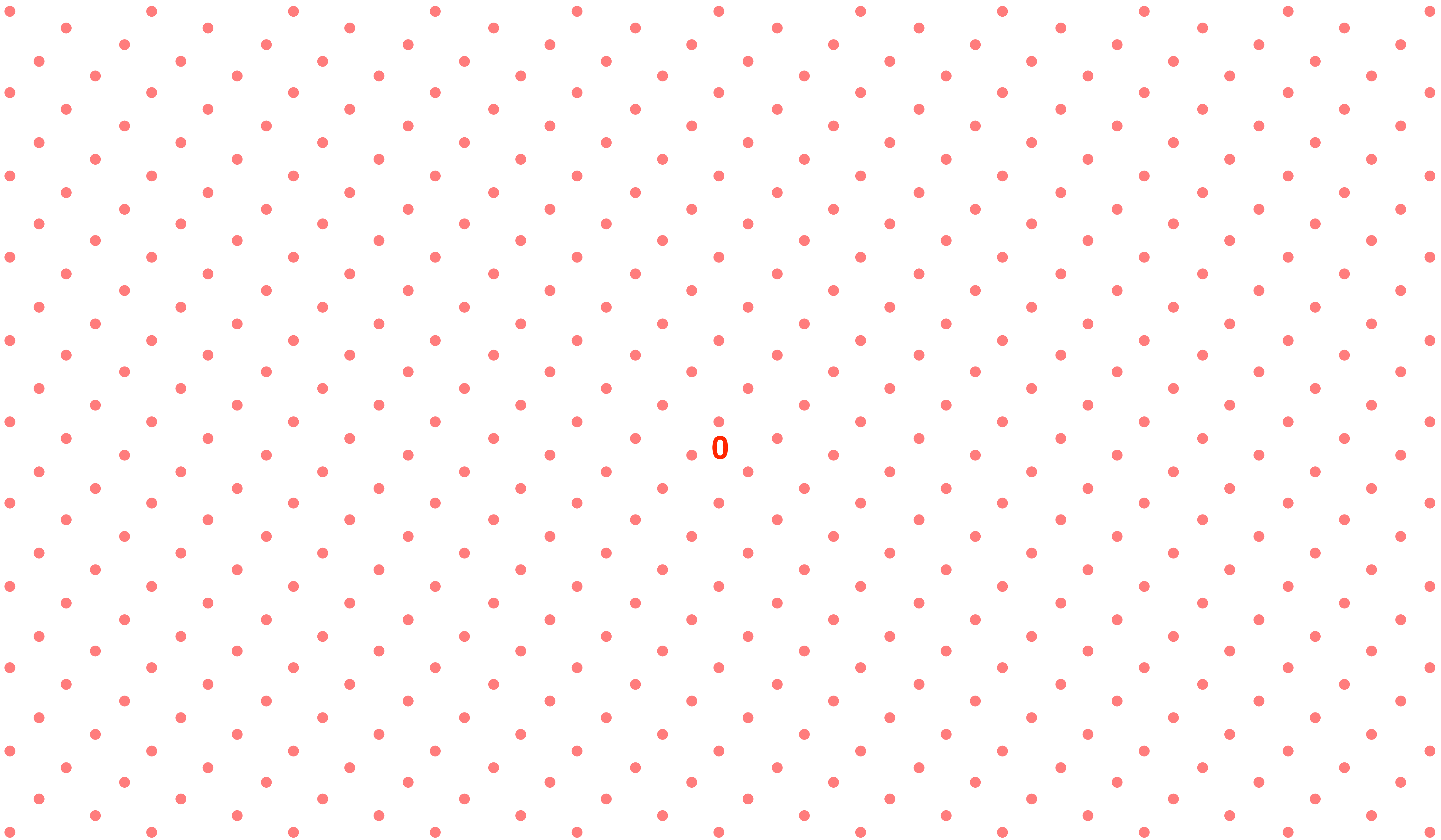

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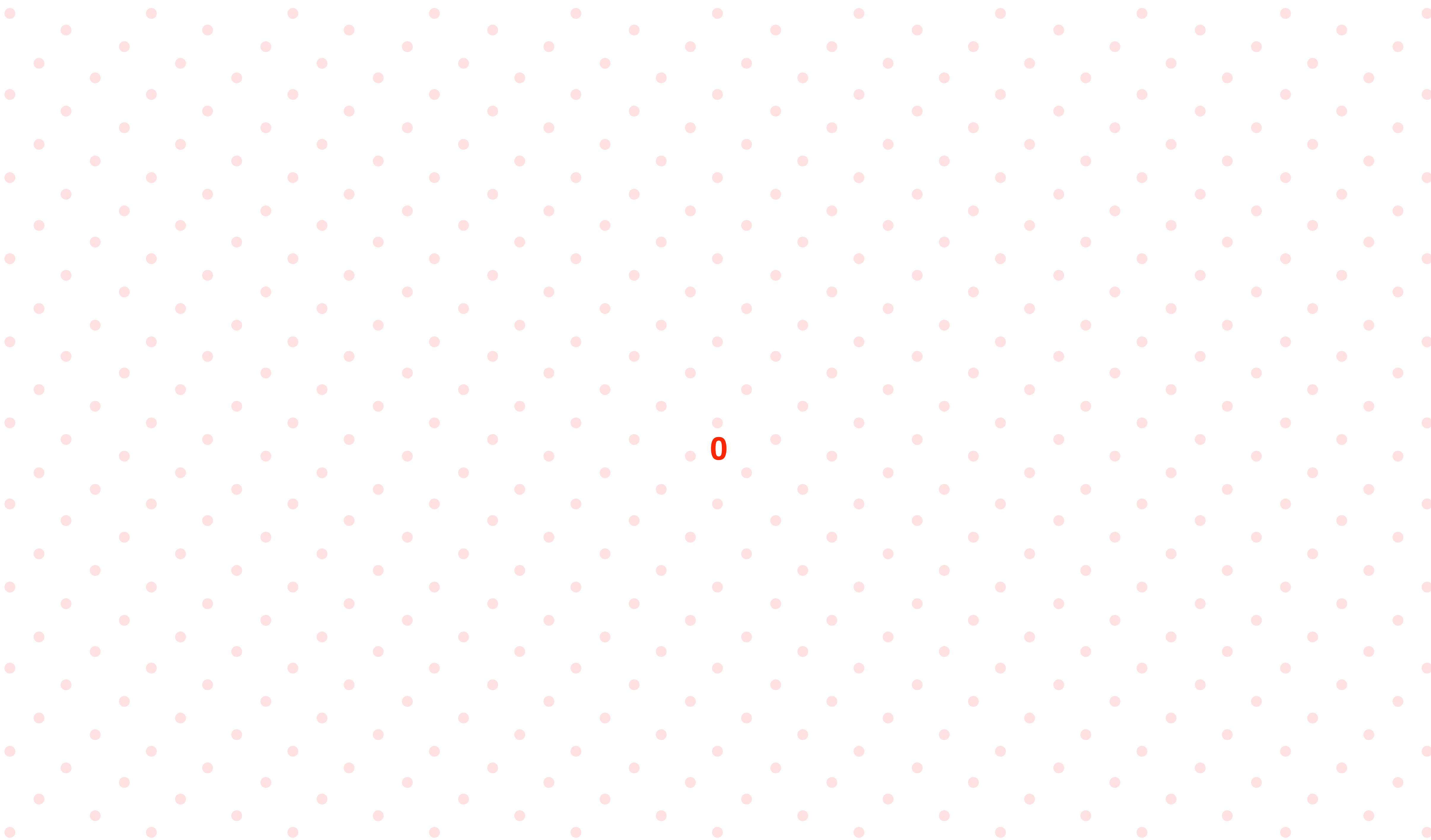

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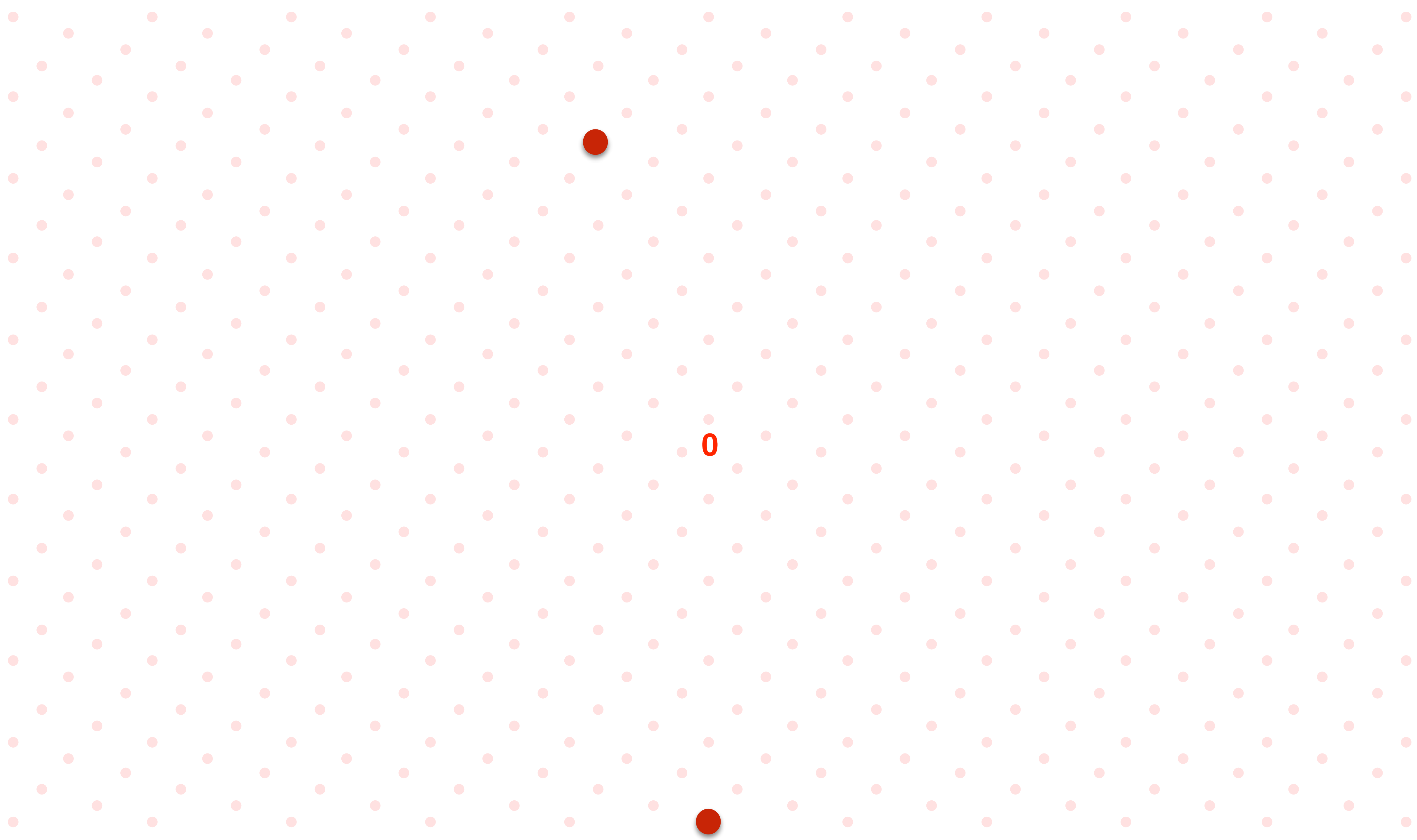

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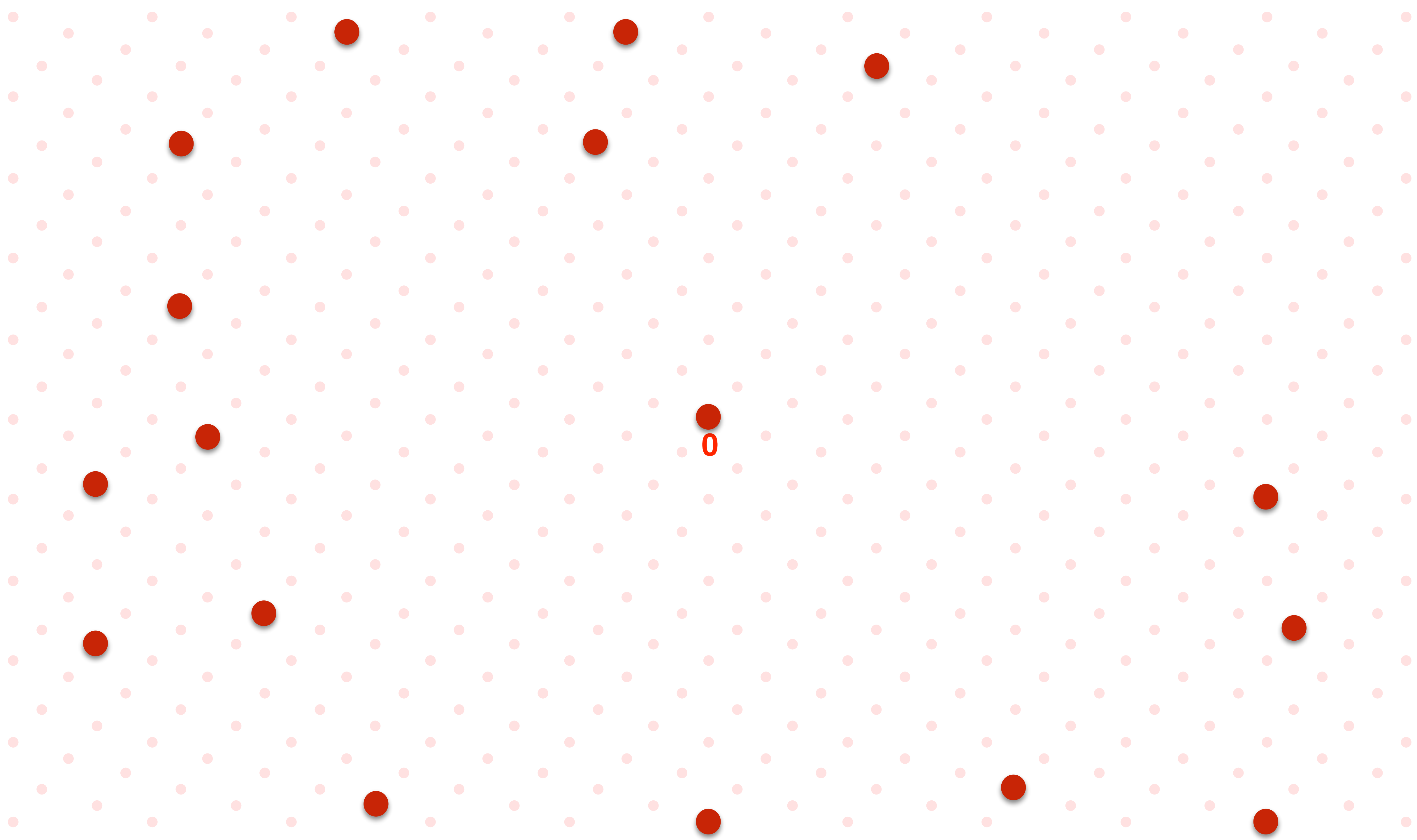

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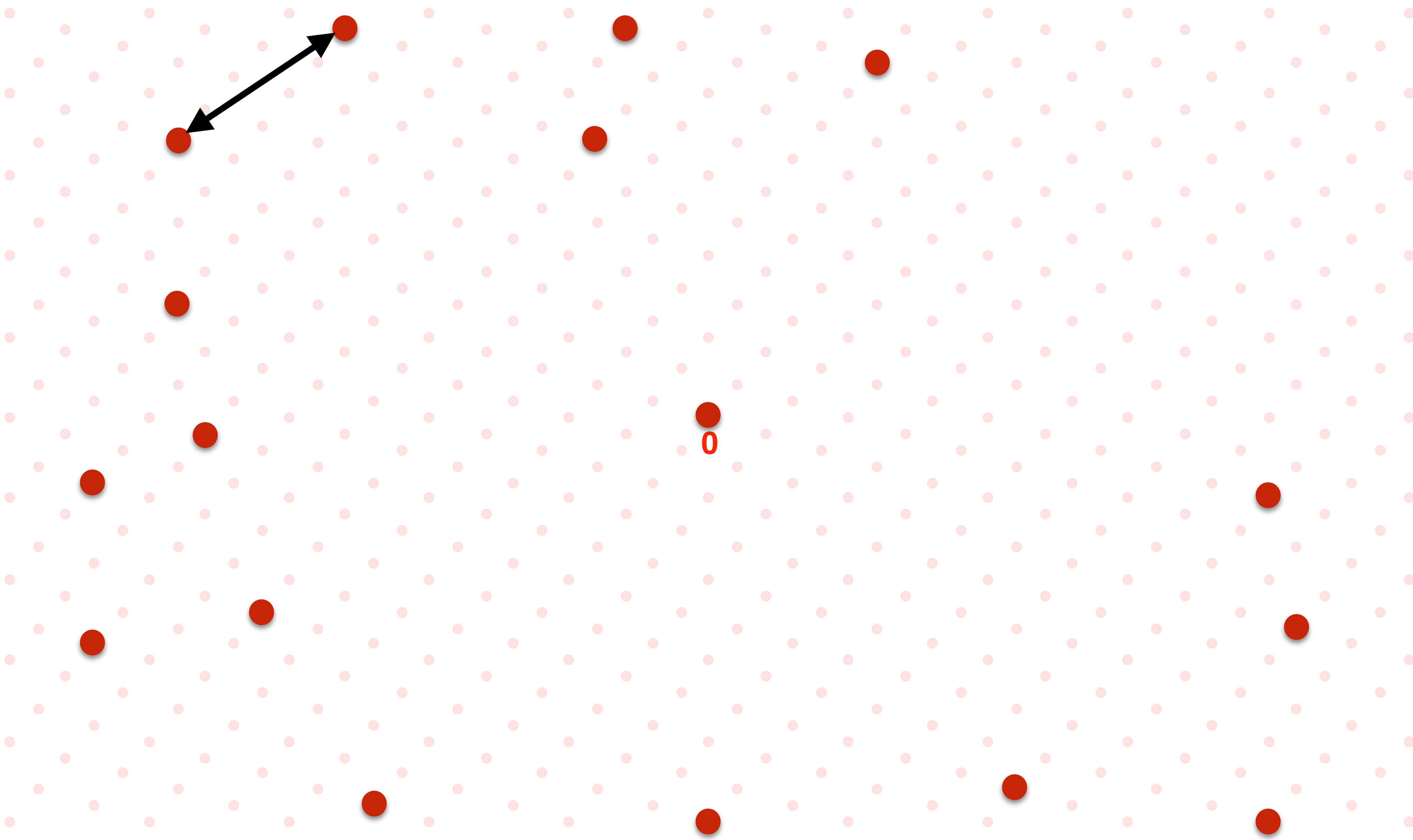

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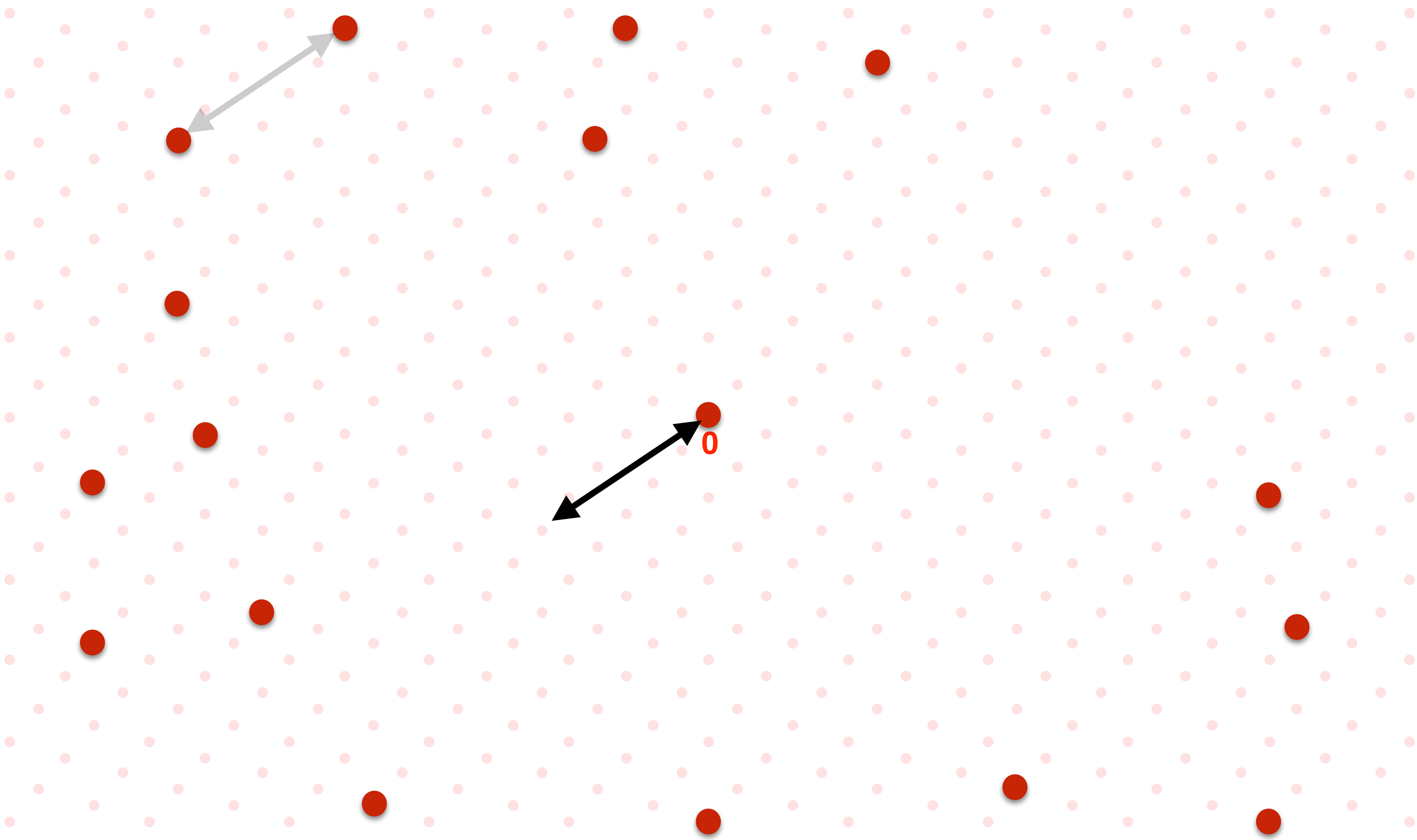

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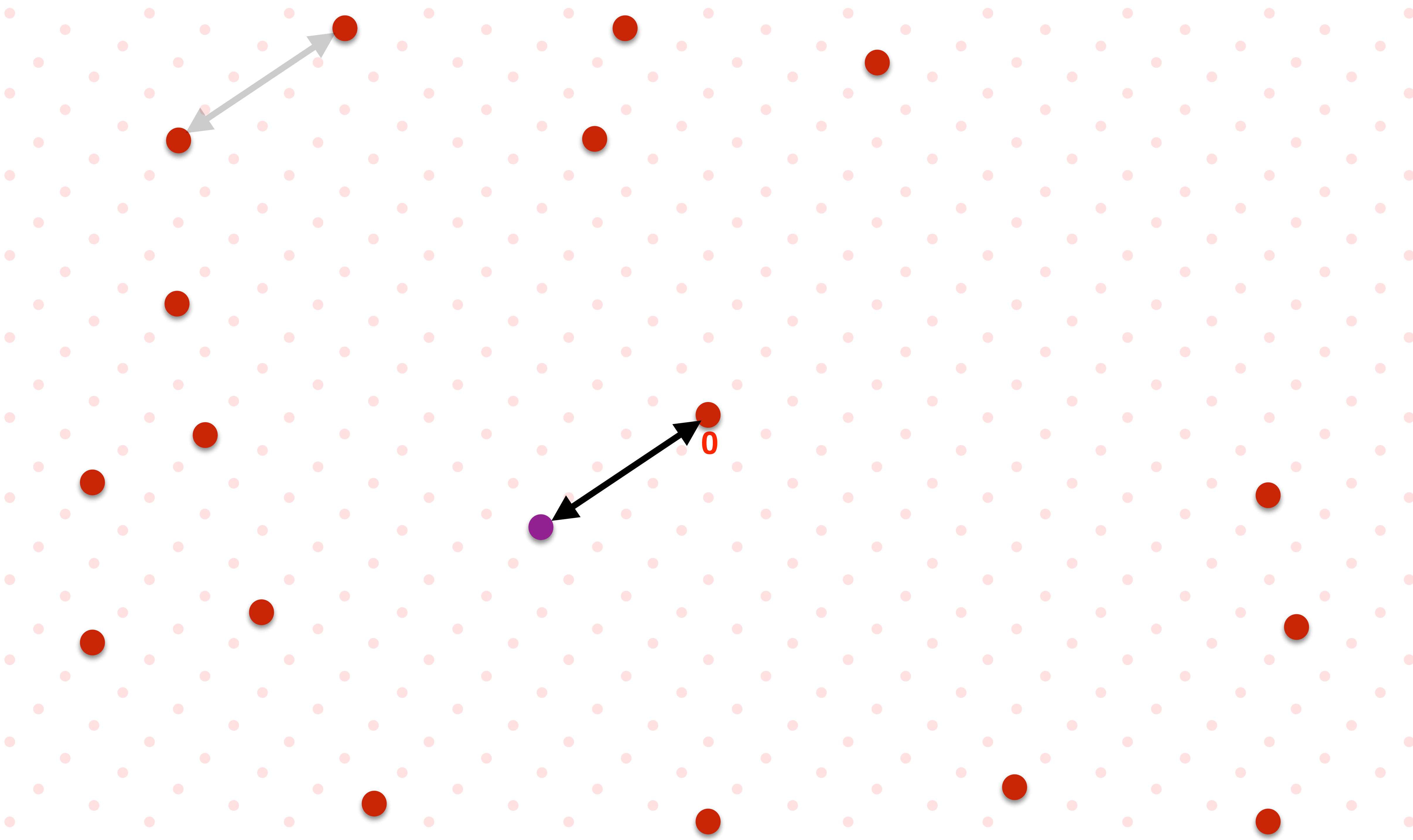

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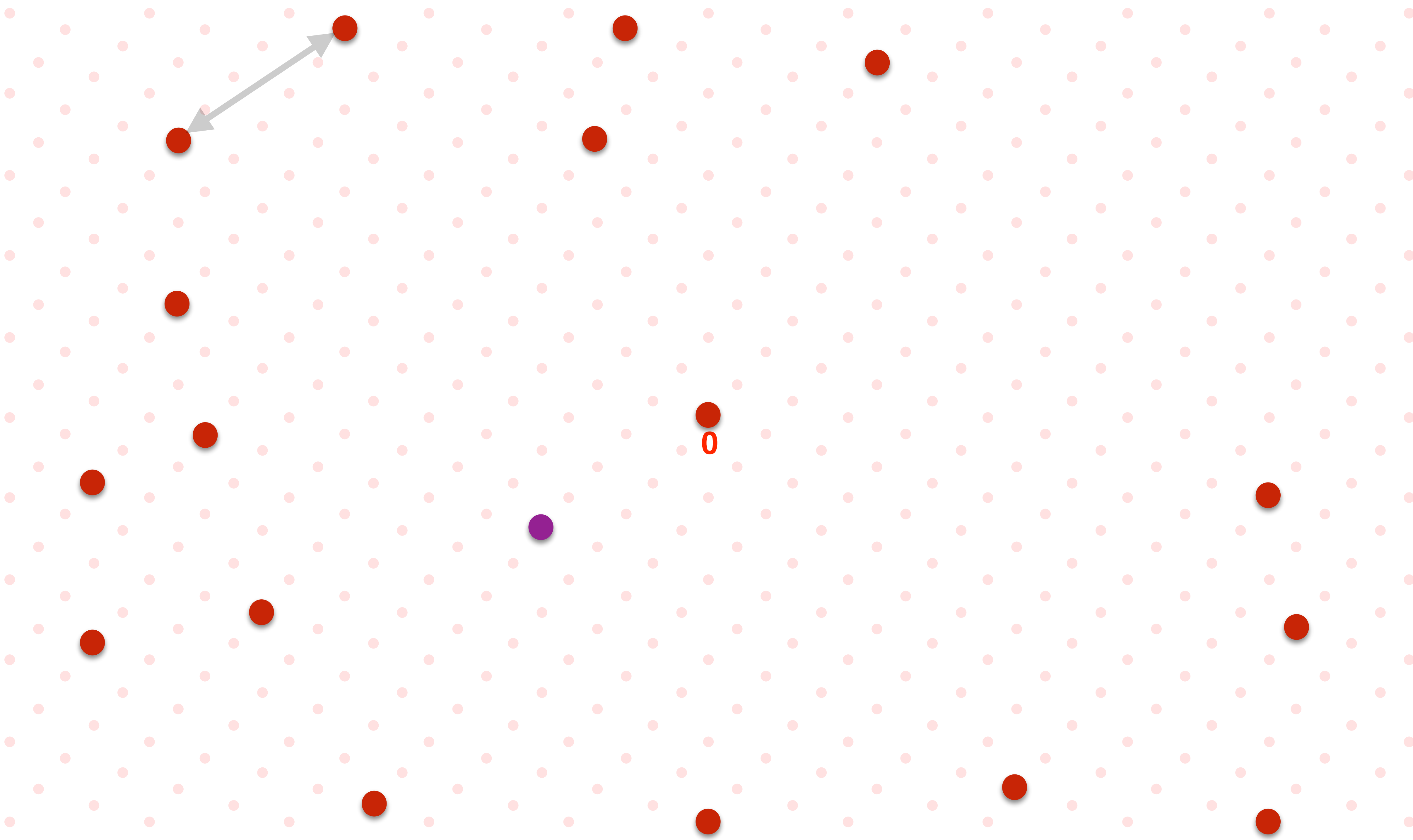

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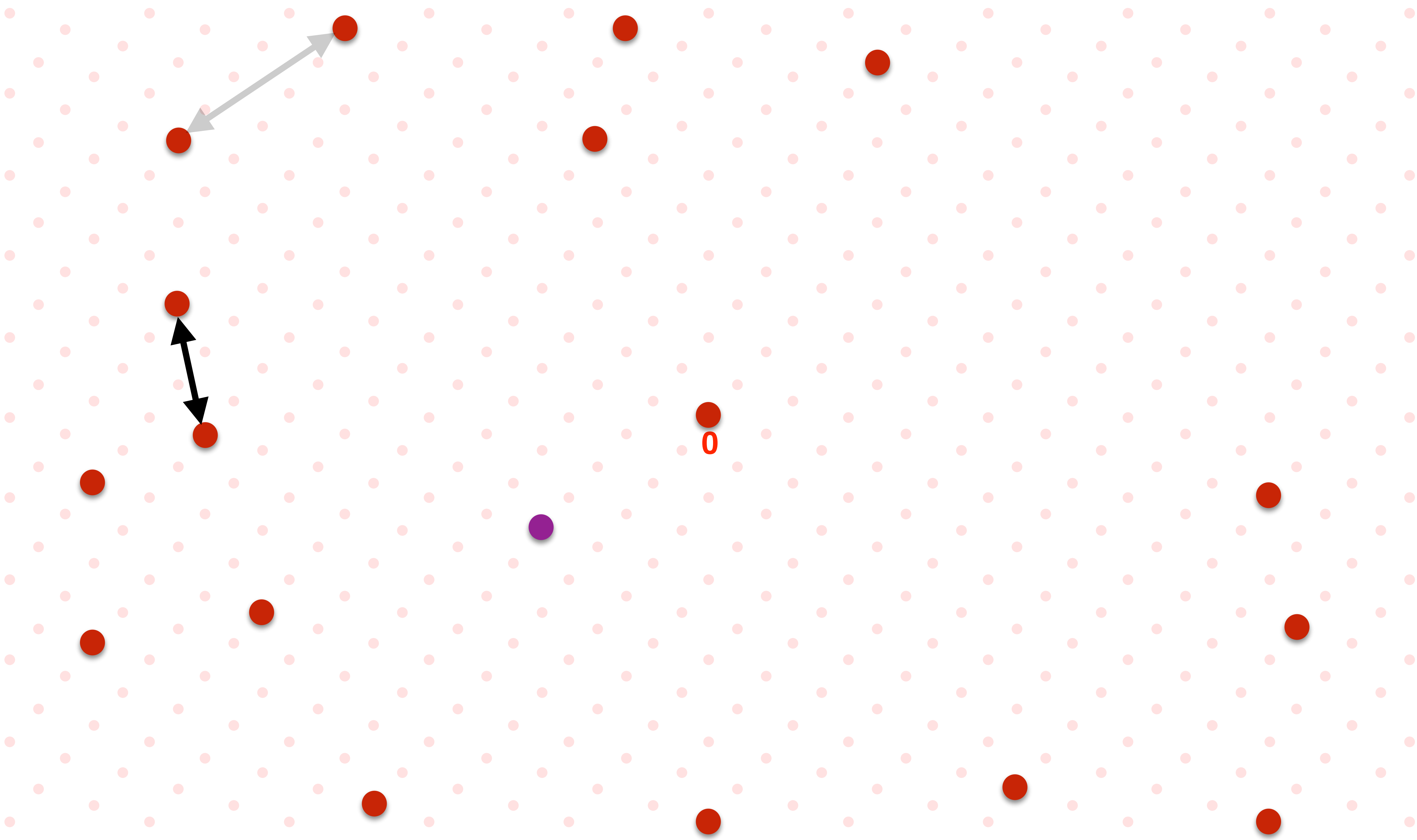

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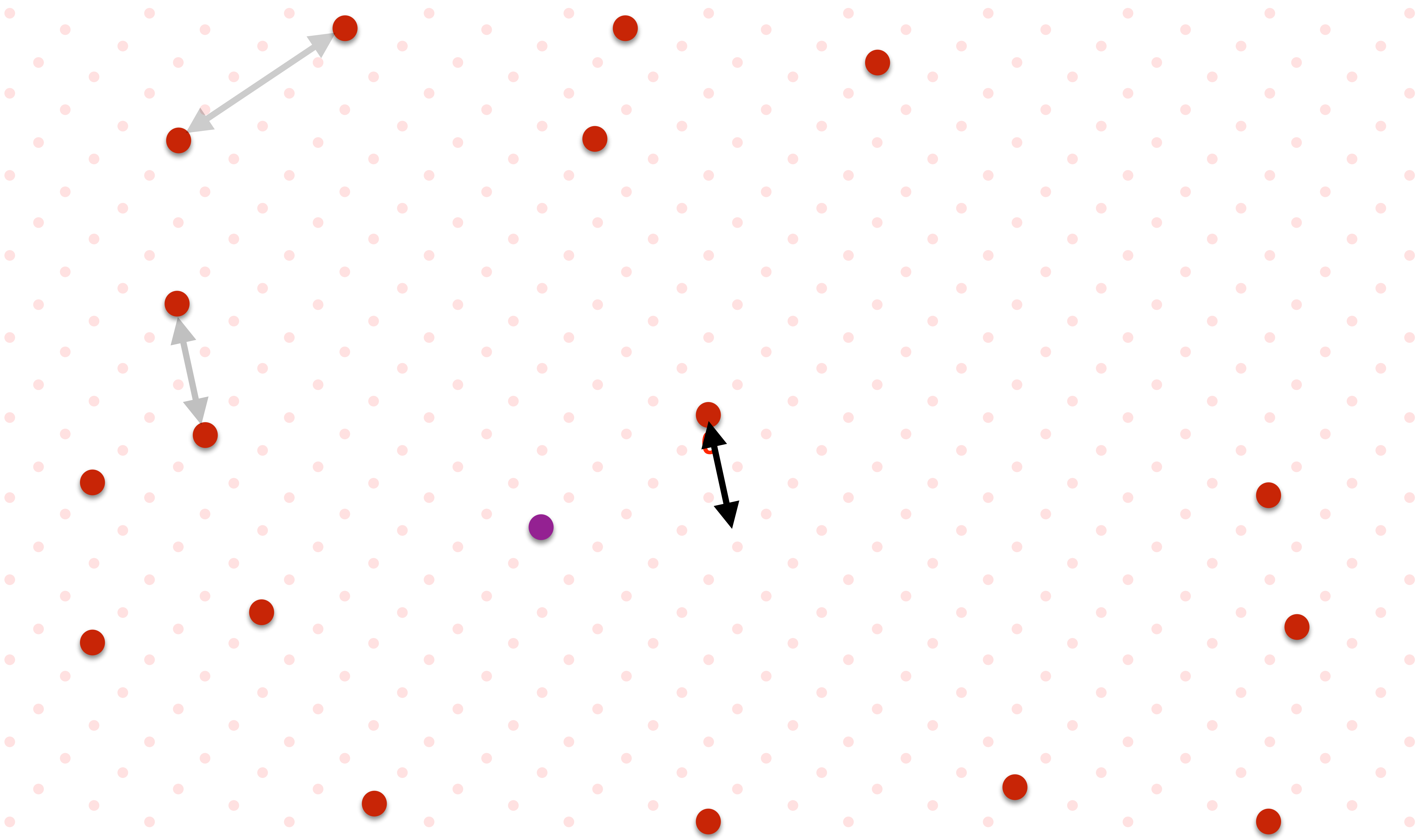

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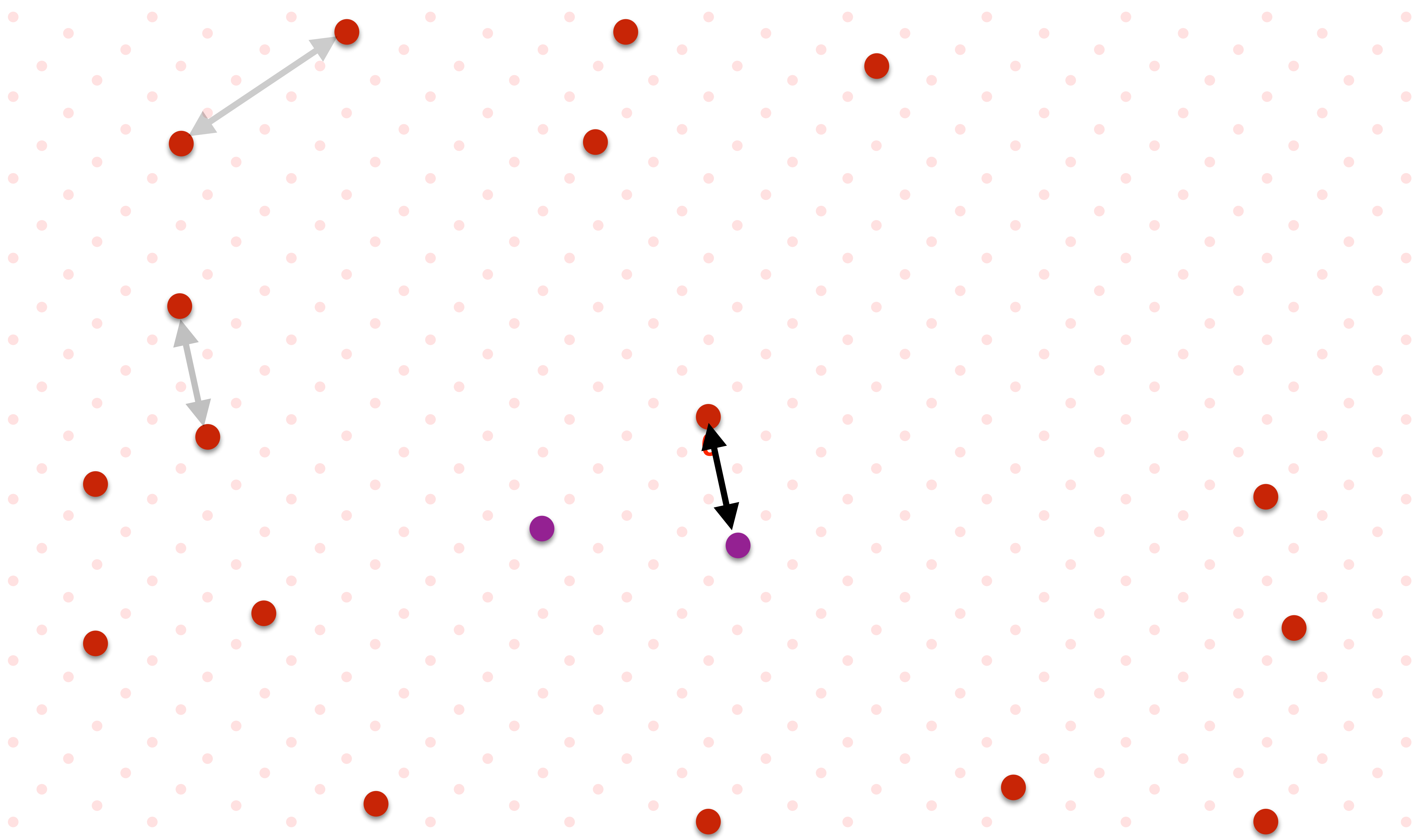

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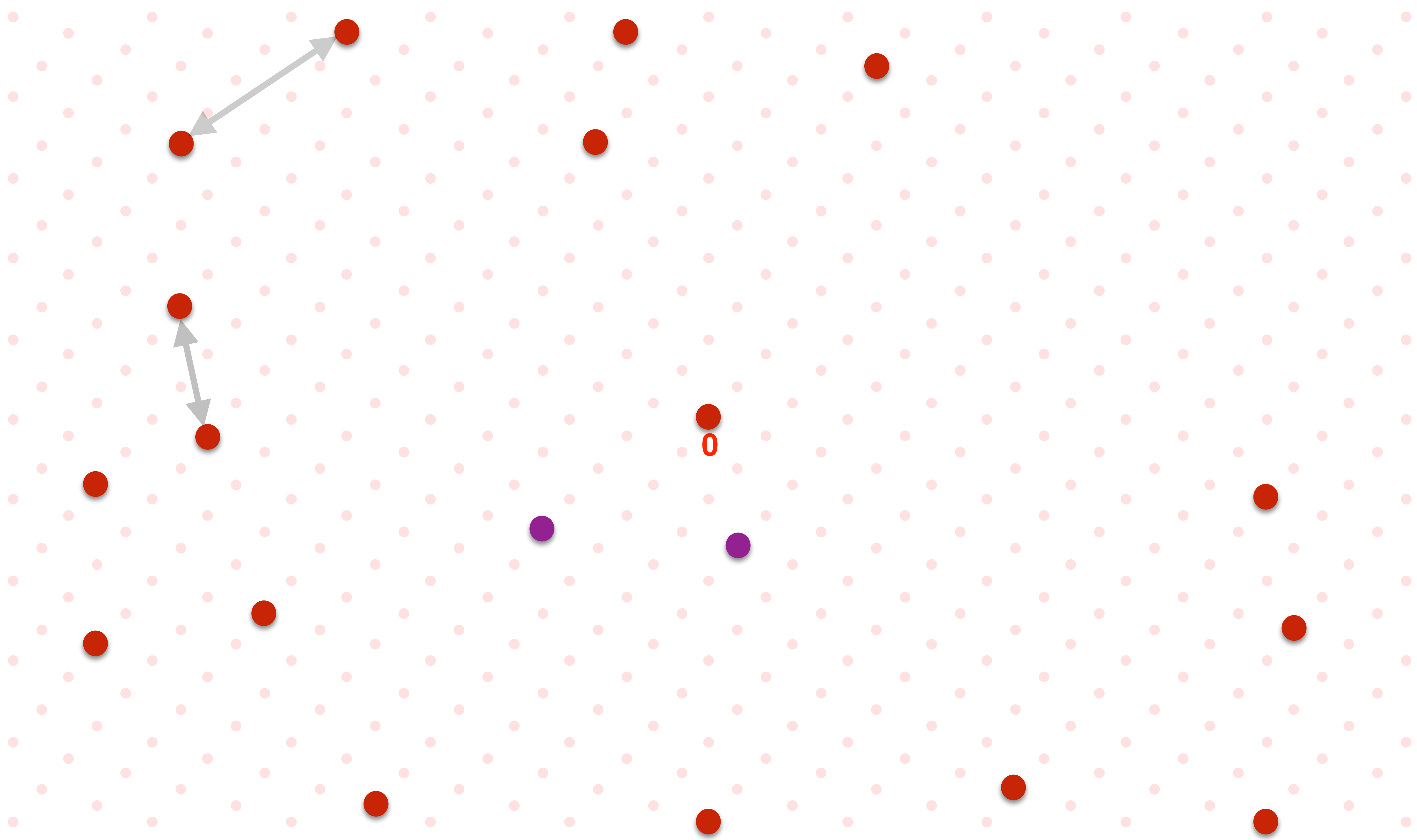

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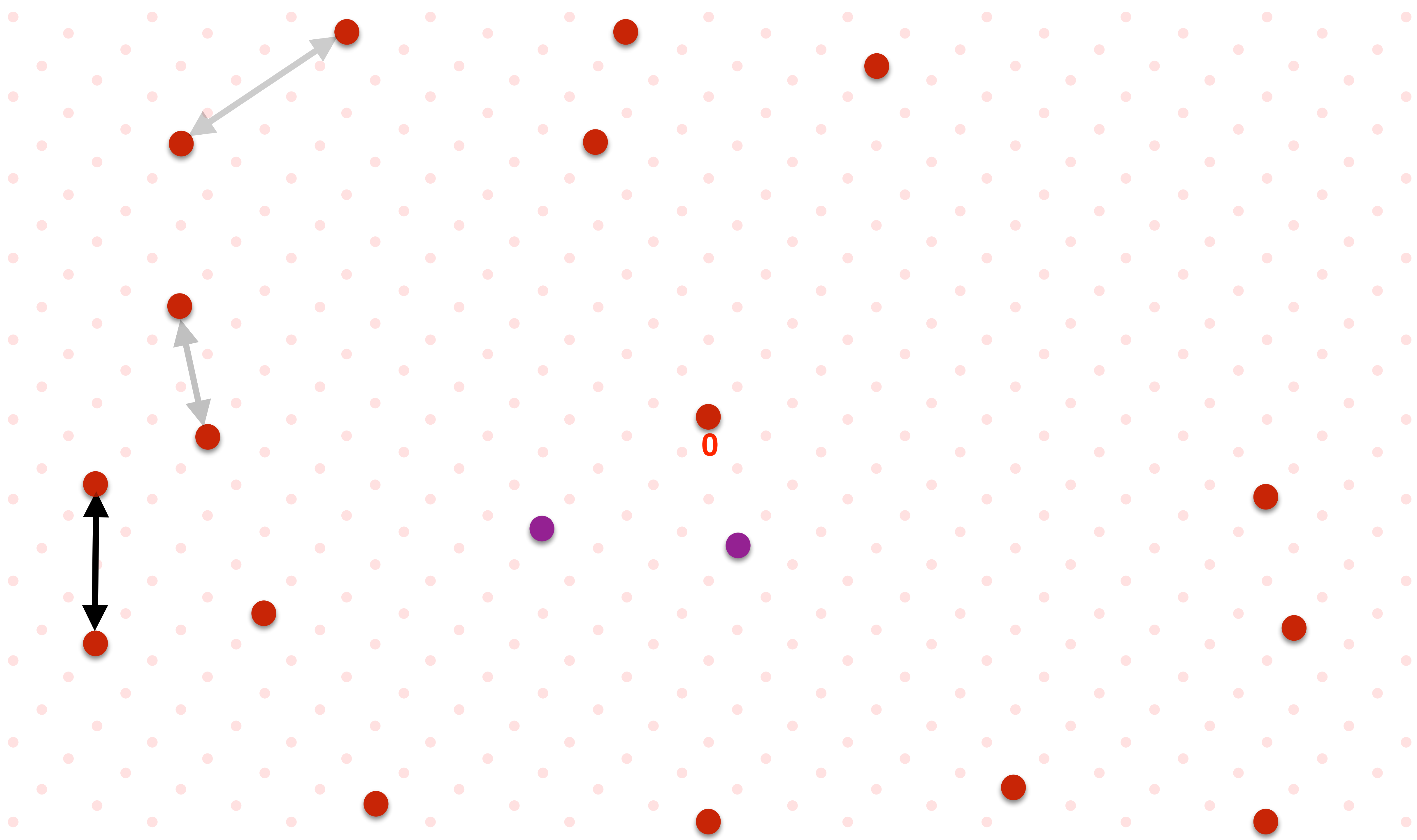

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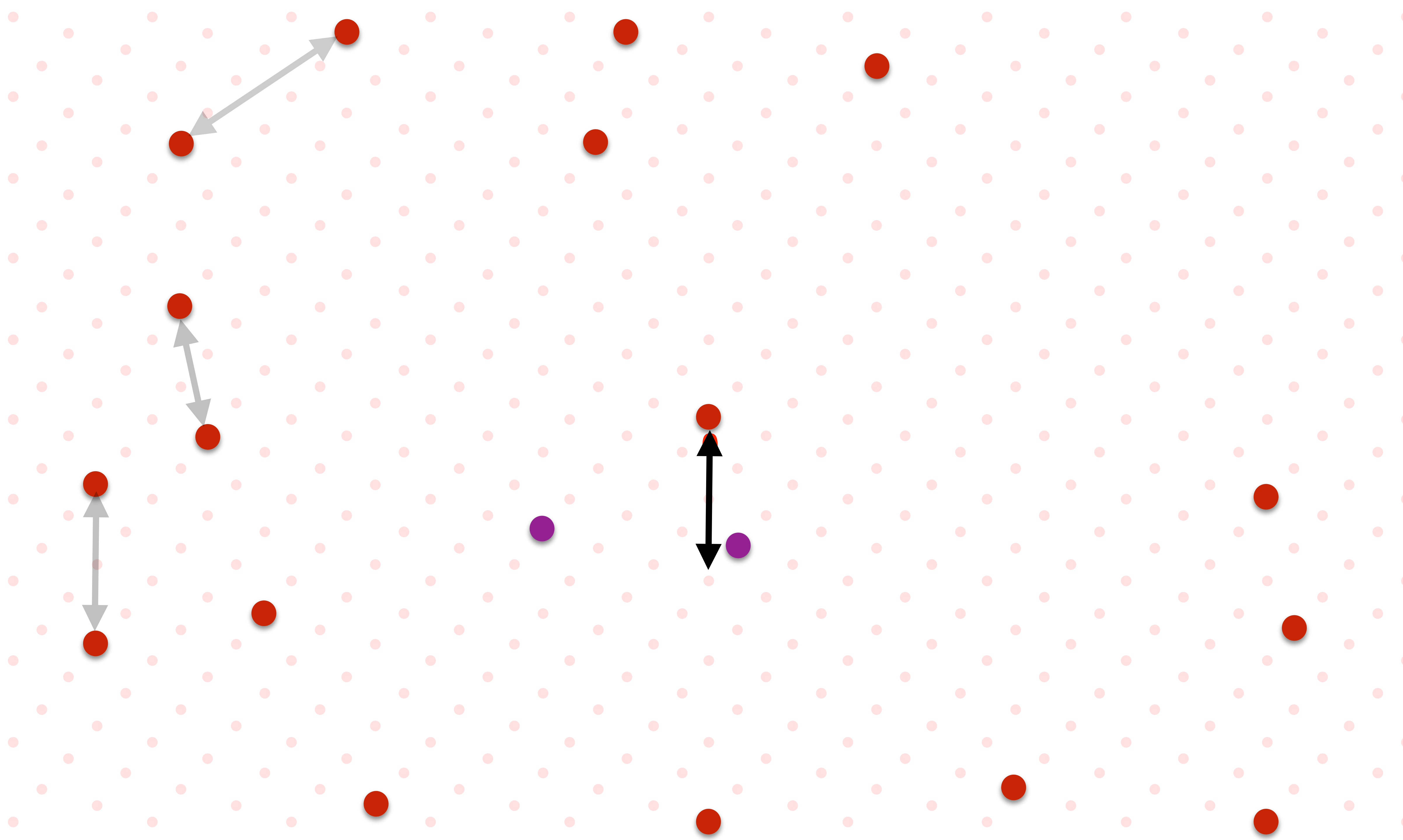

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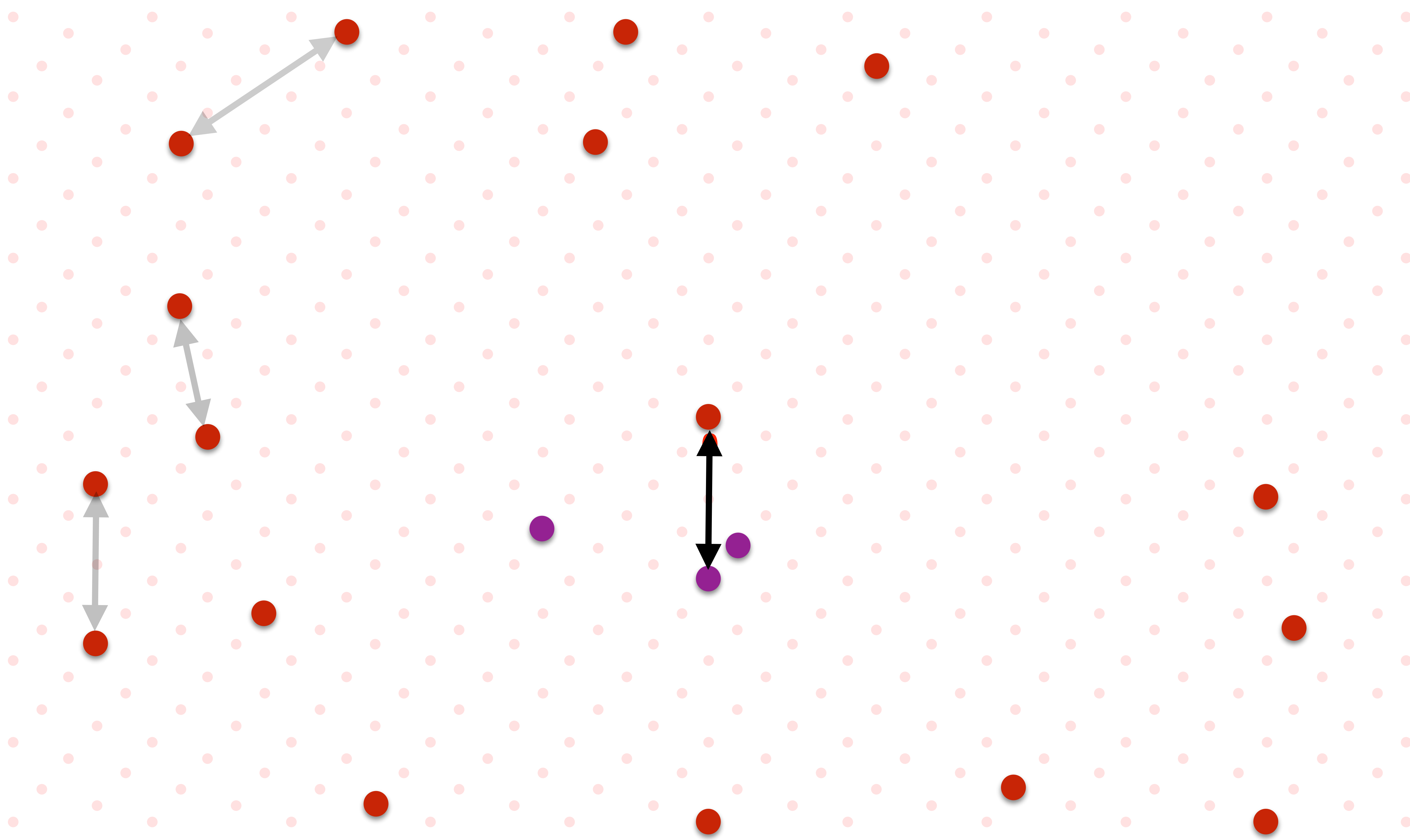

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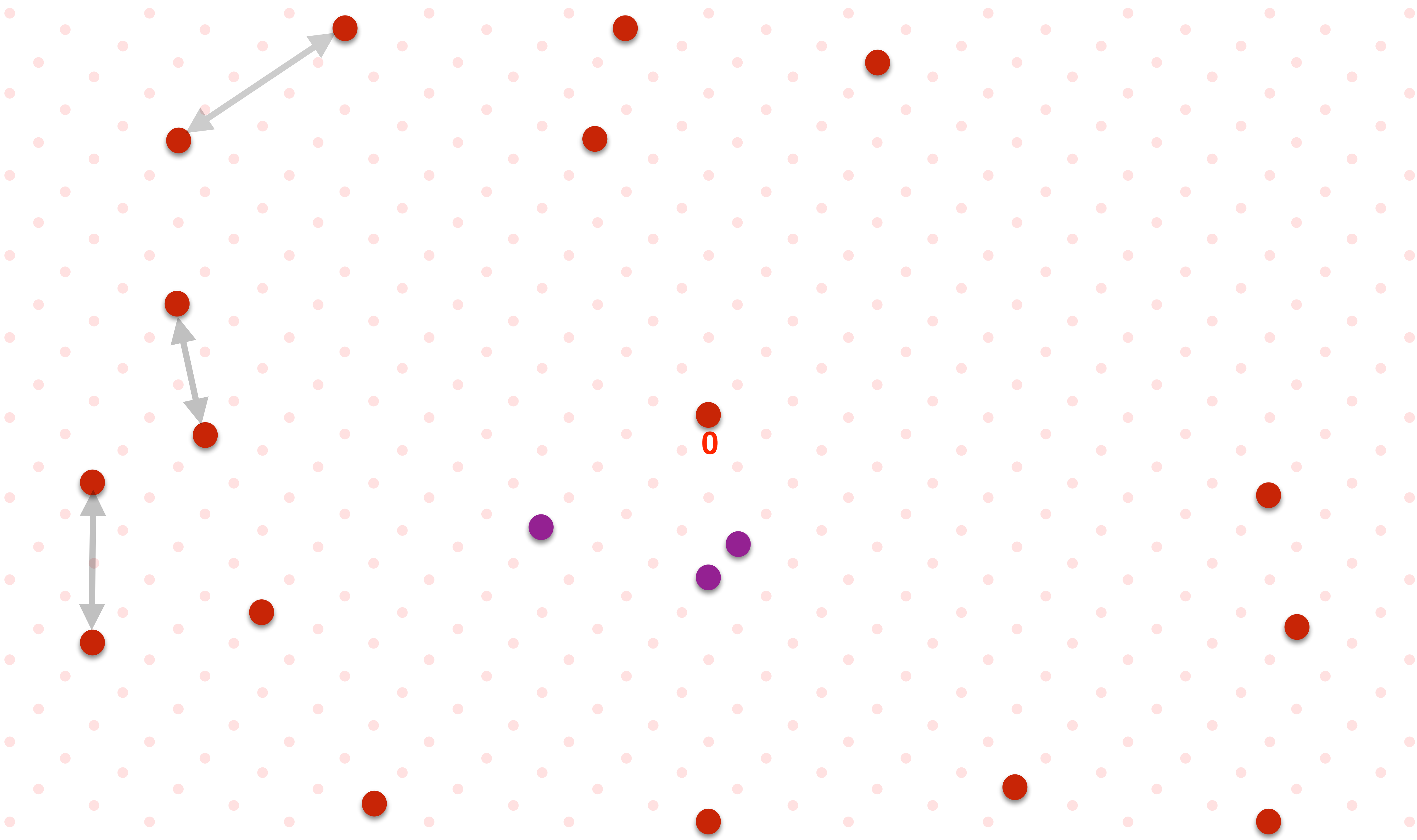

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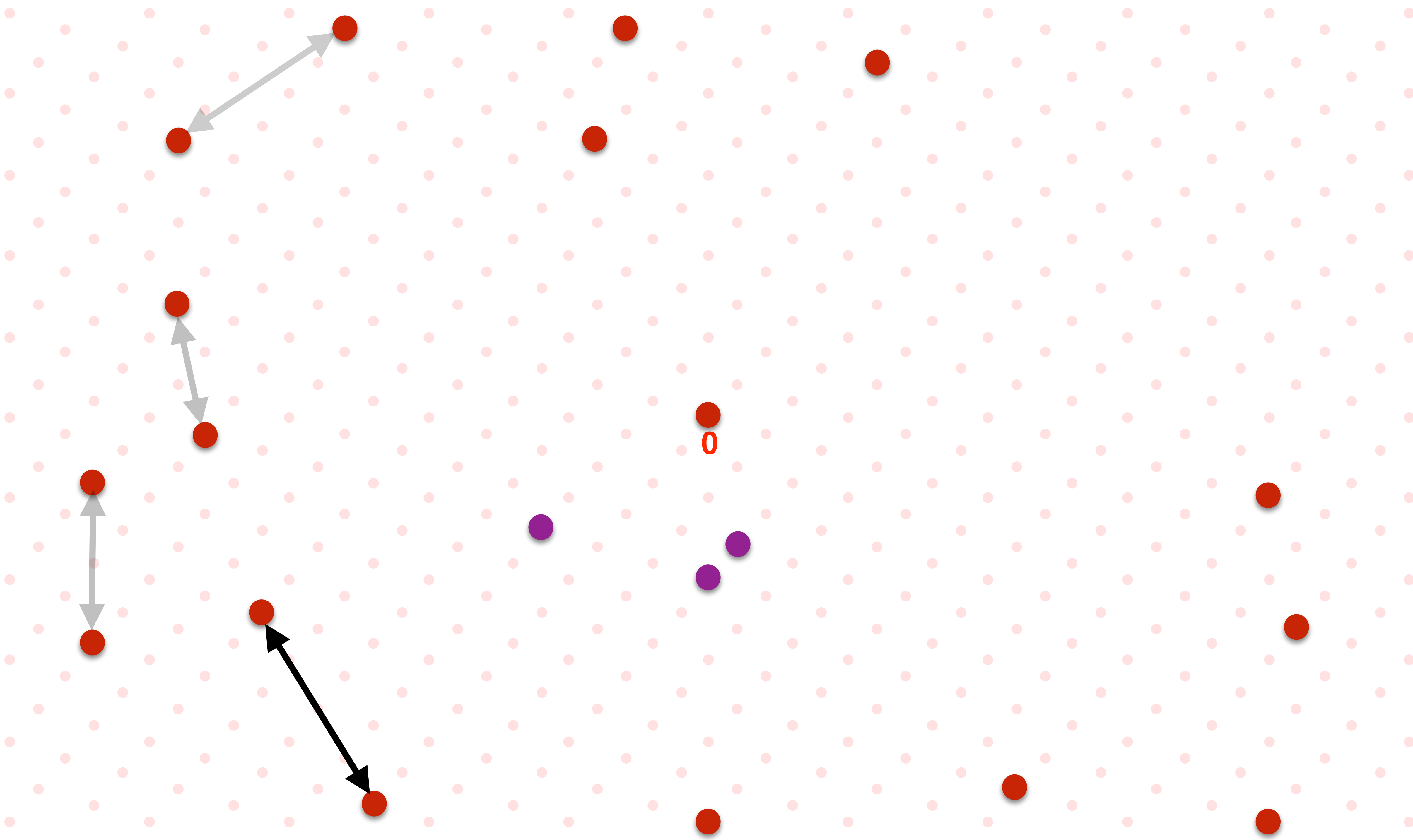

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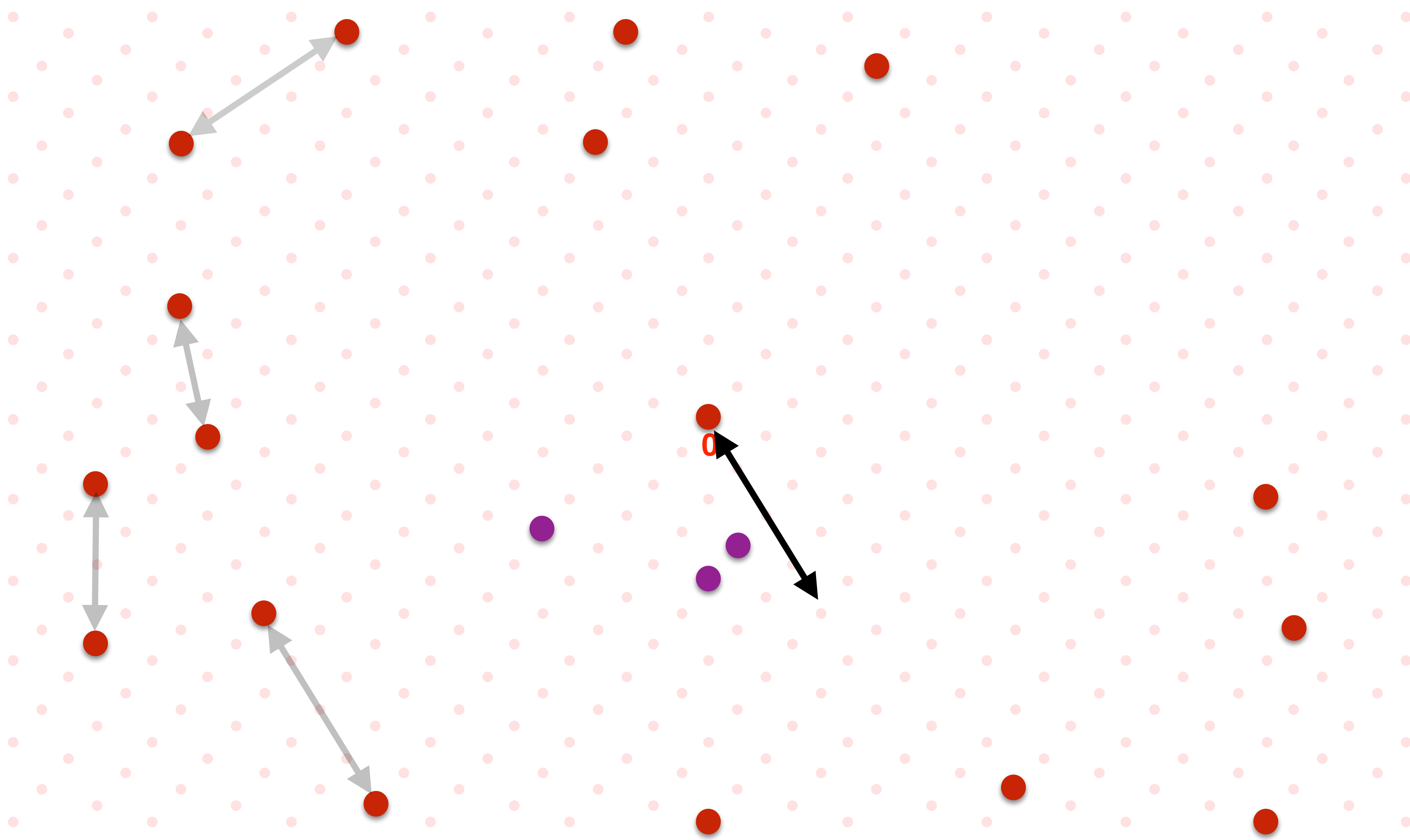

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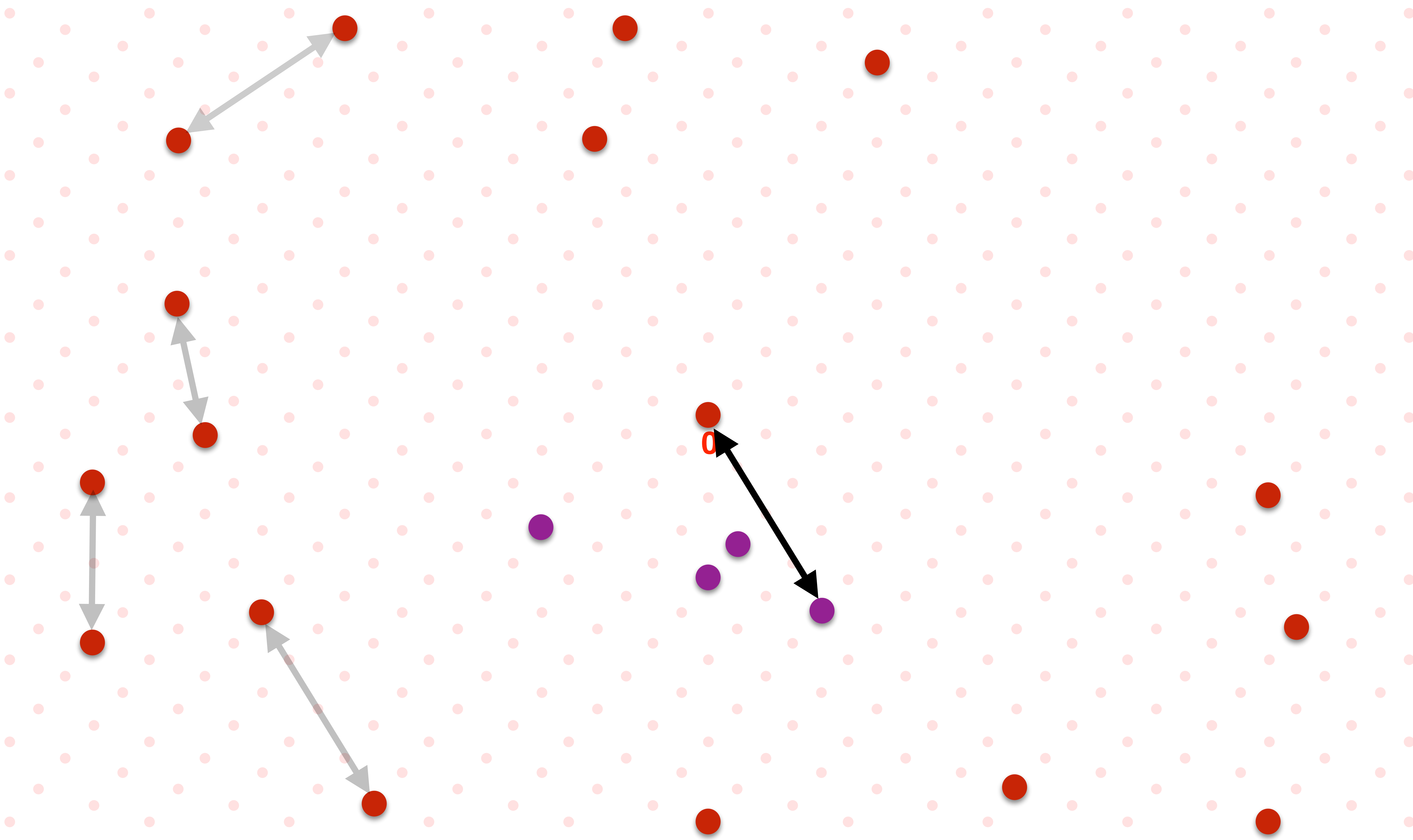

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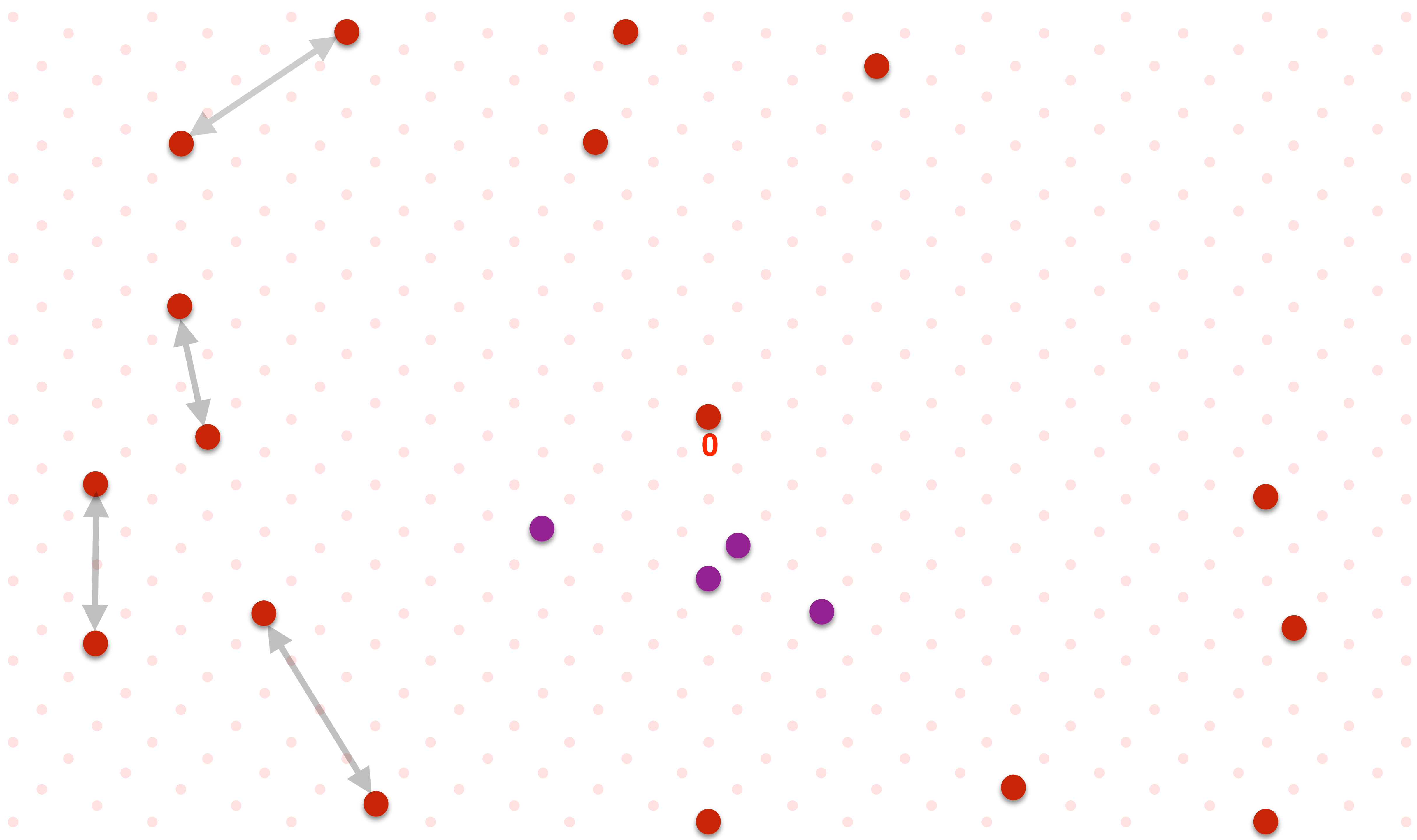

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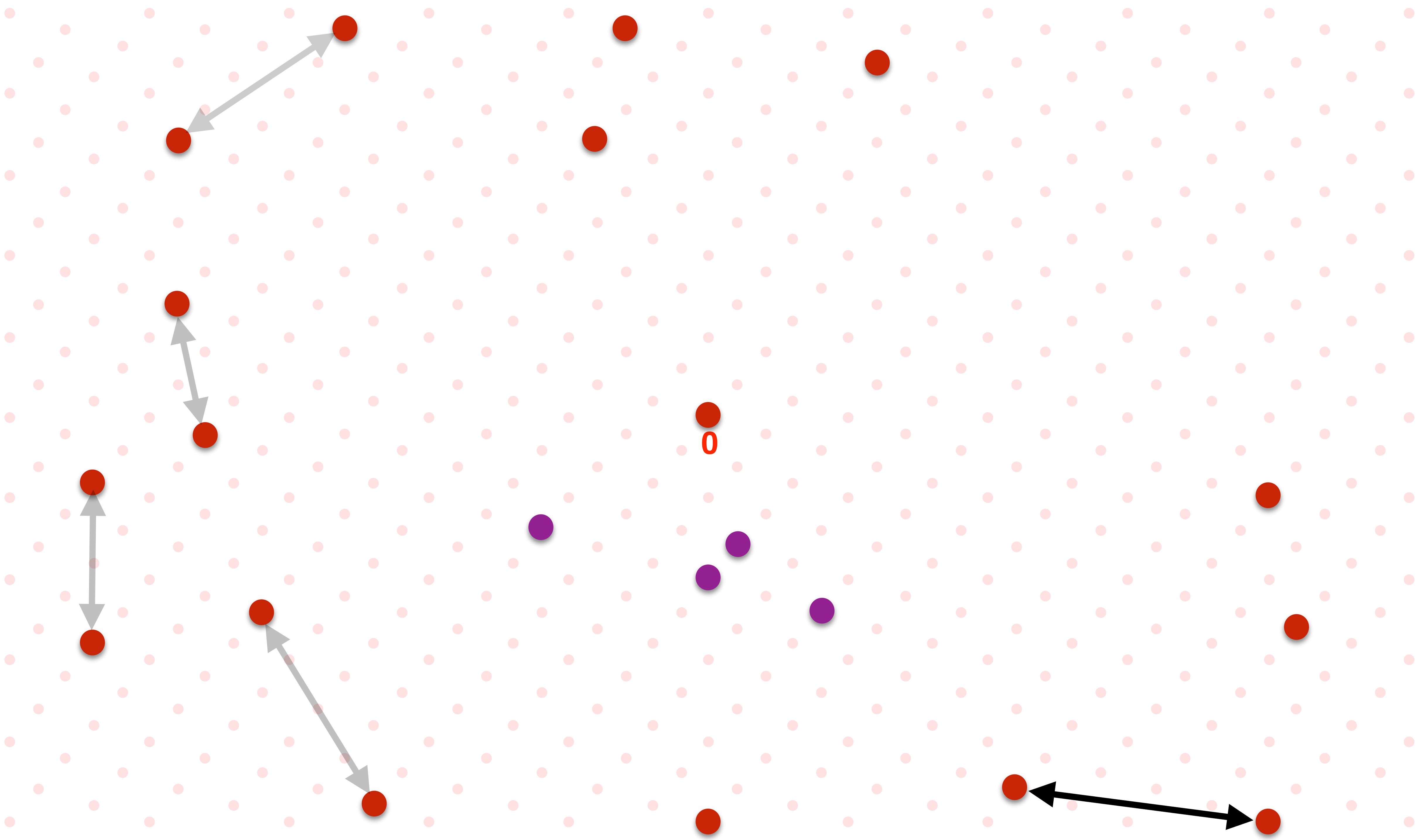

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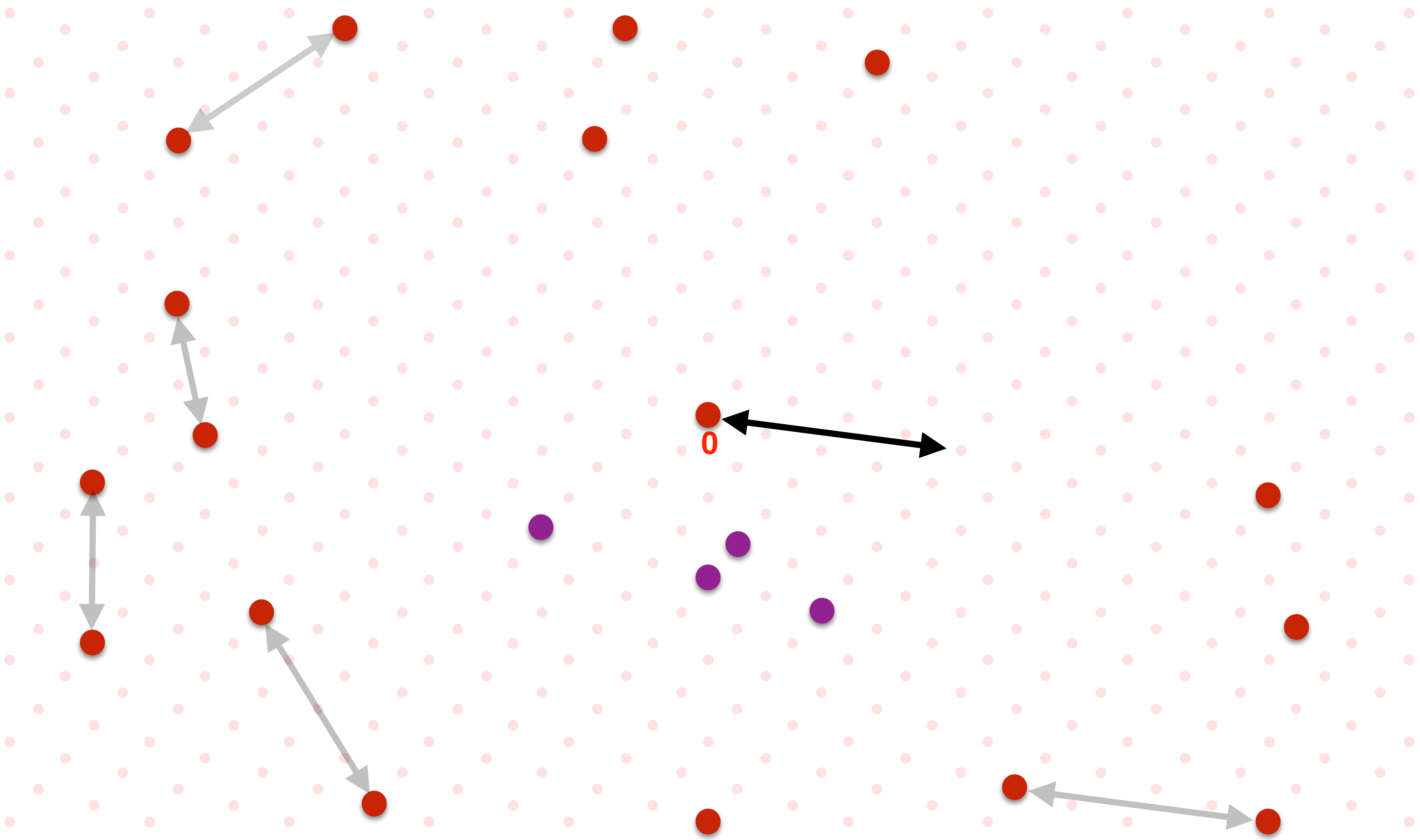

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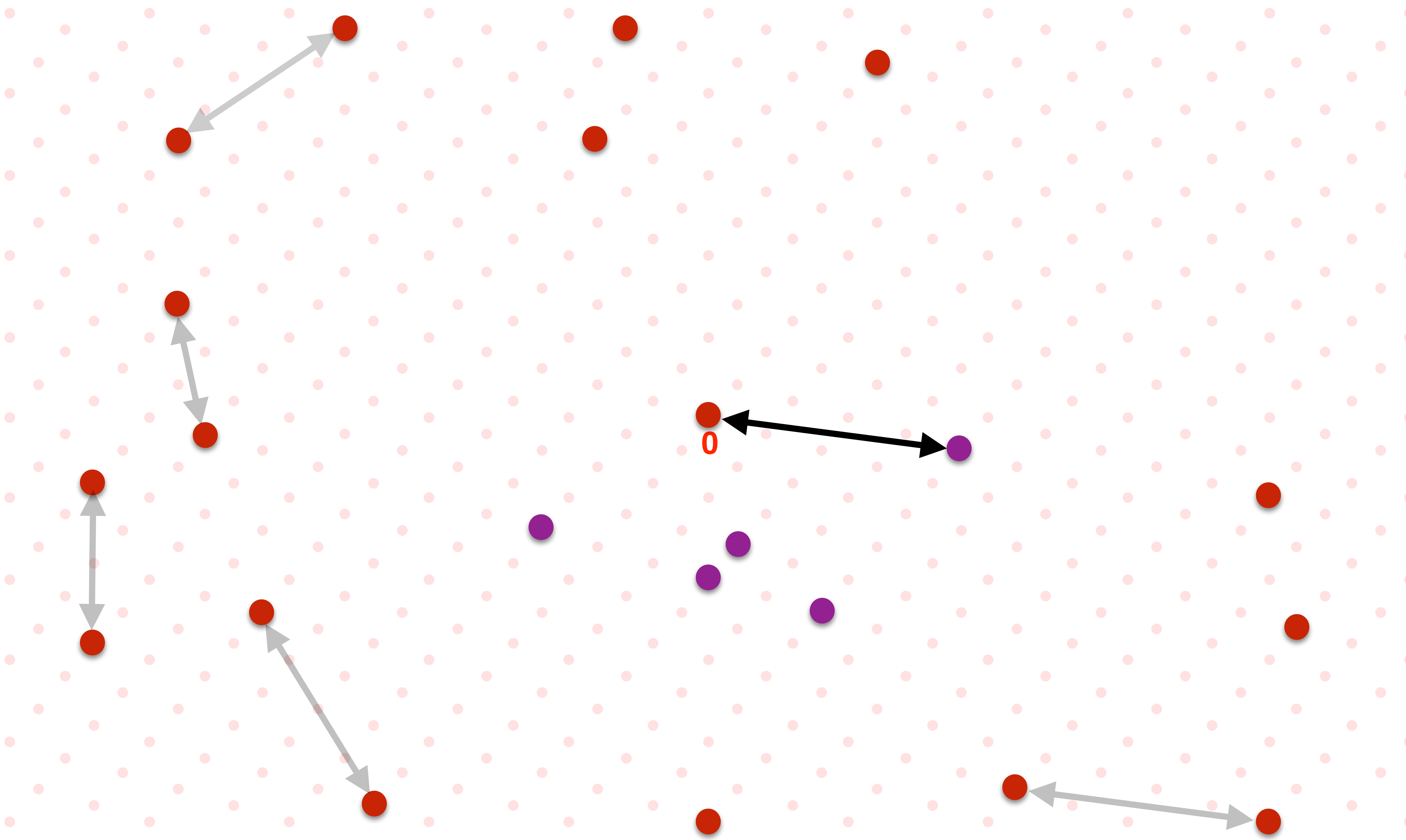

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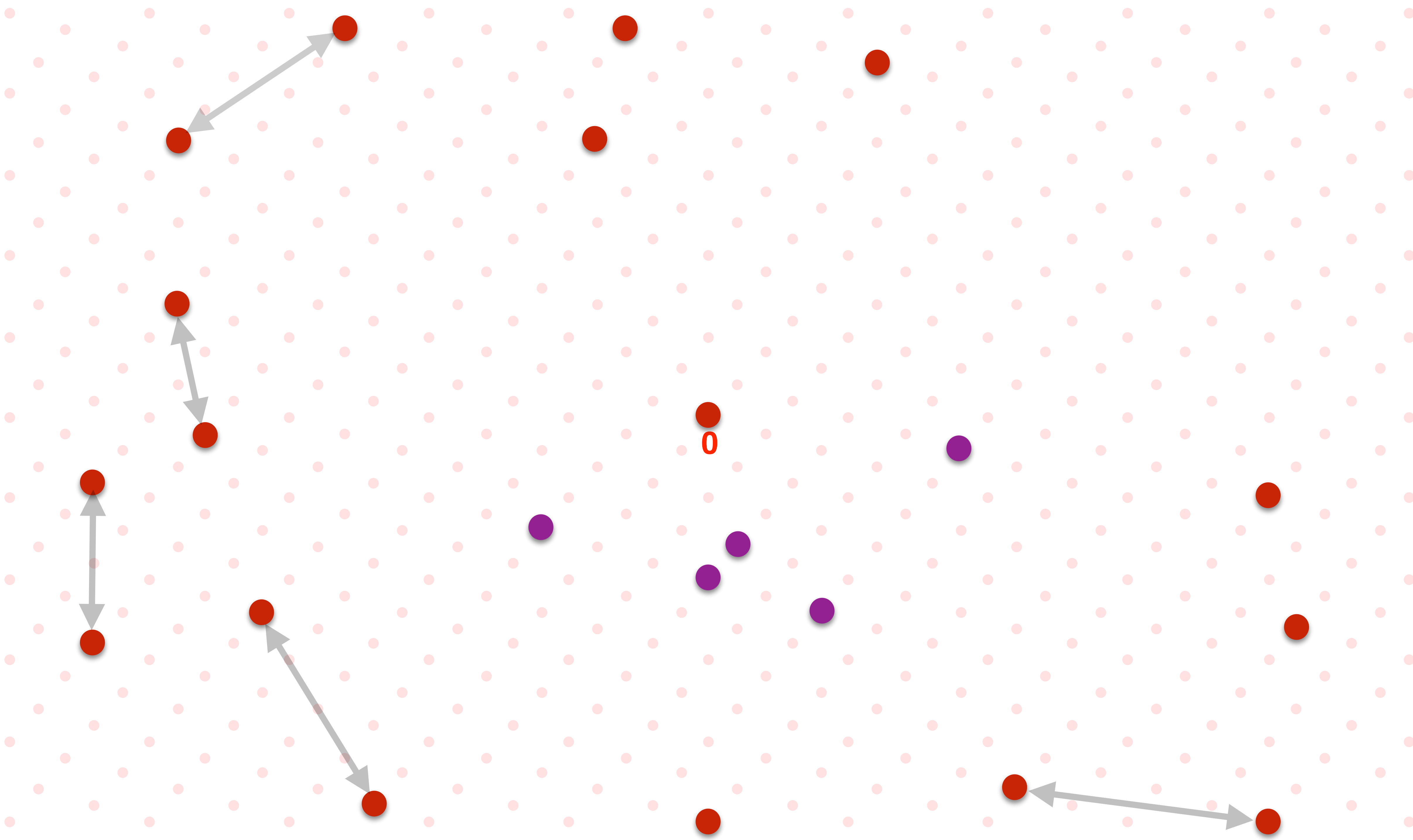

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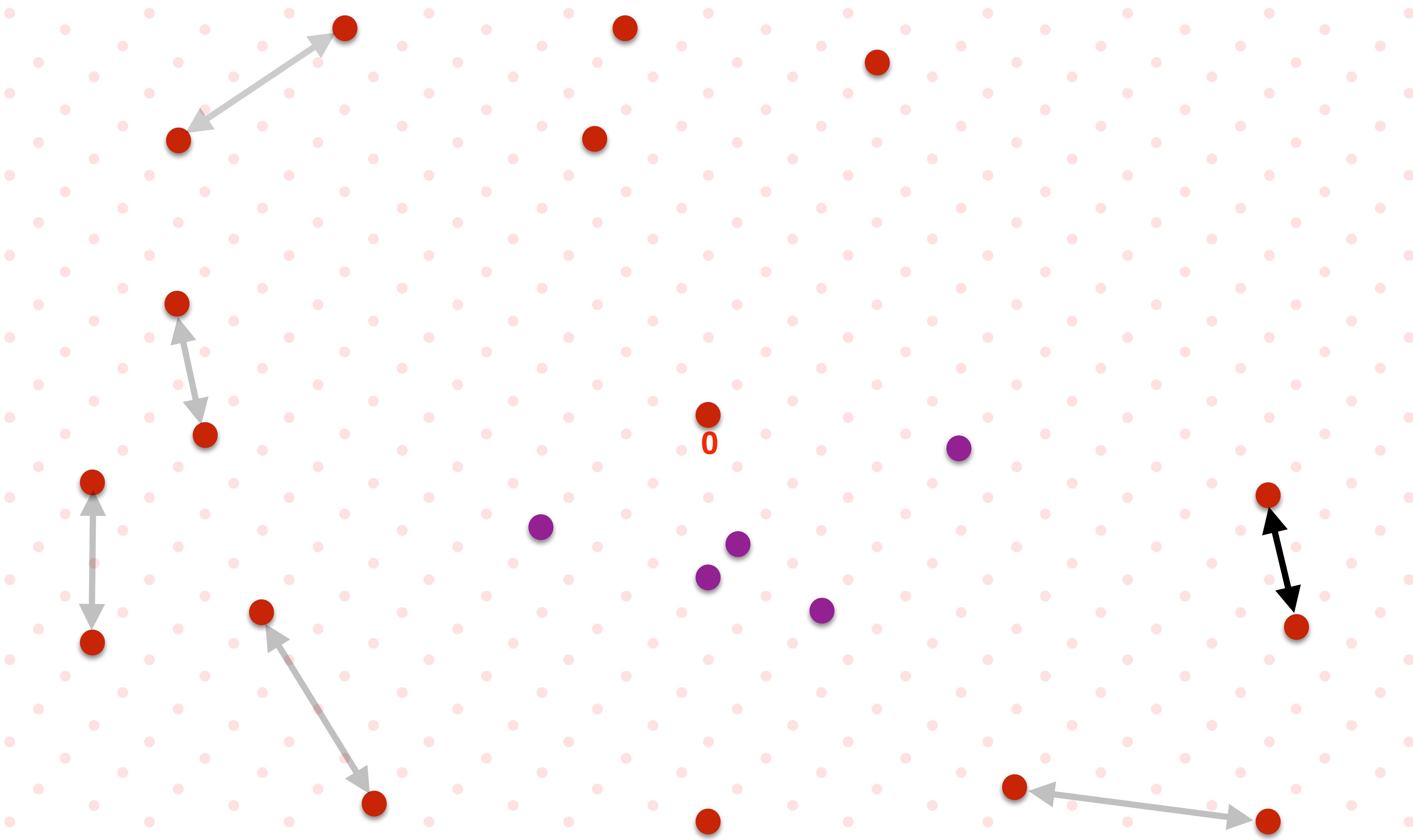

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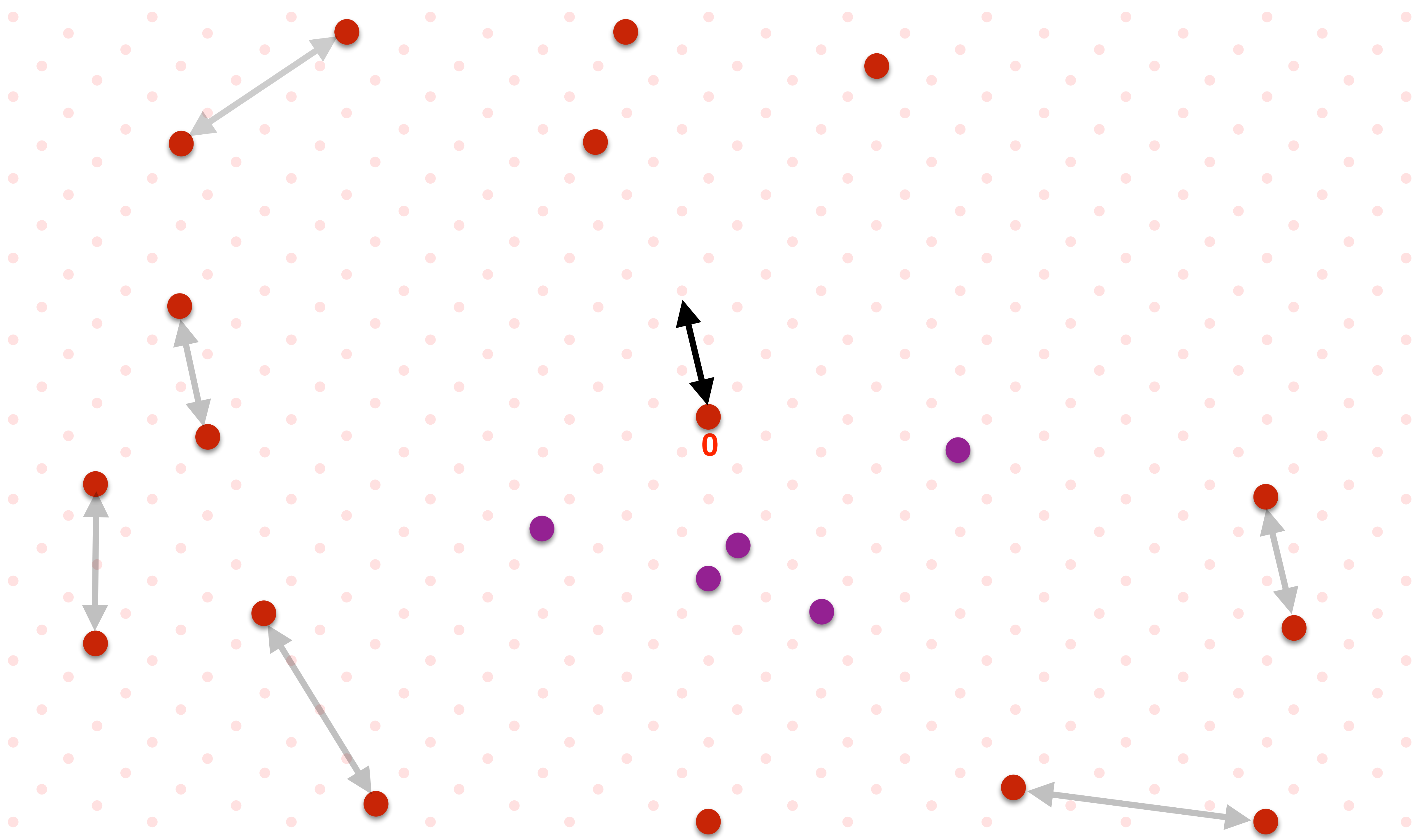

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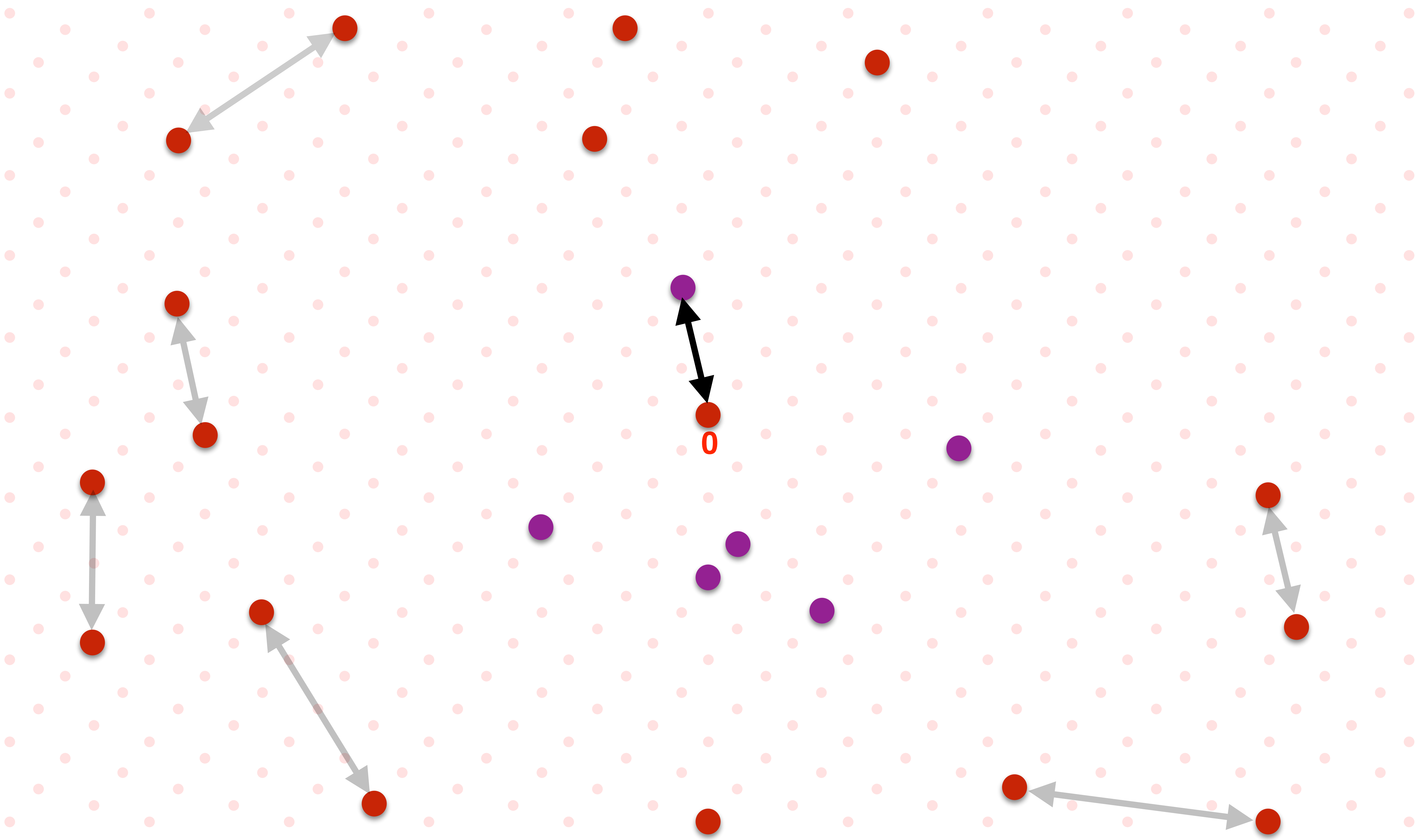

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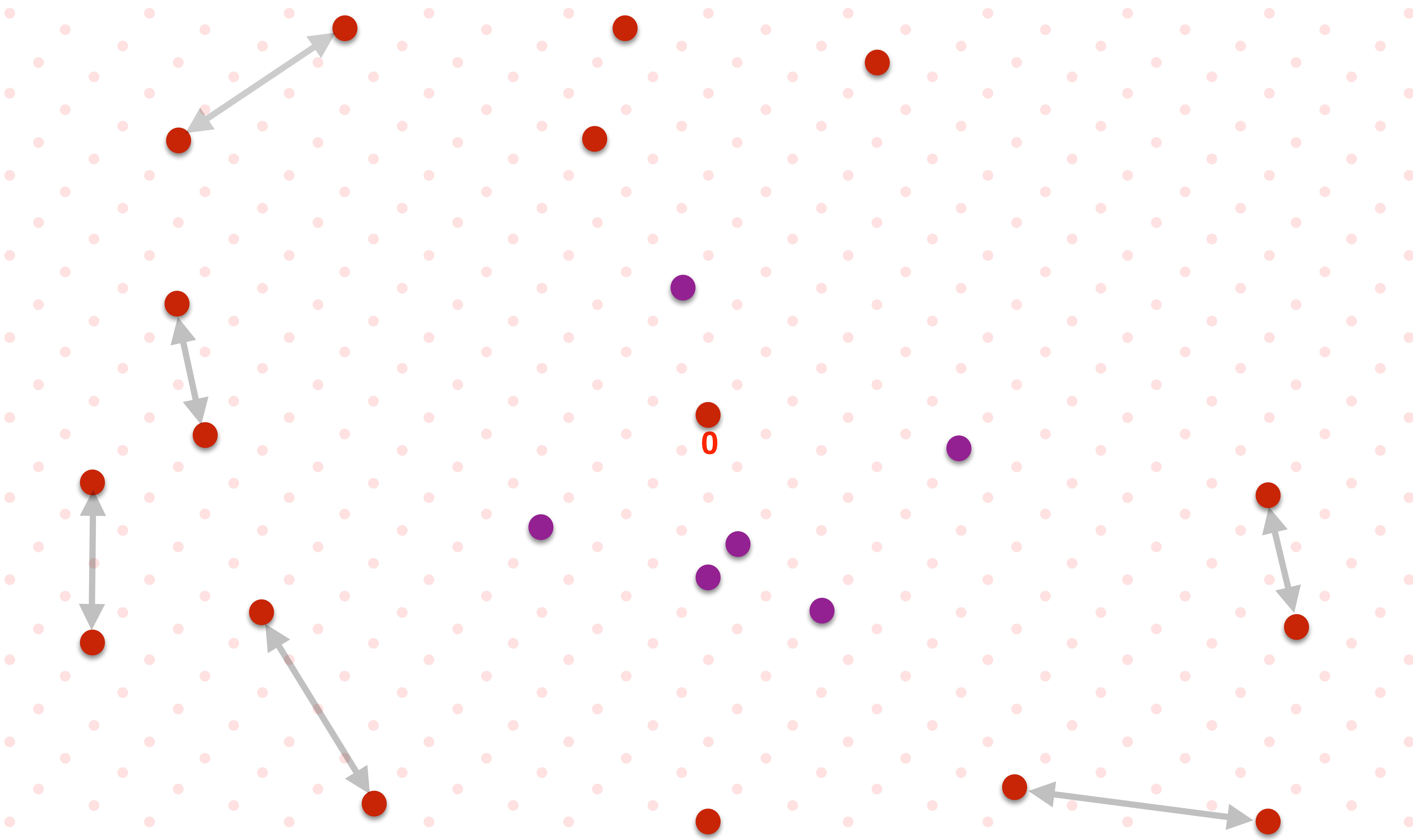

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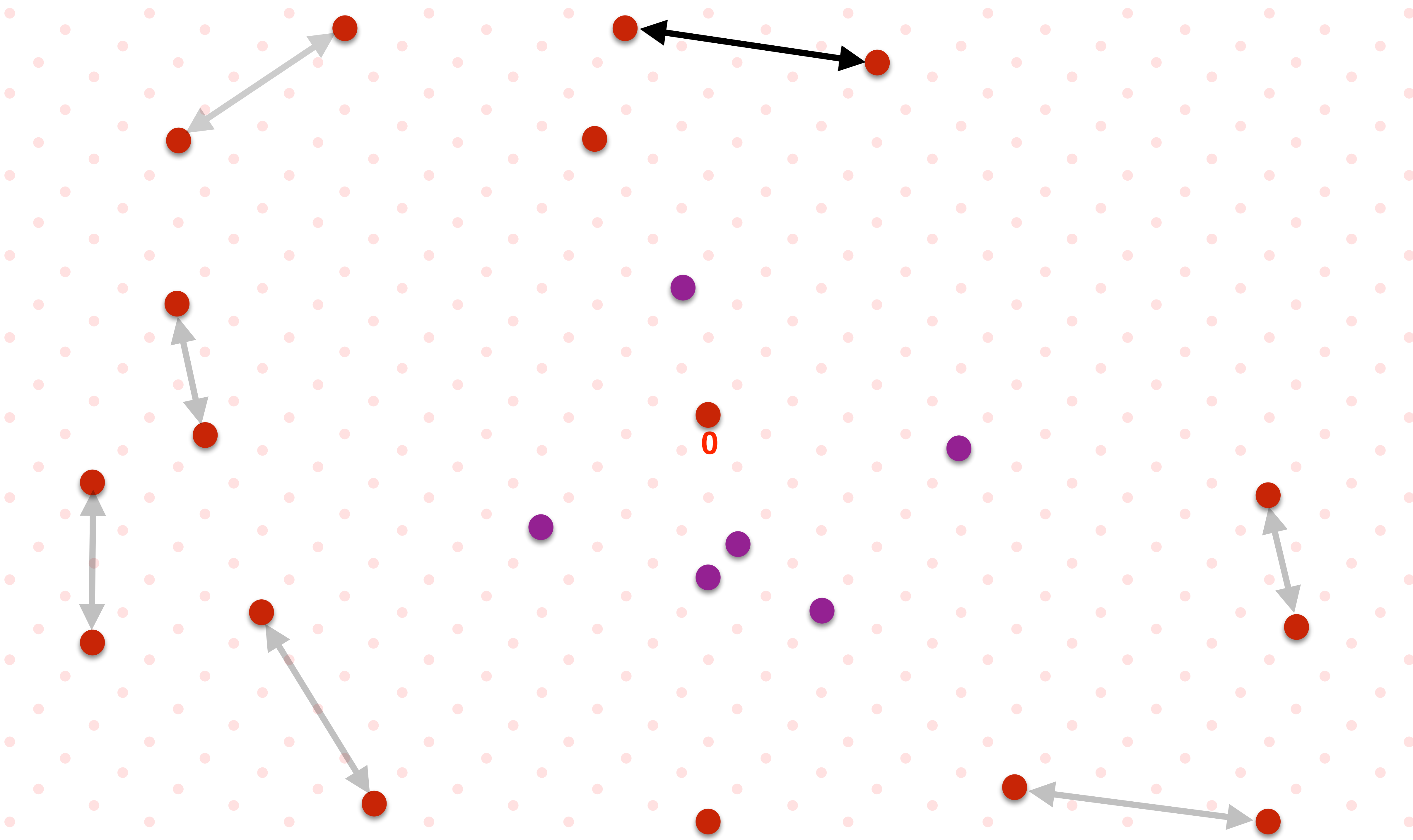

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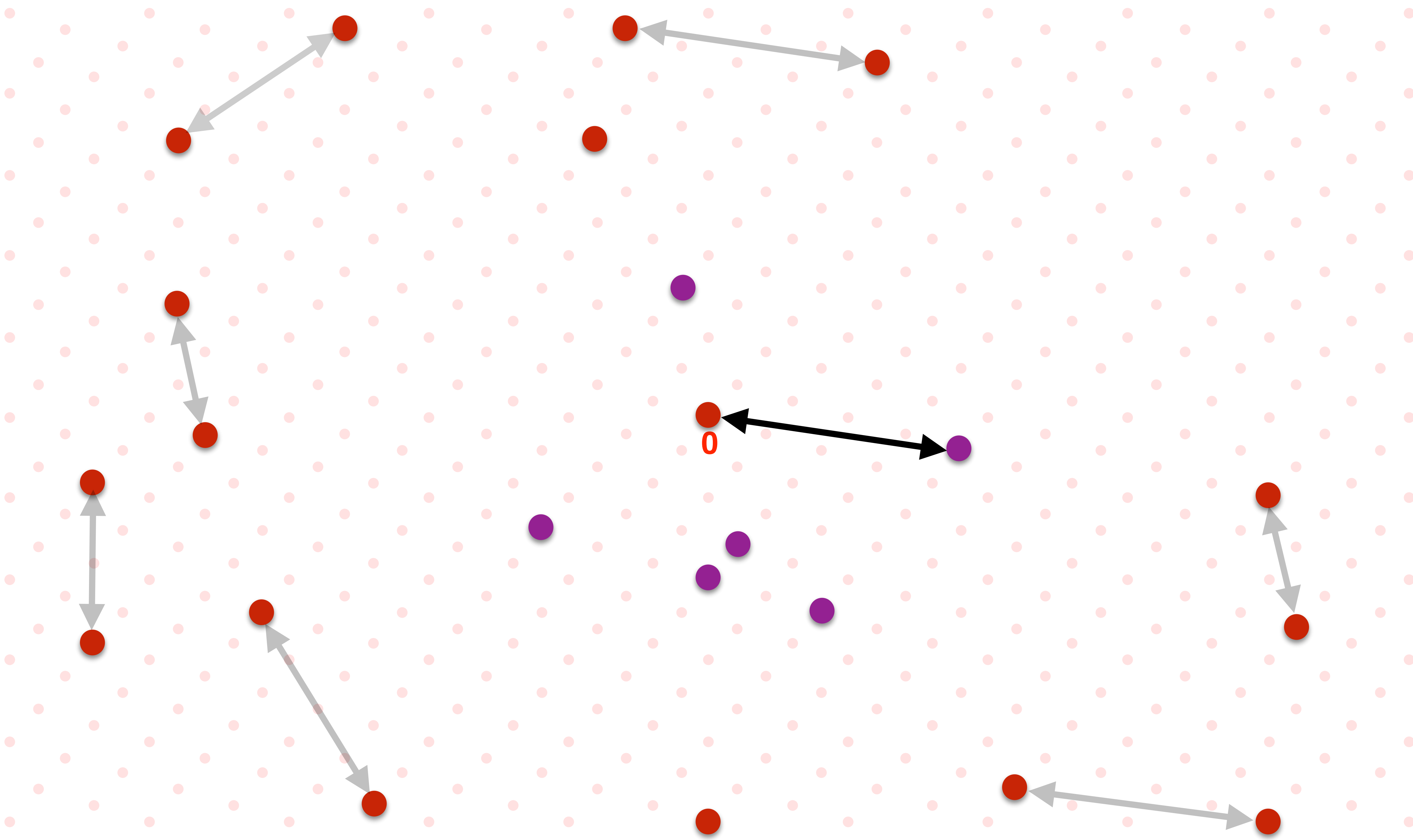

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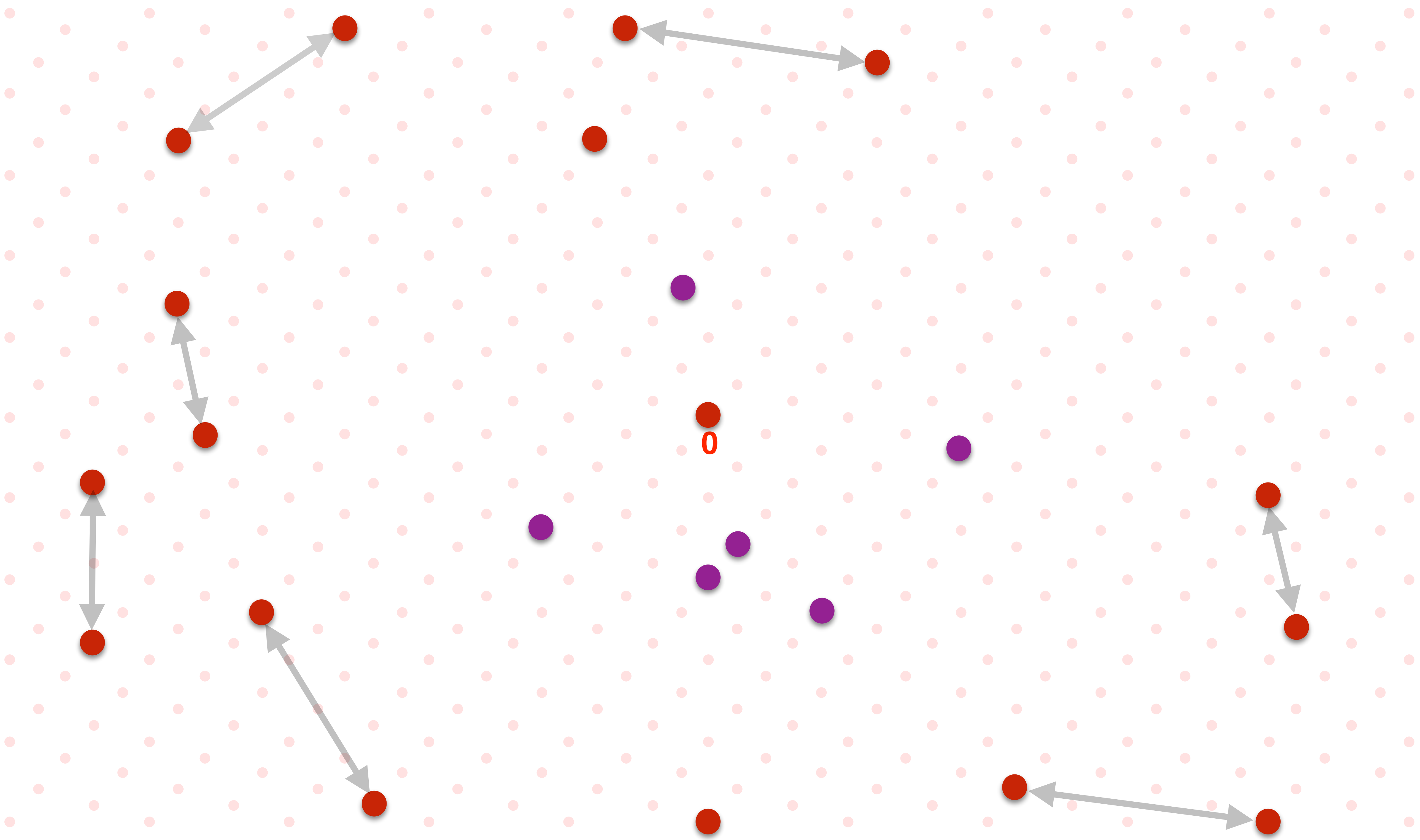

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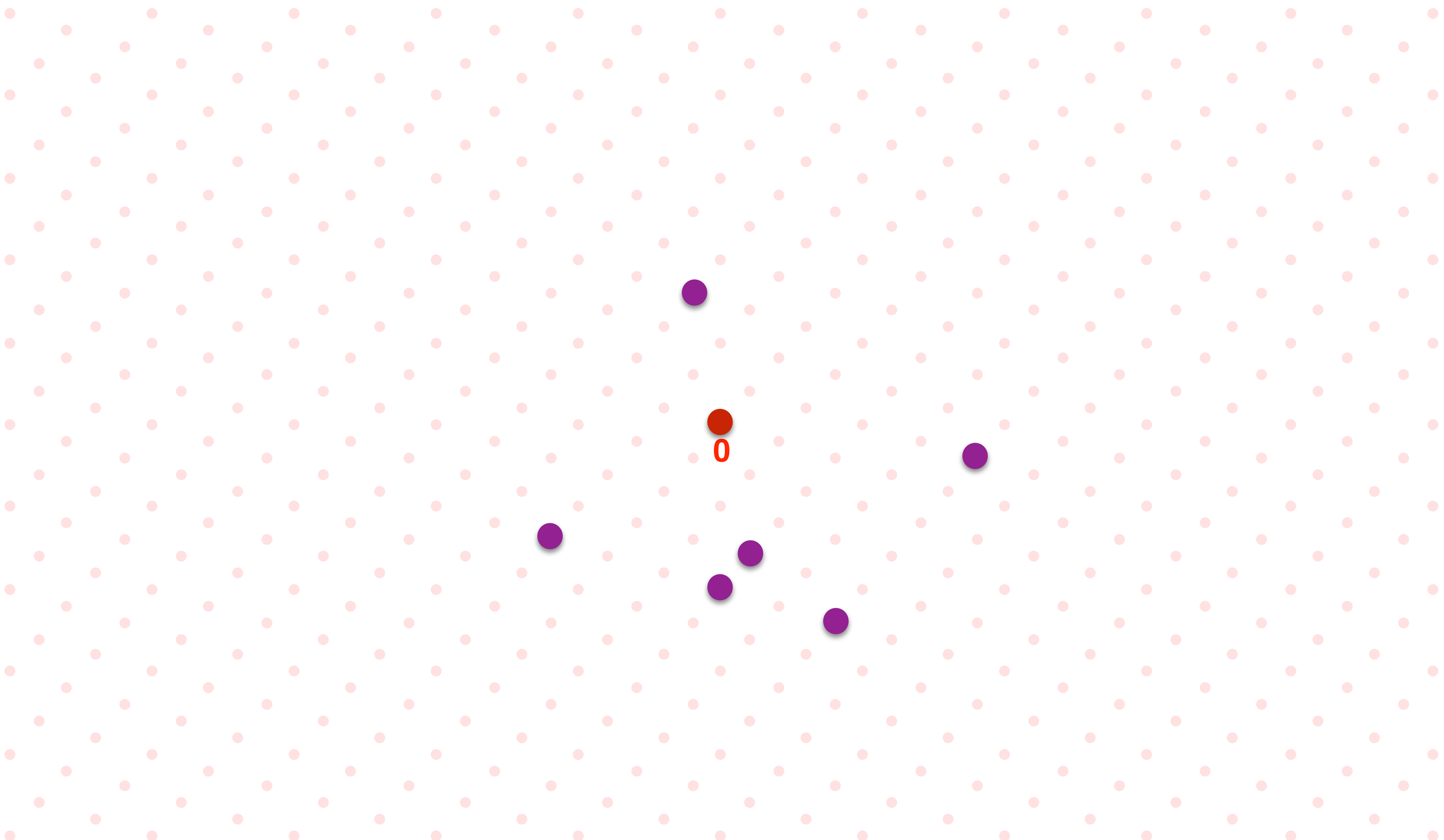

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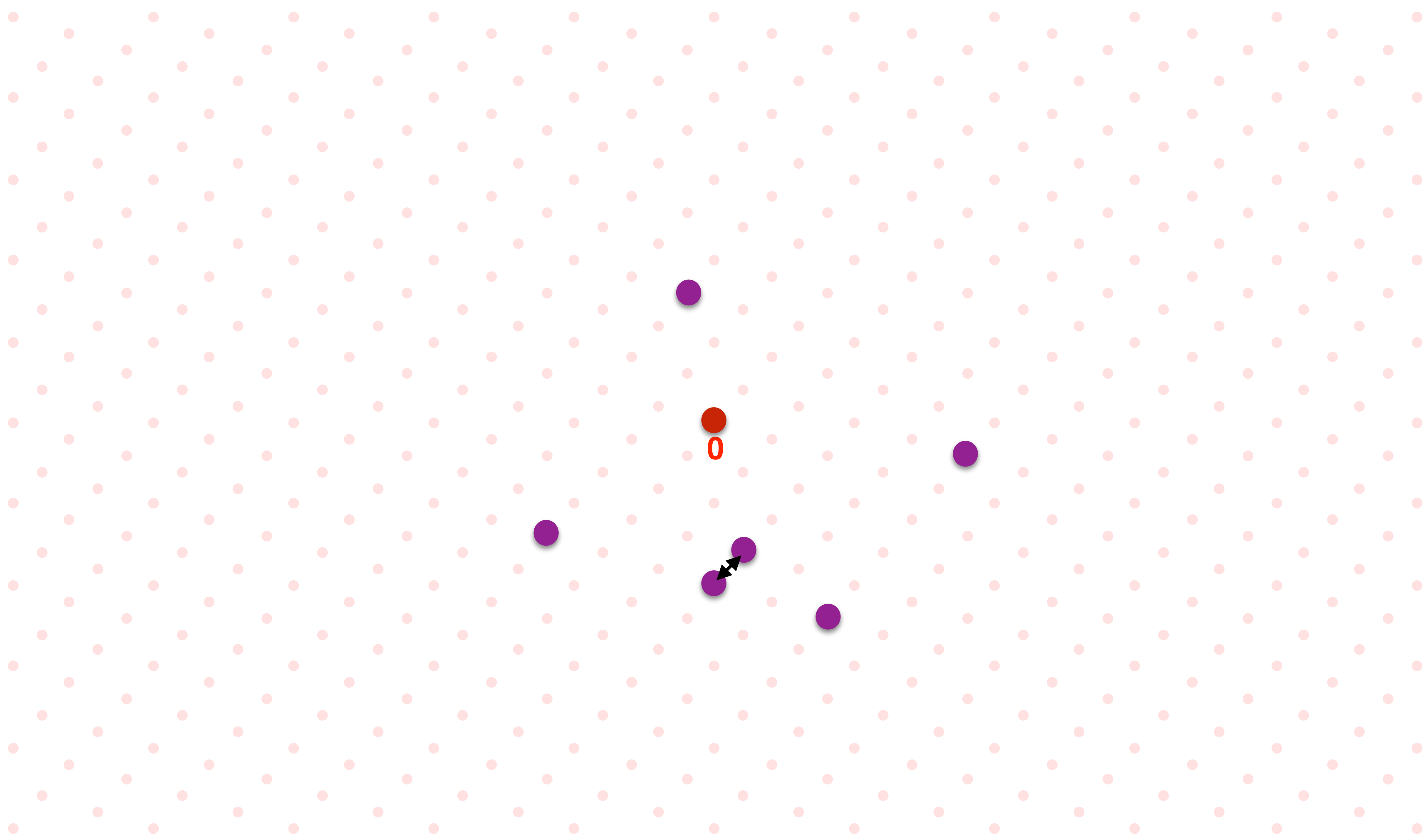

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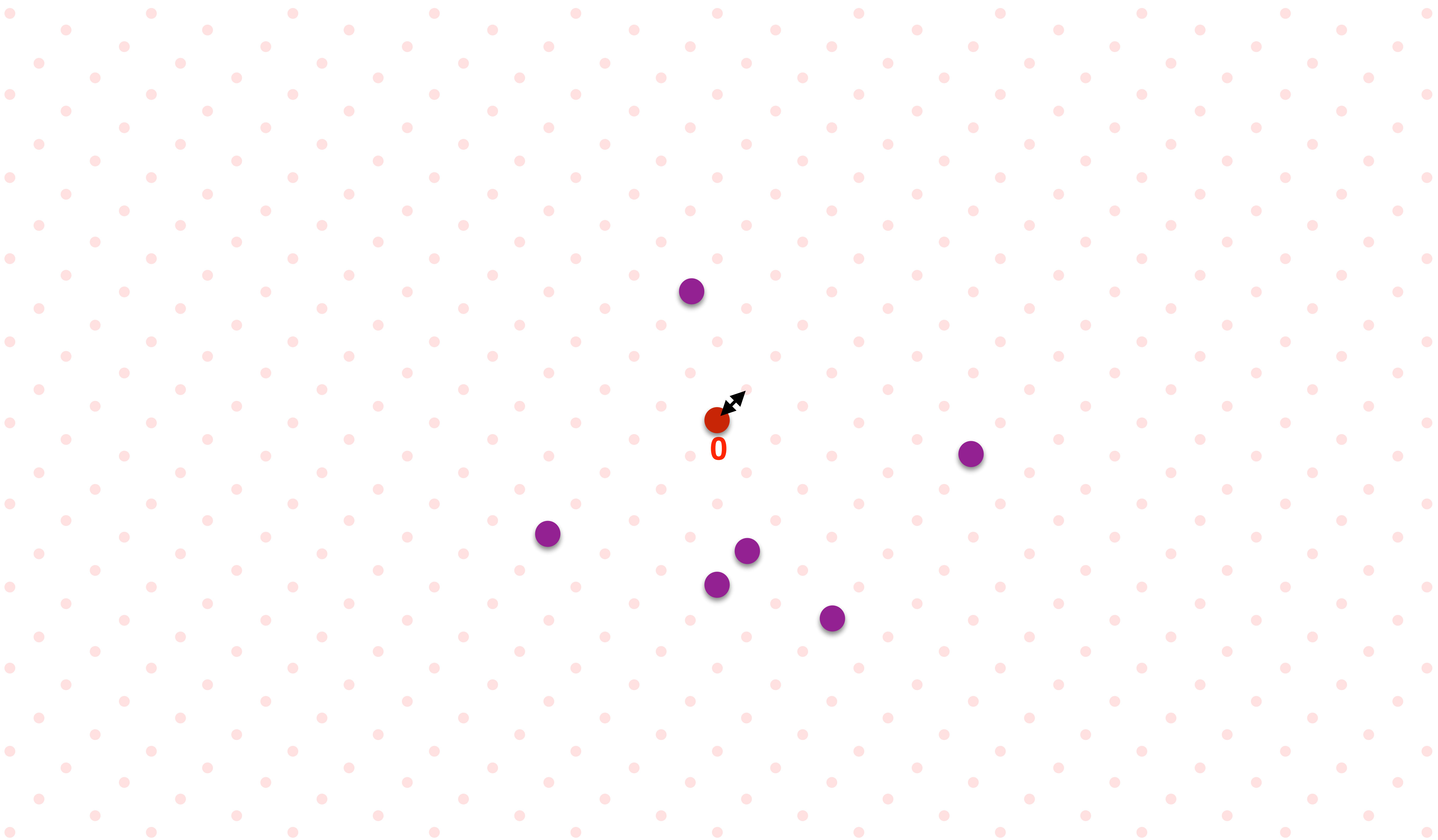

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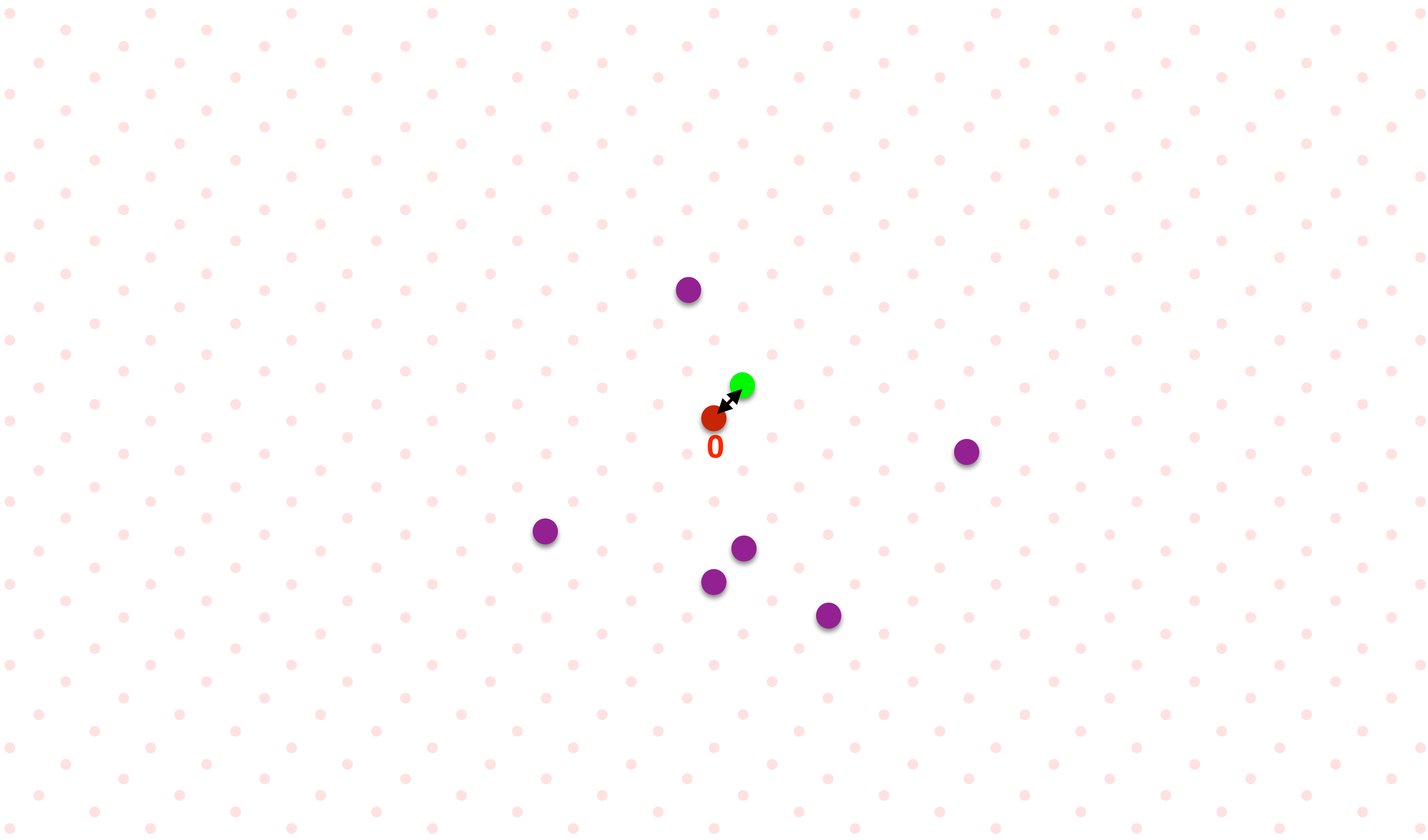

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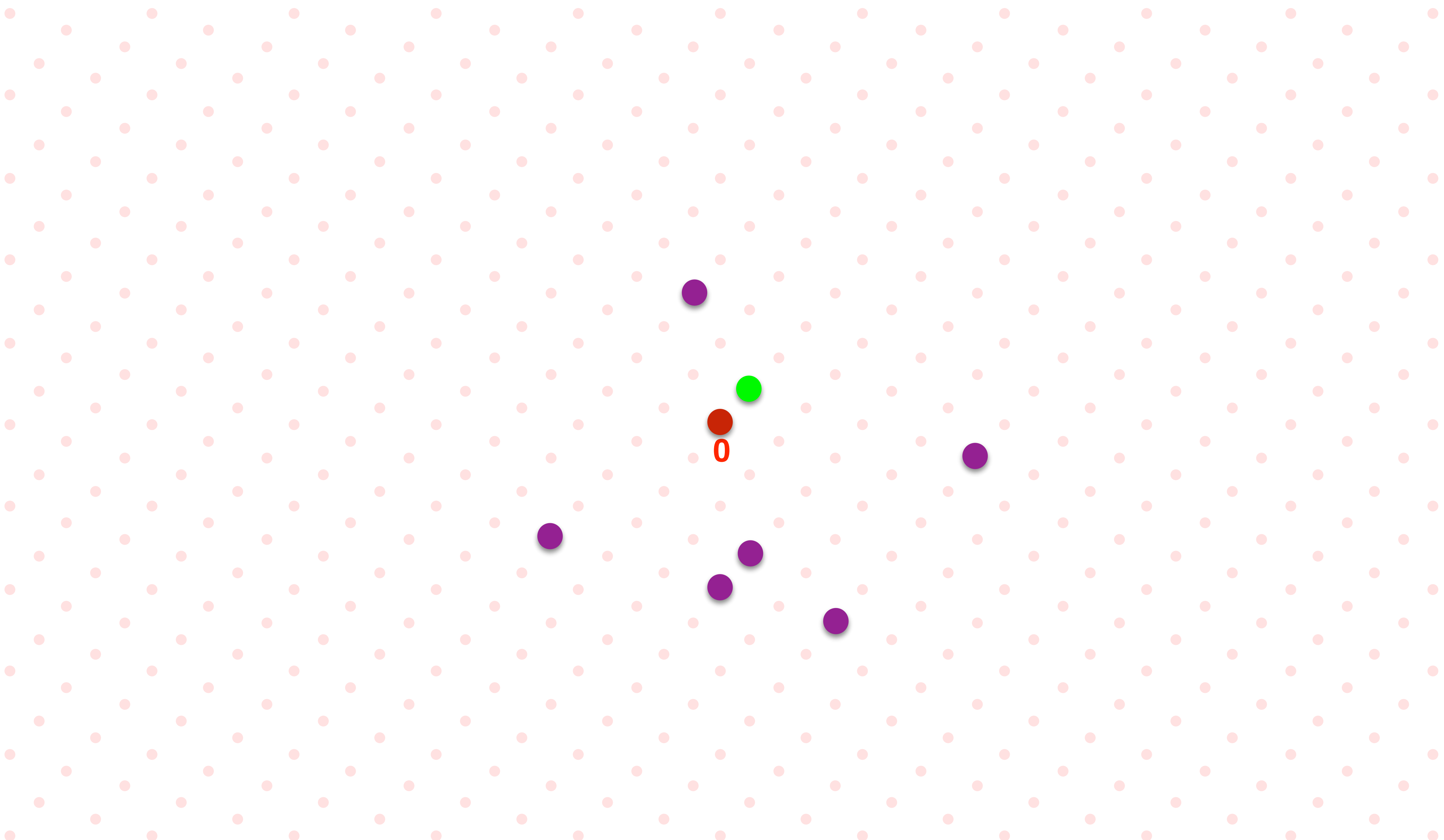

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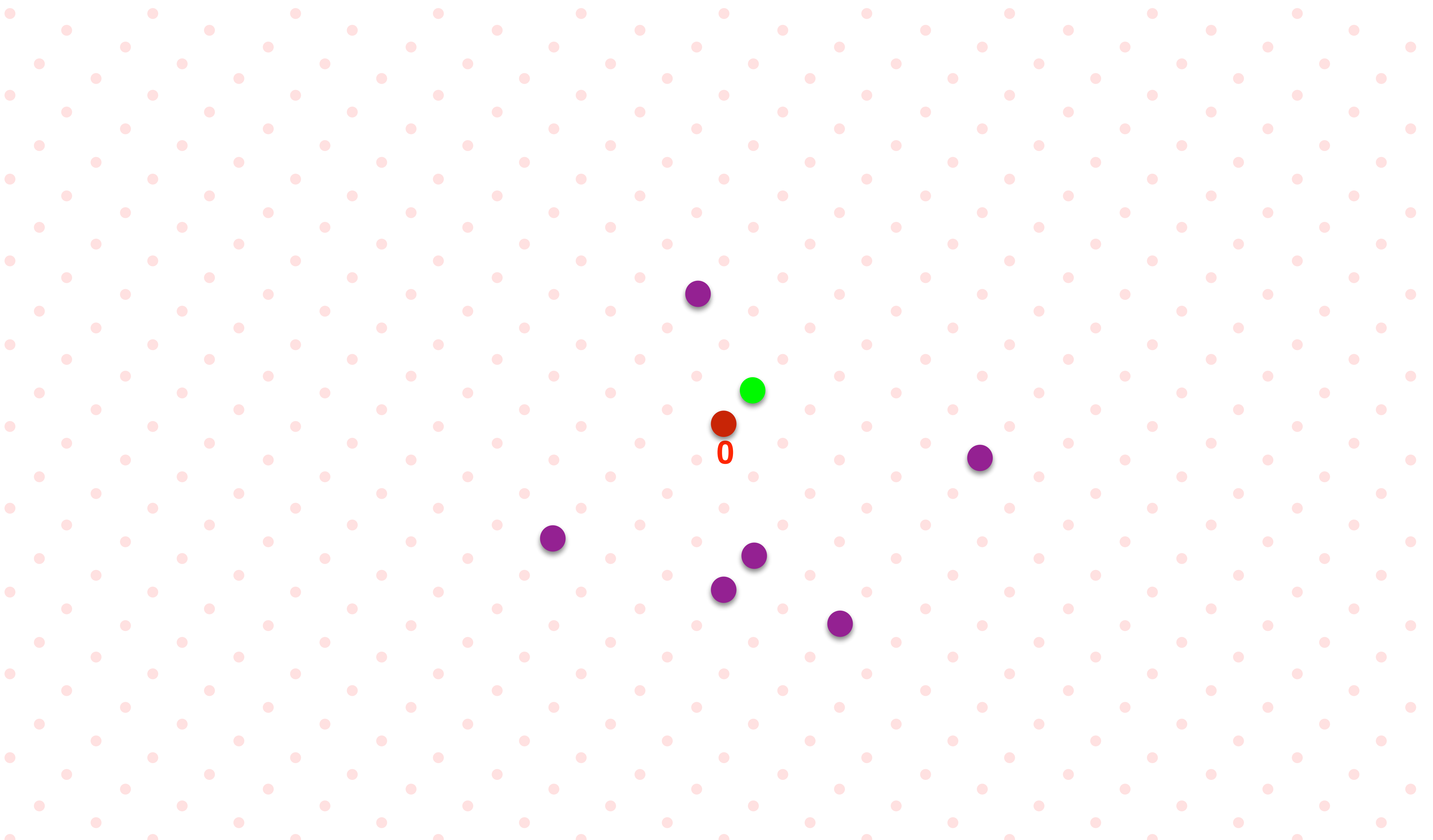

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- Hard to analyze.
- What is the distribution of the vectors at each step?
- How common are collisions?

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????	Fast!	[FuturePeople18]

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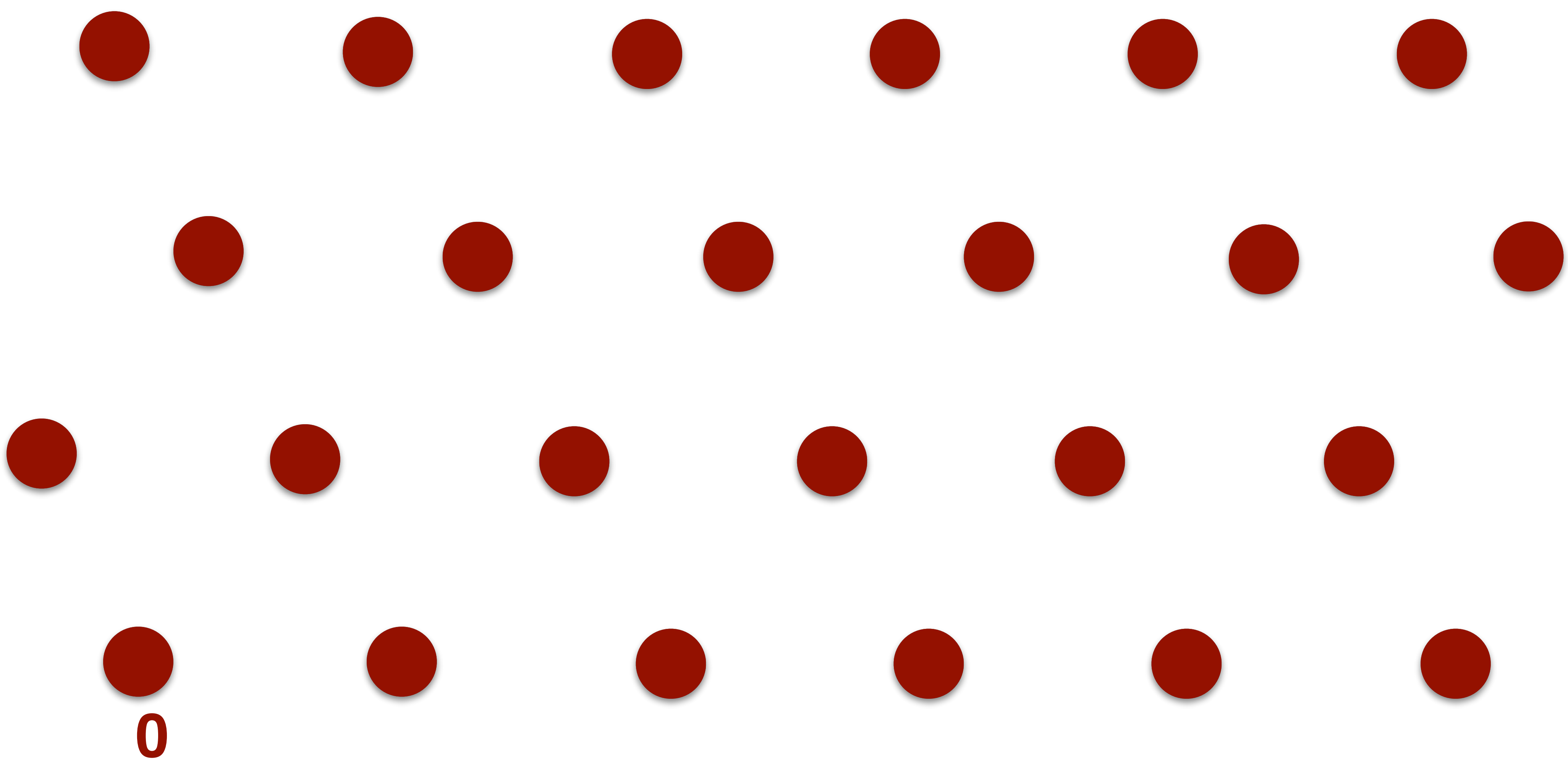

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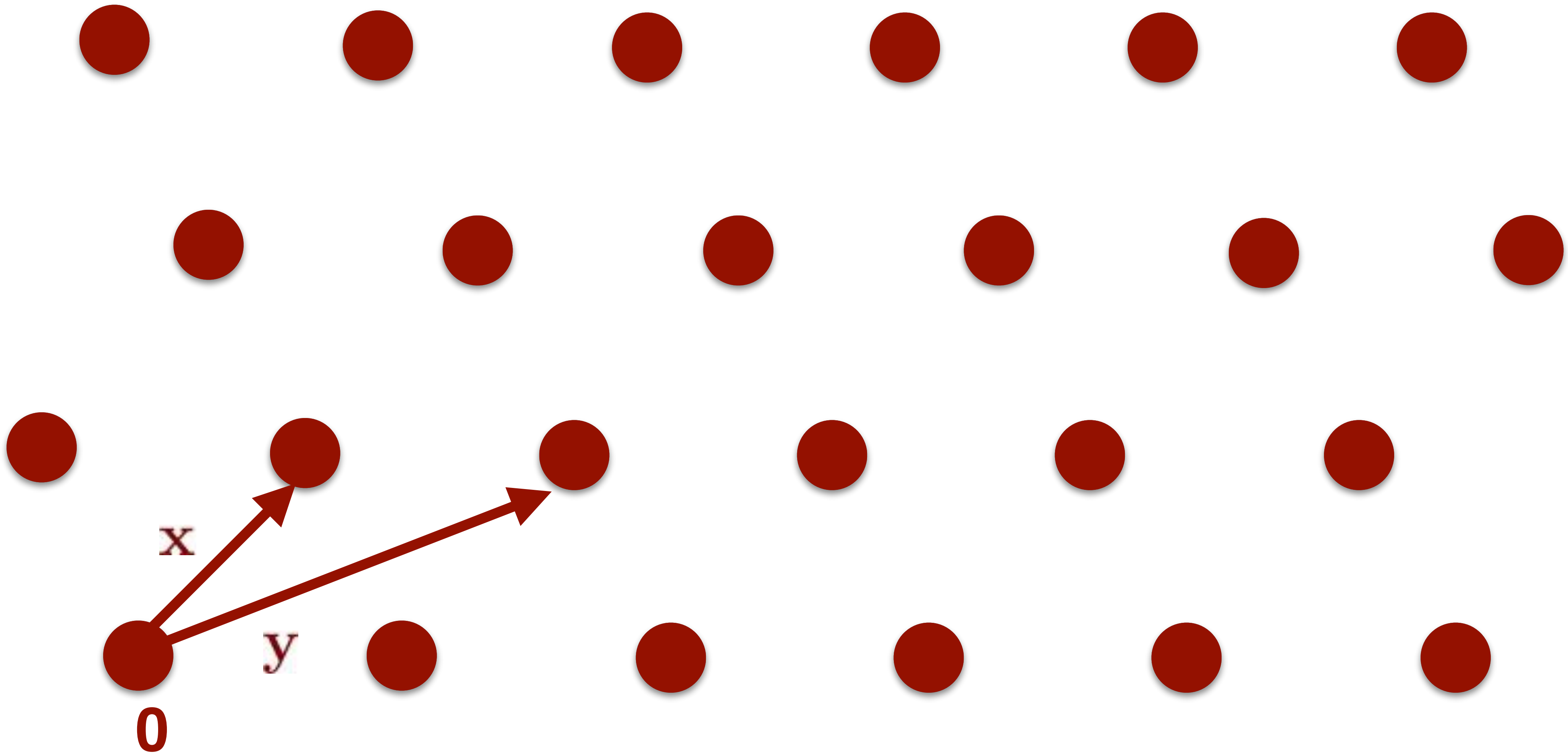

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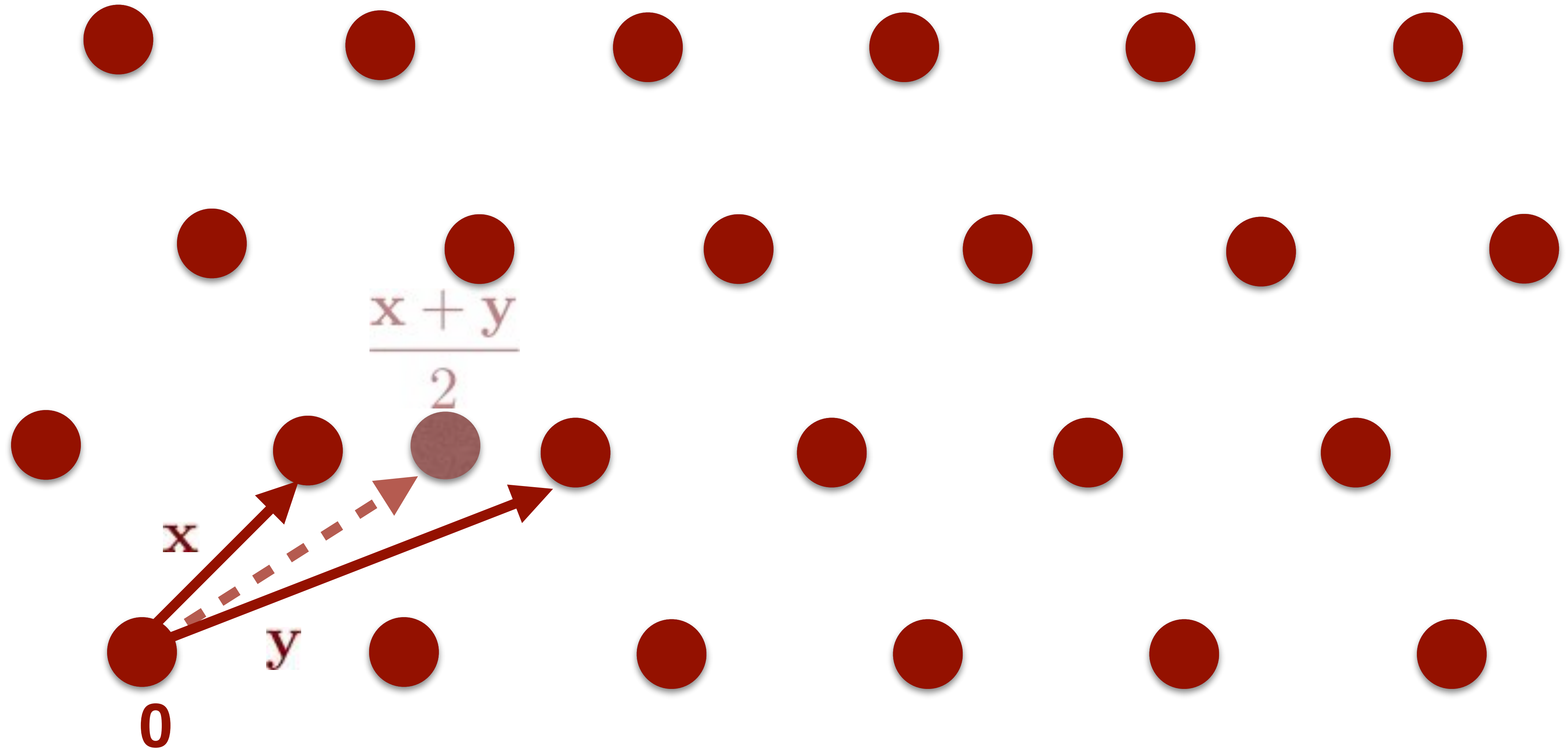

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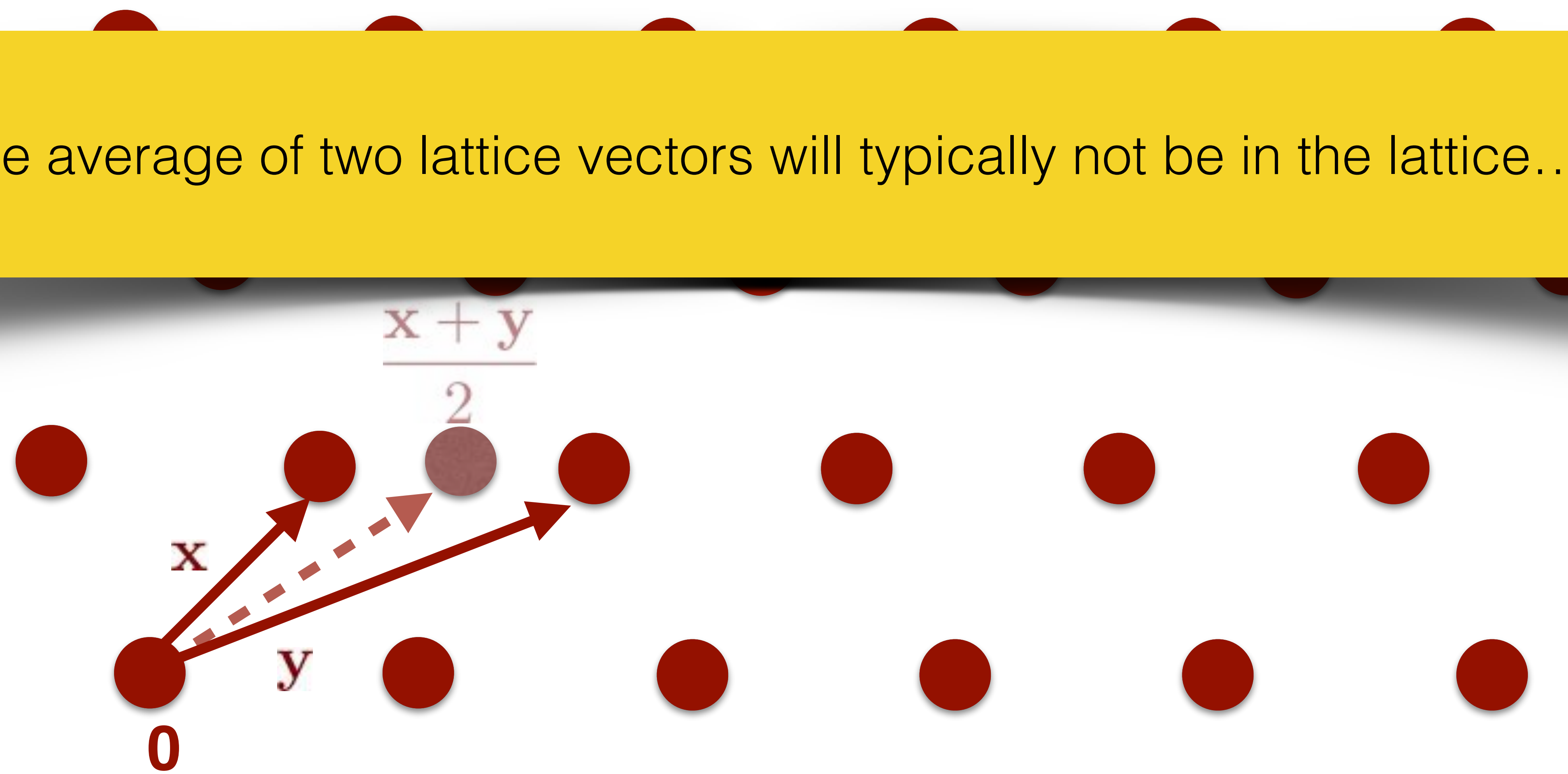
The average of two lattice vectors will typically not be in the lattice…



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$$\iff \mathbf{y}_1 \equiv \mathbf{y}_2 \mod 2\mathcal{L}$$

Sieving by Averages

When do we have $\dfrac{\mathbf{y}_1 + \mathbf{y}_2}{2} \in \mathcal{L}$?

$$\mathbf{y}_1 = a_{1,1}\mathbf{b}_1 + \cdots + a_{1,n}\mathbf{b}_n \qquad \mathbf{y}_2 = a_{2,1}\mathbf{b}_1 + \cdots + a_{2,n}\mathbf{b}_n$$

We have $\dfrac{\mathbf{y}_1 + \mathbf{y}_2}{2} \in \mathcal{L}$ if and only if $\mathbf{y}_1, \mathbf{y}_2$ are in the same ***coset*** *of $2\mathcal{L}$.*

(Note that there are 2^n cosets.)

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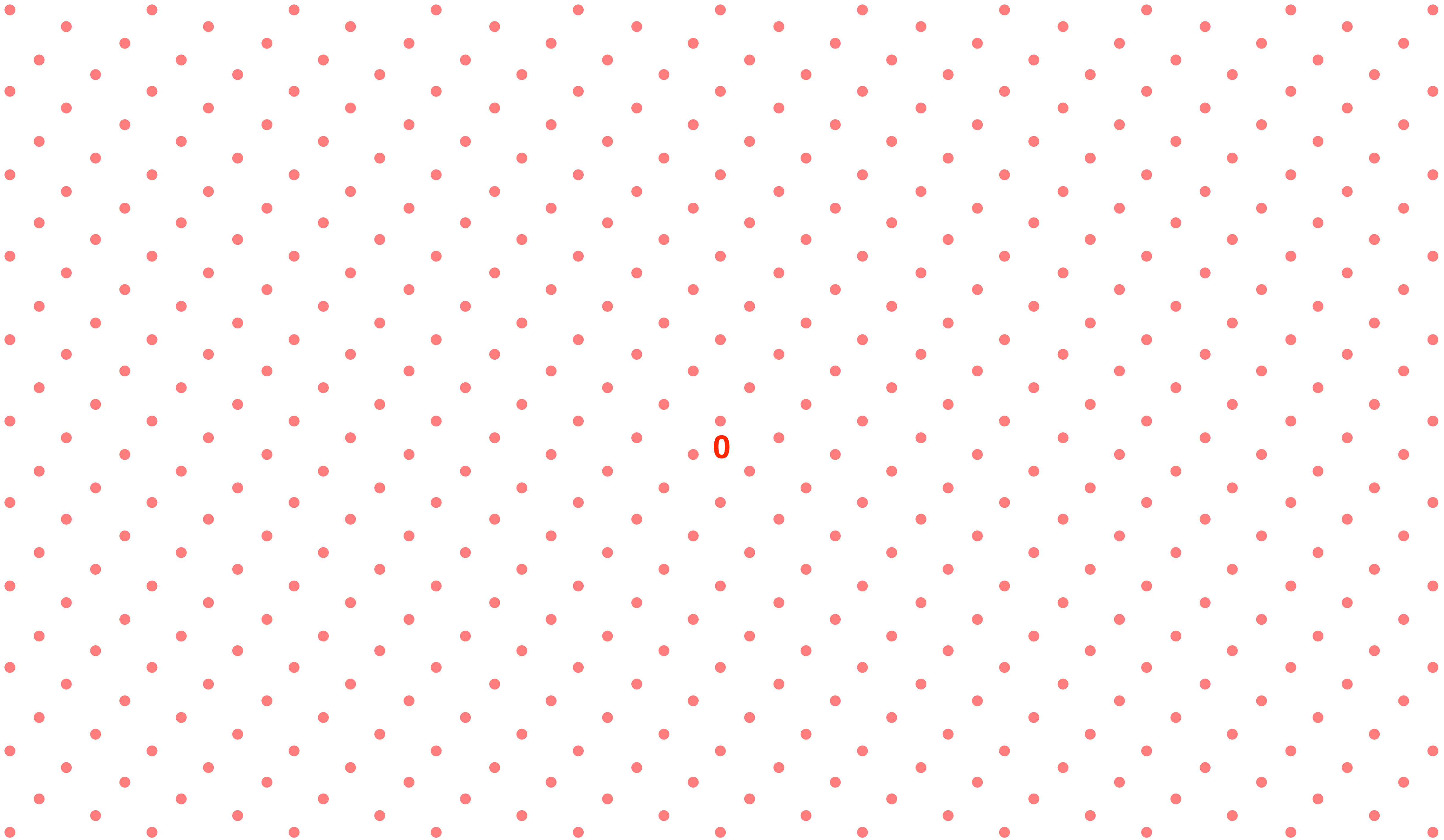

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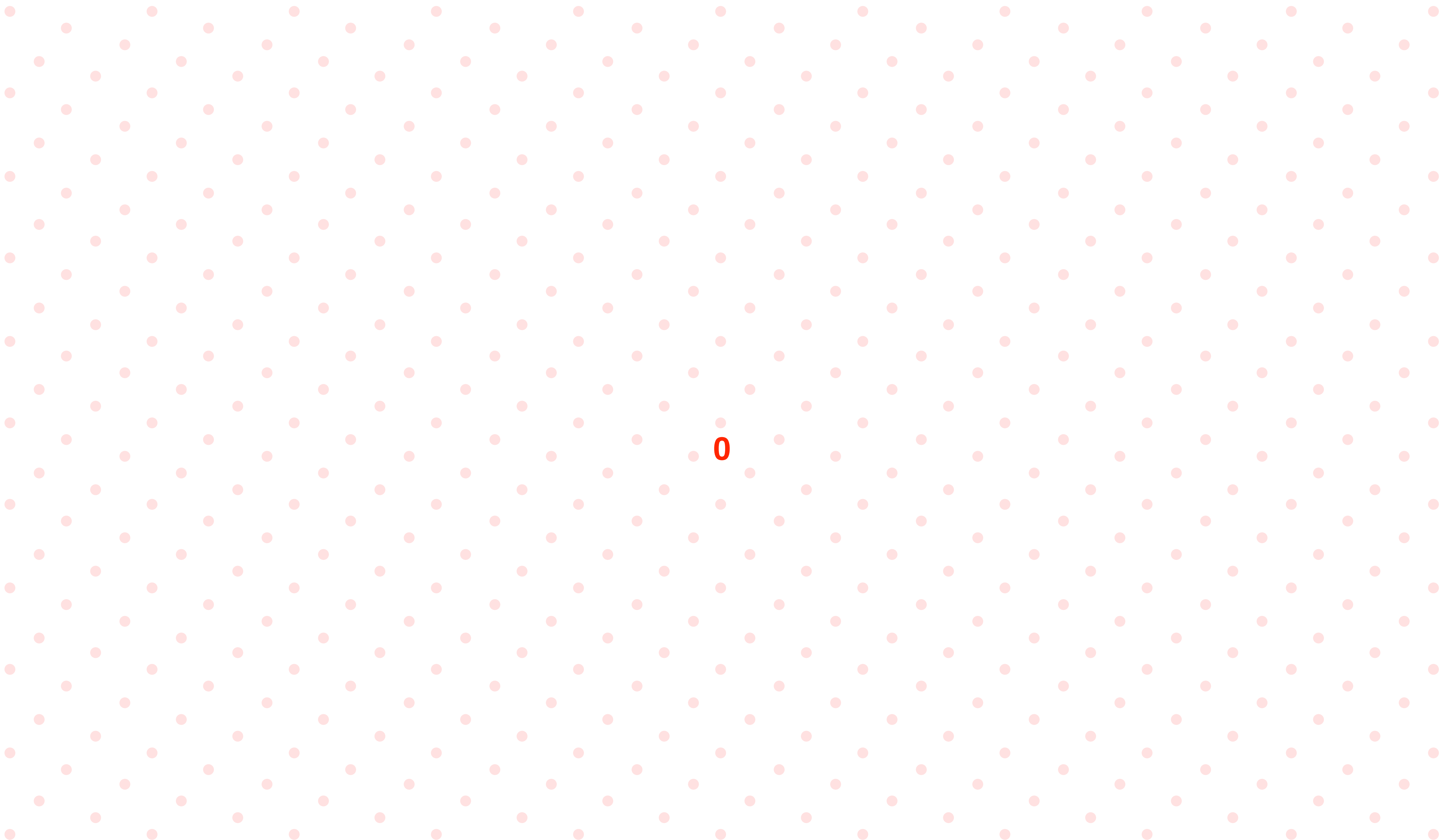

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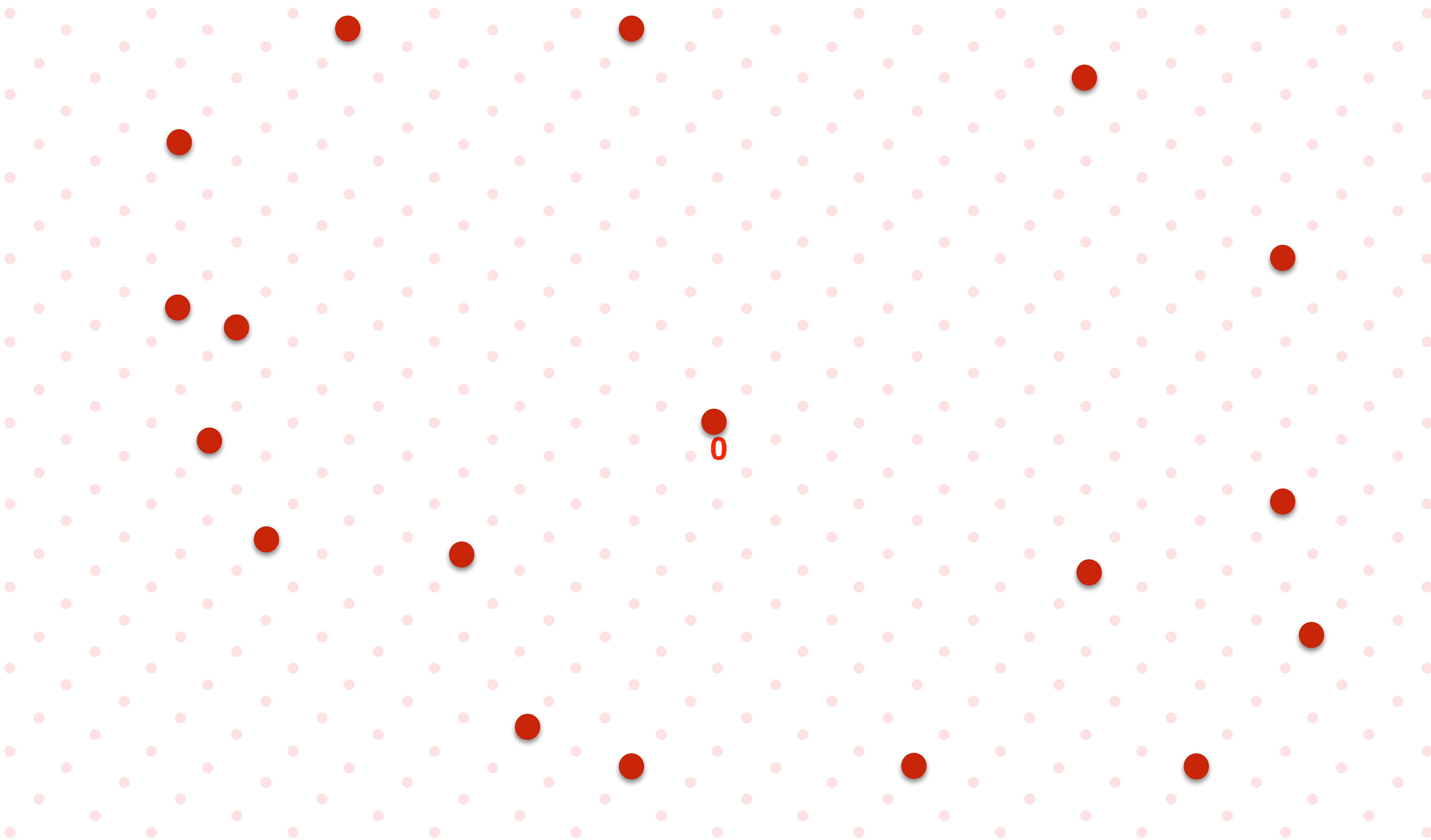

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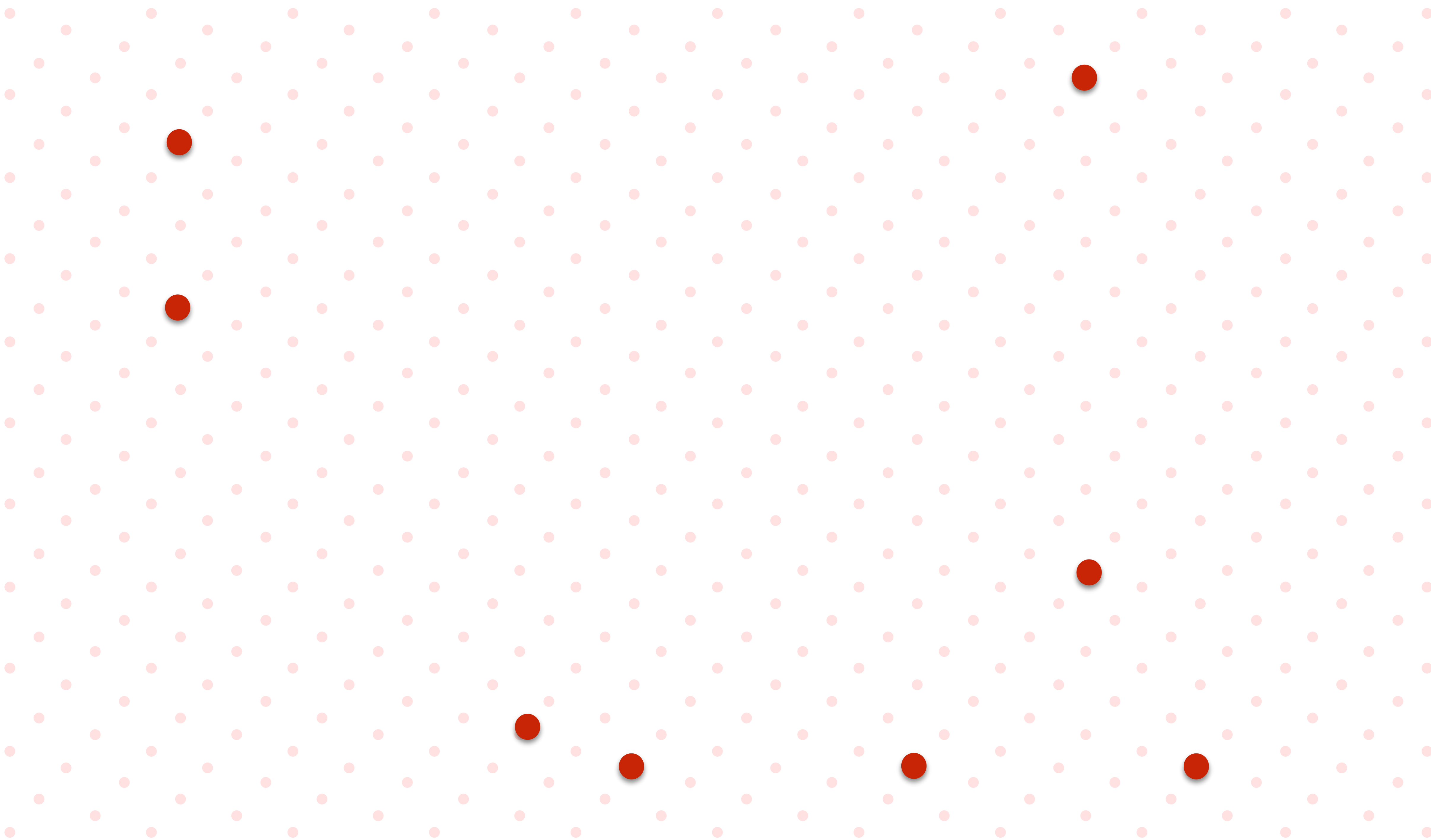

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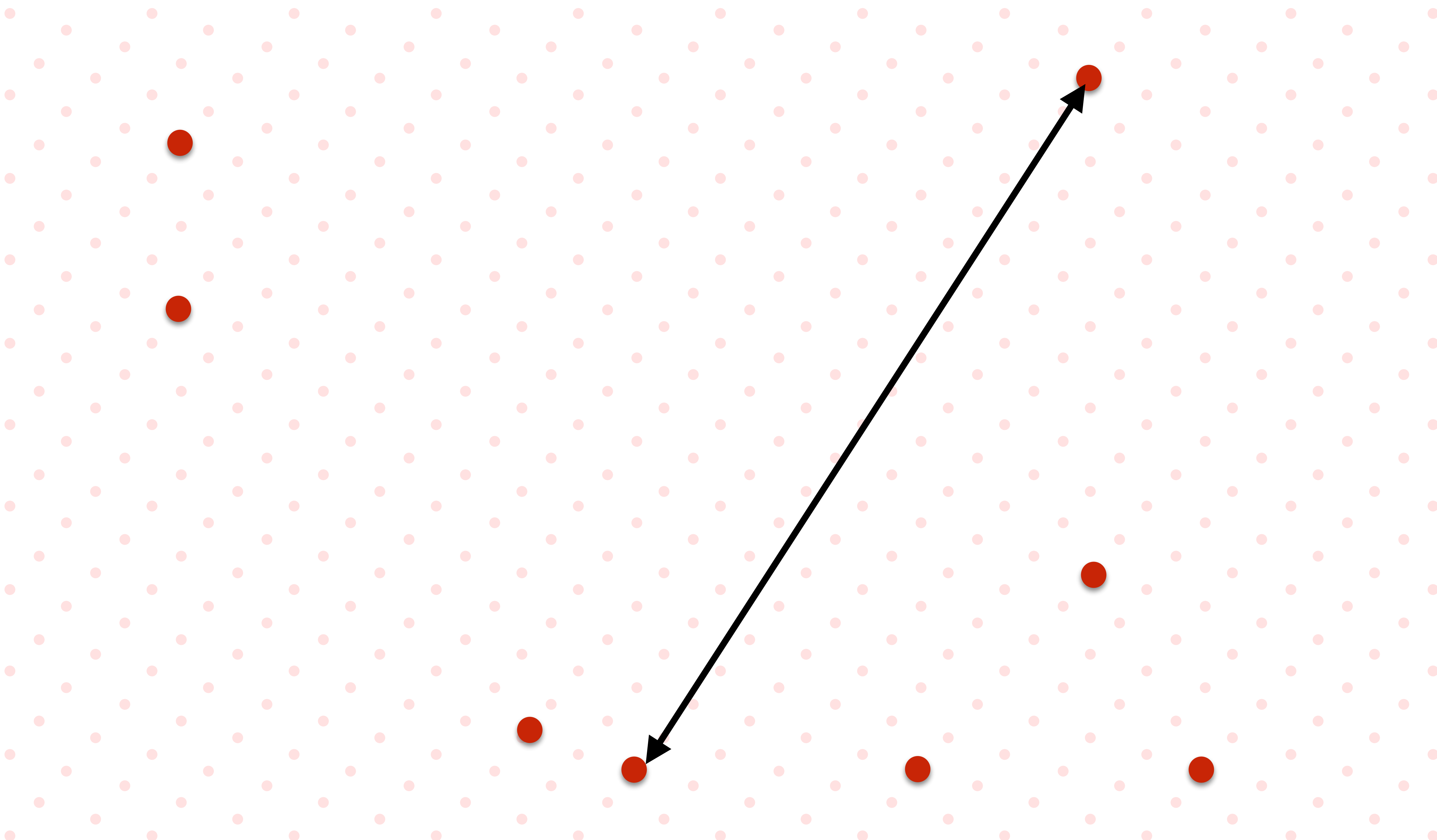

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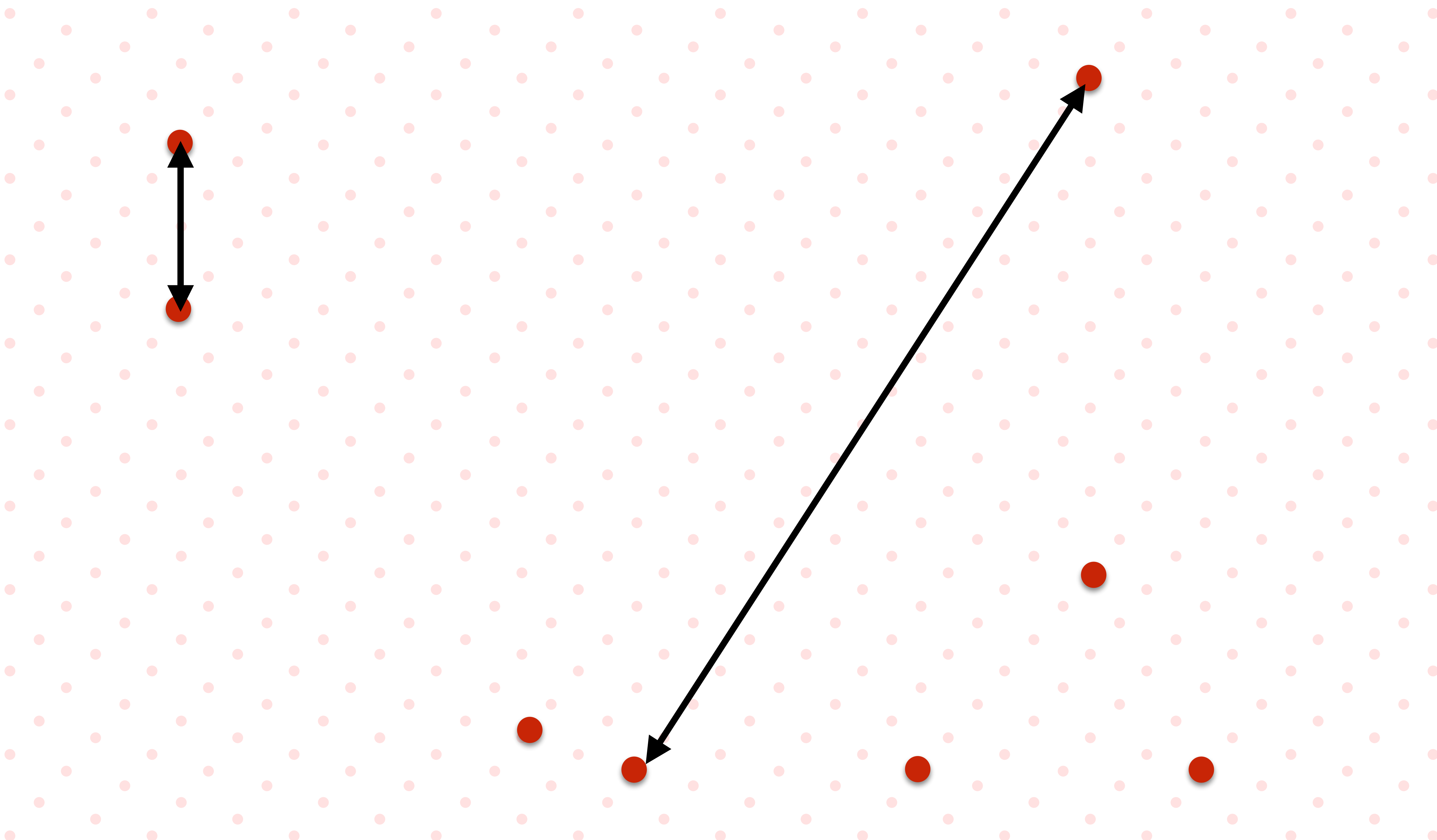

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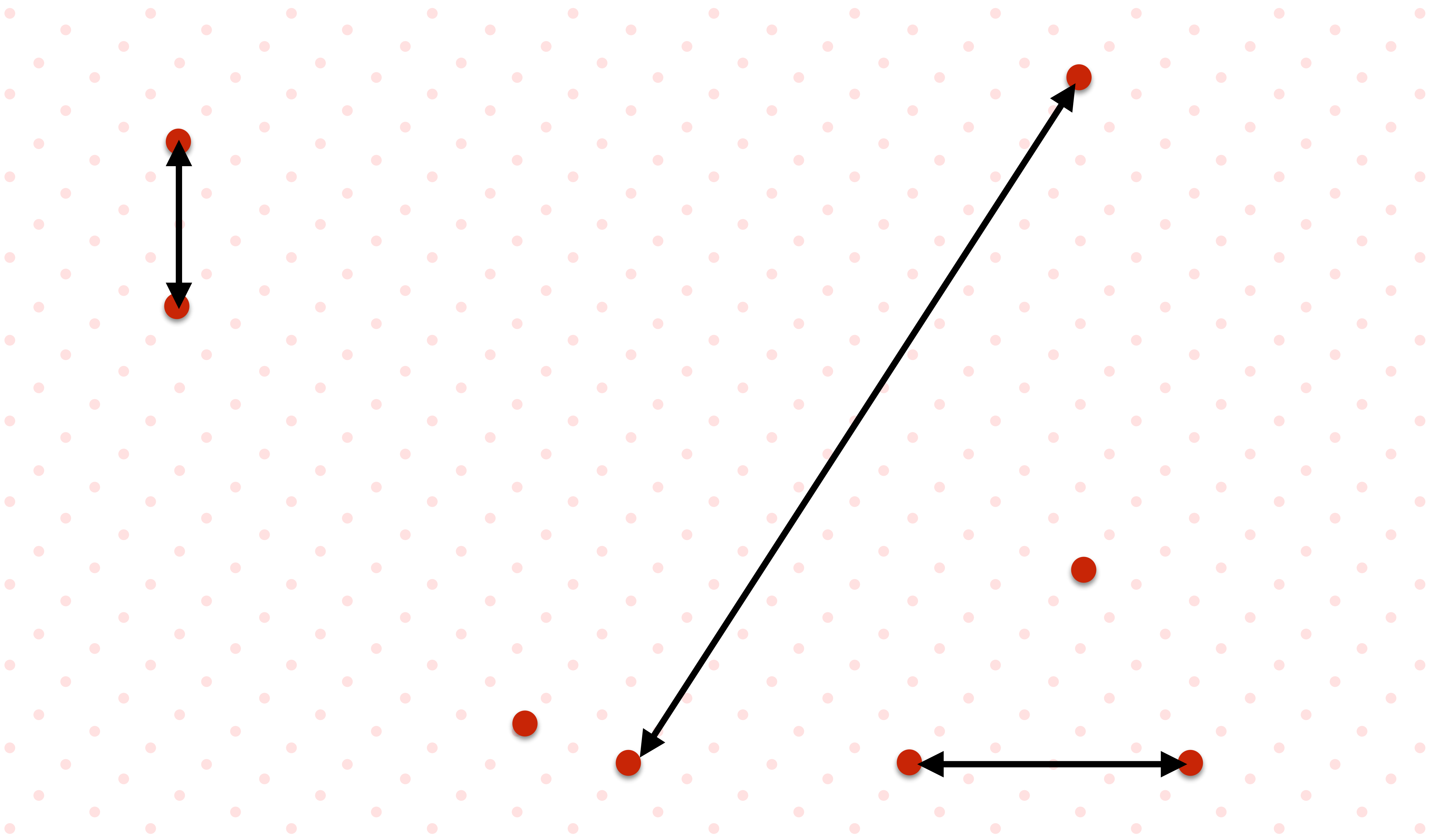

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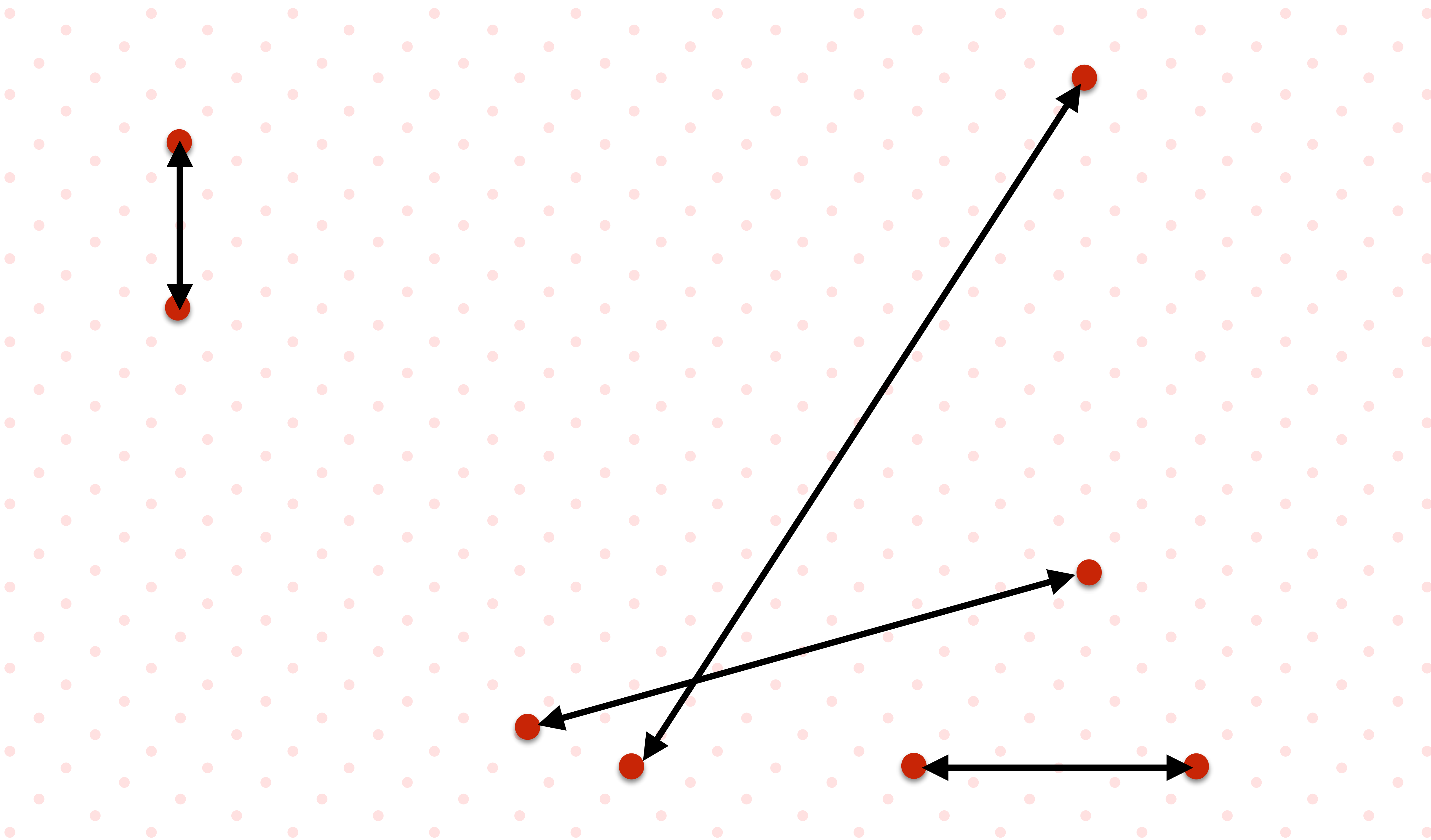

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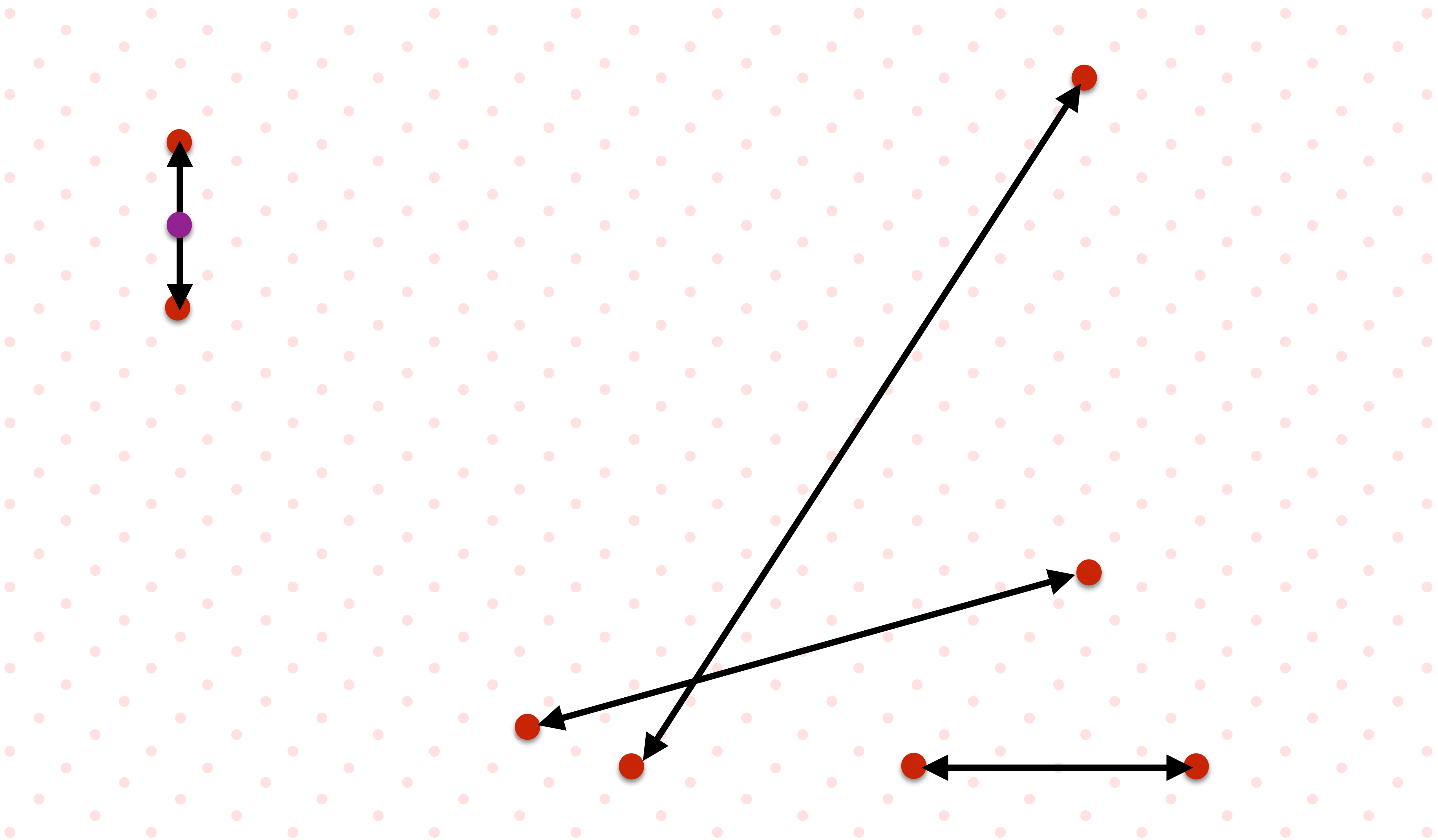

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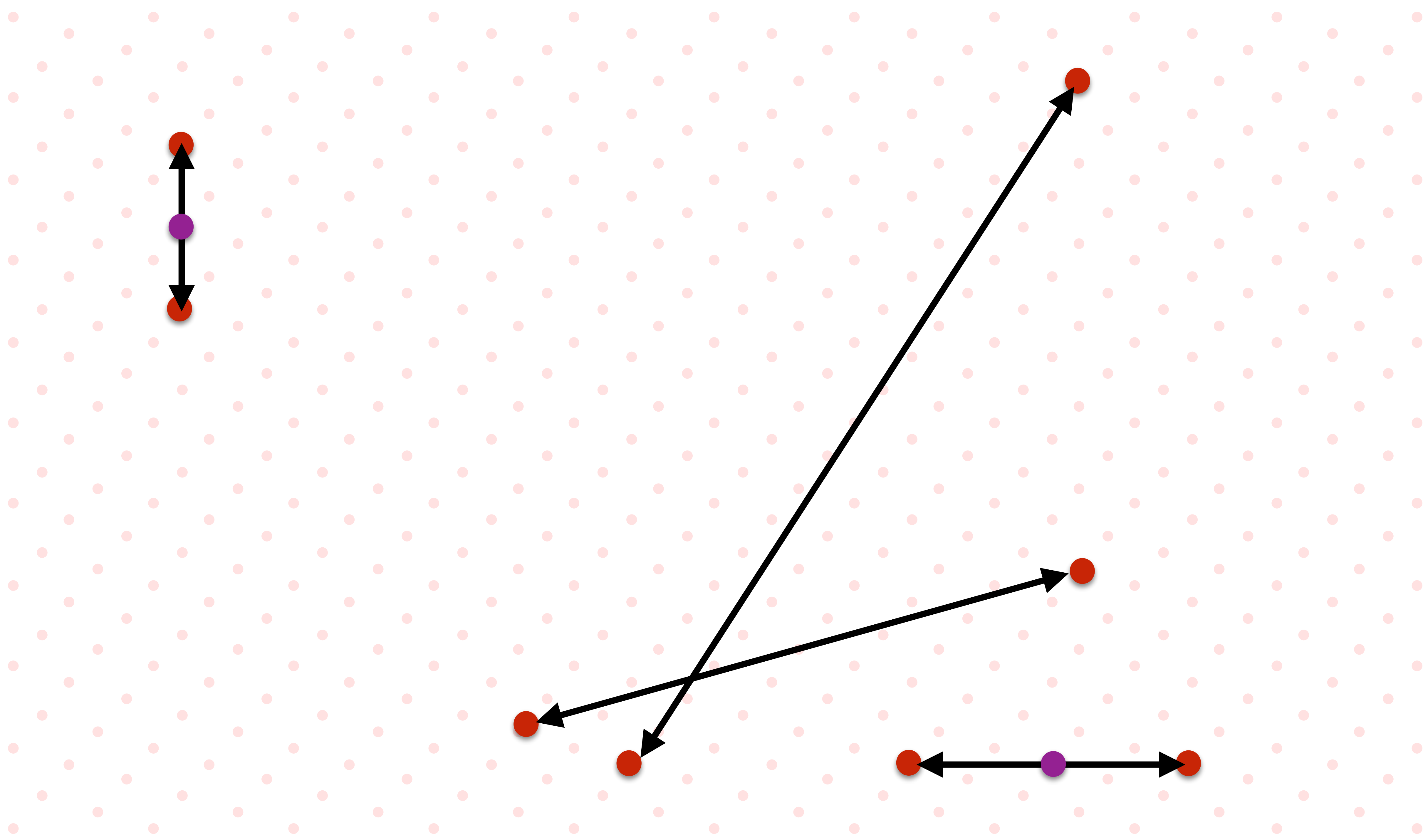

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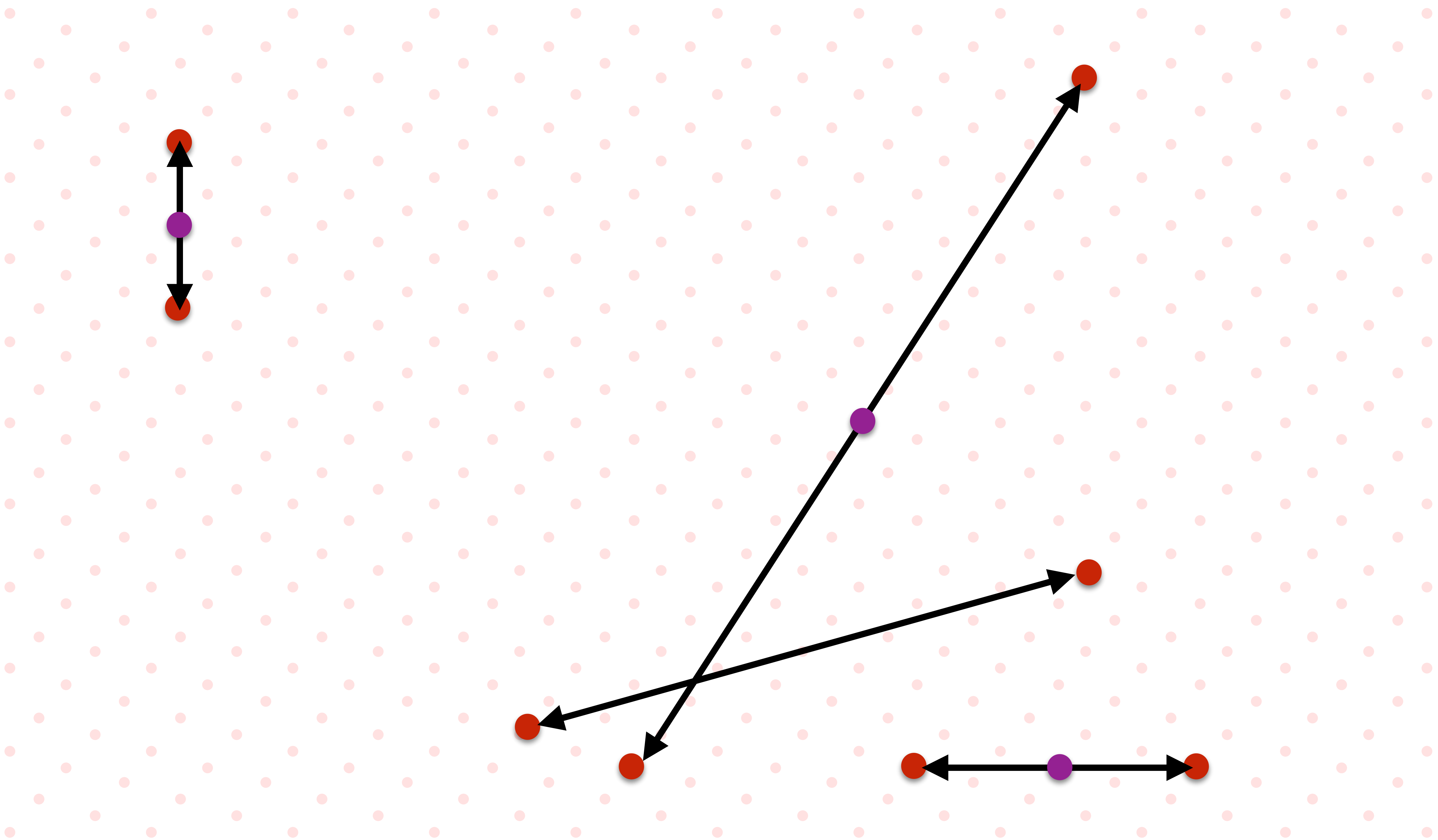

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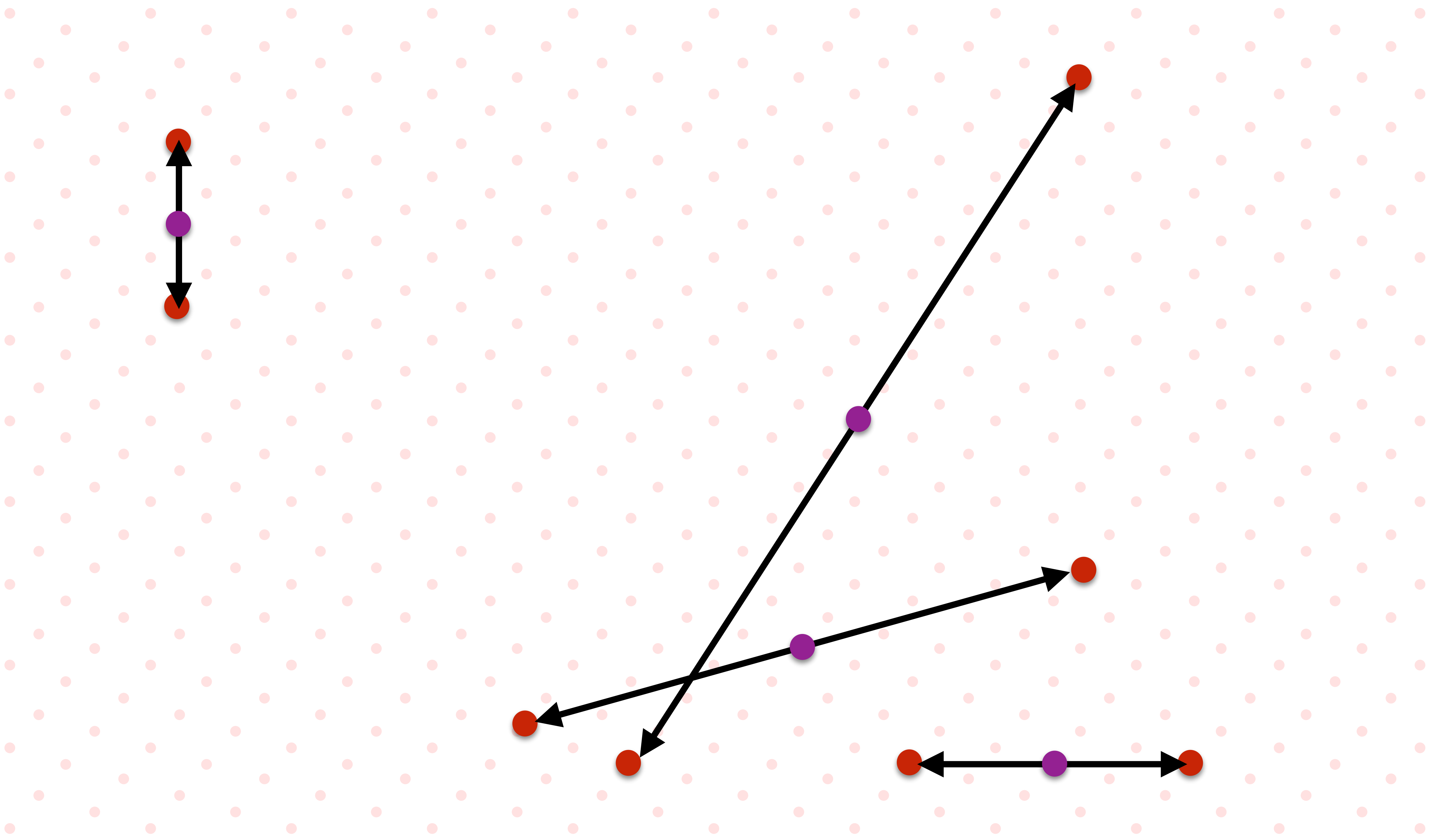

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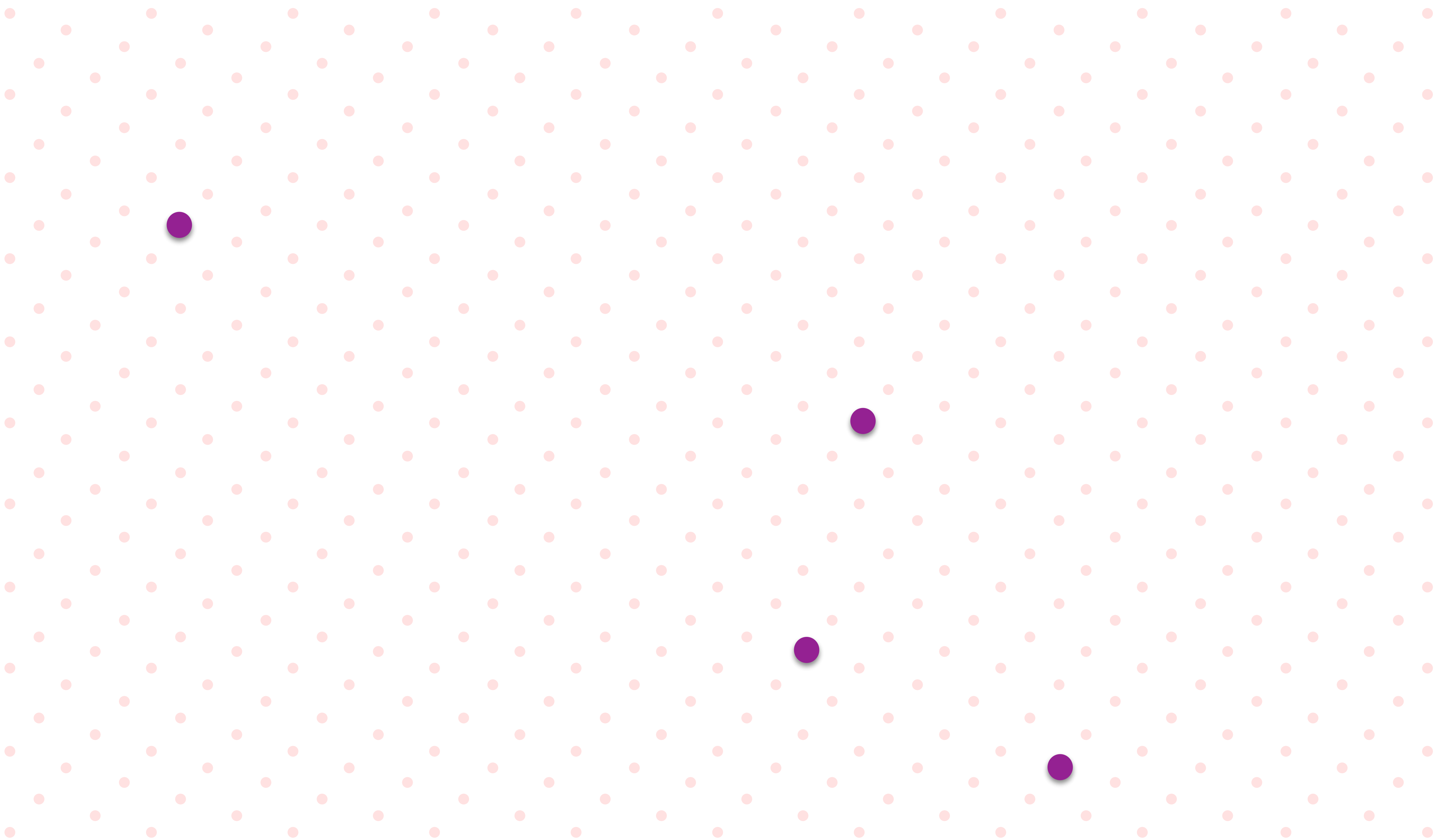

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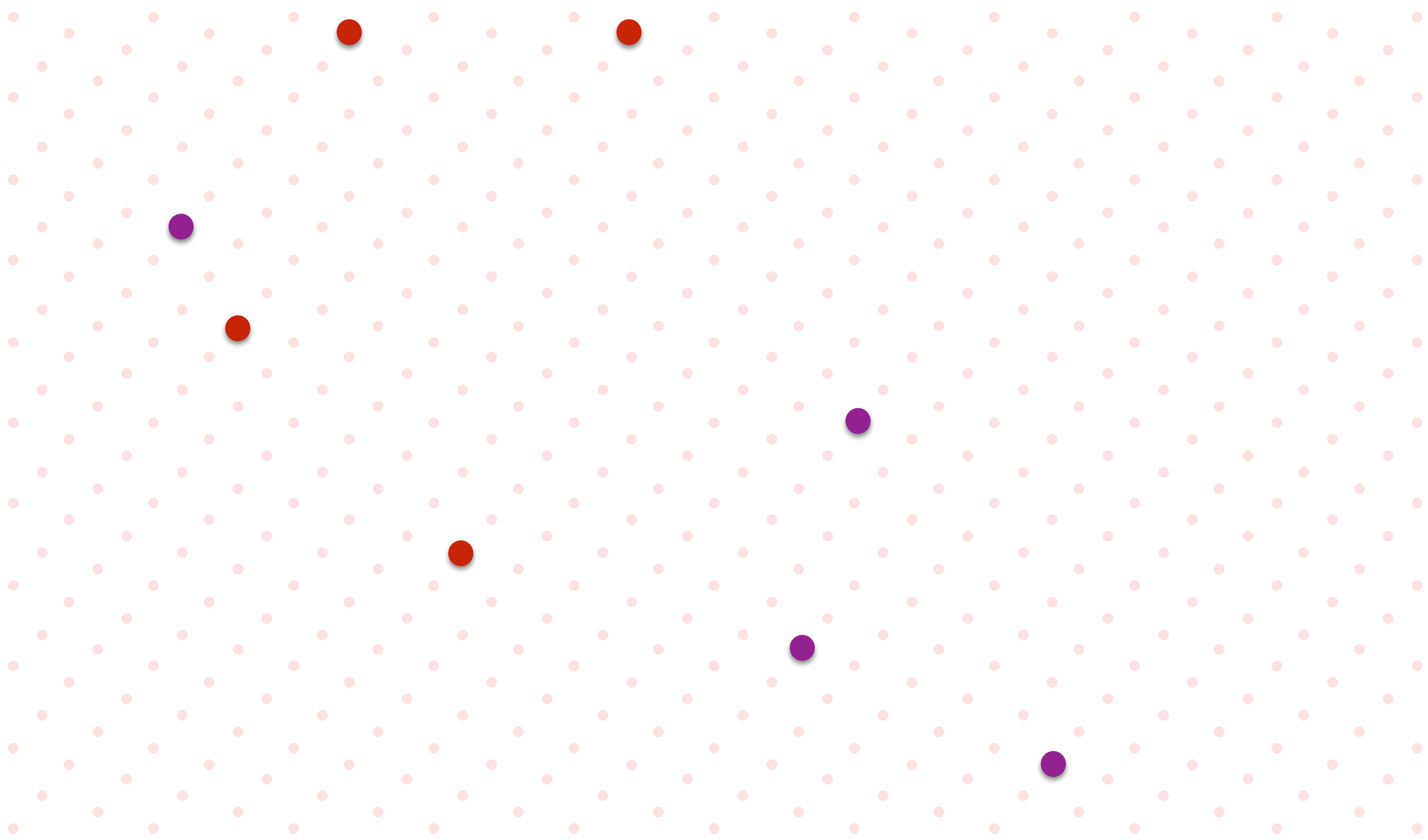

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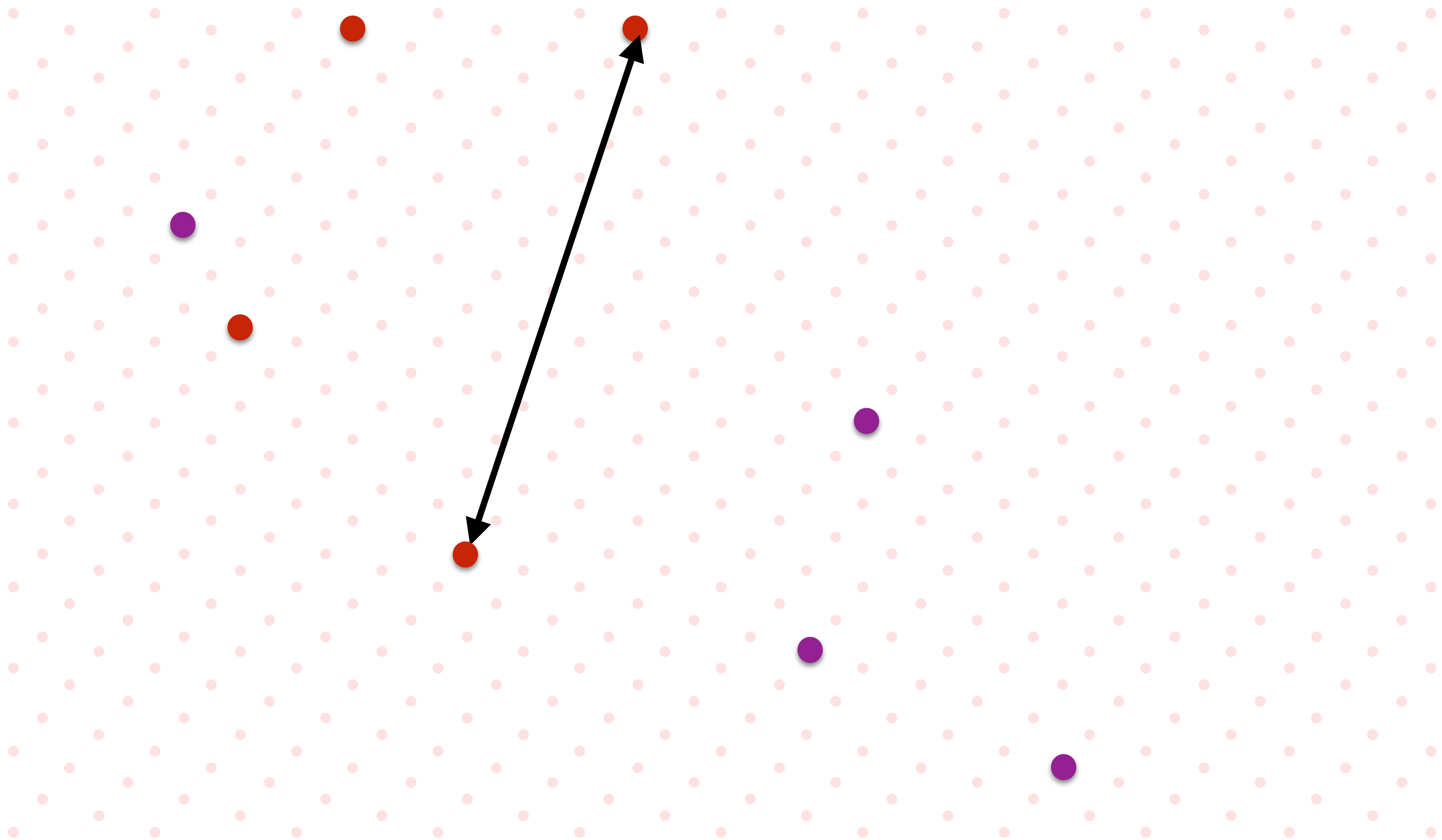

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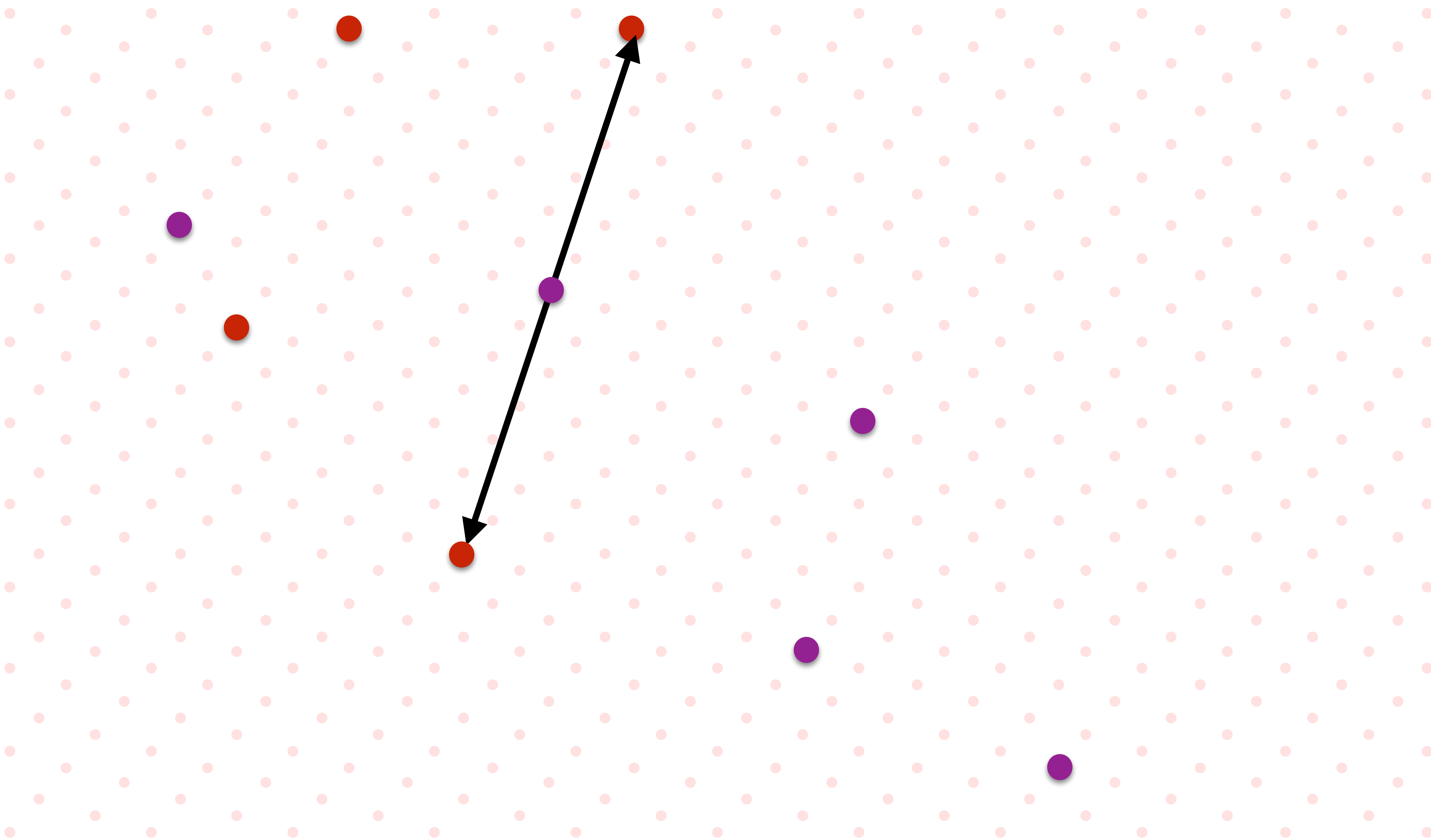

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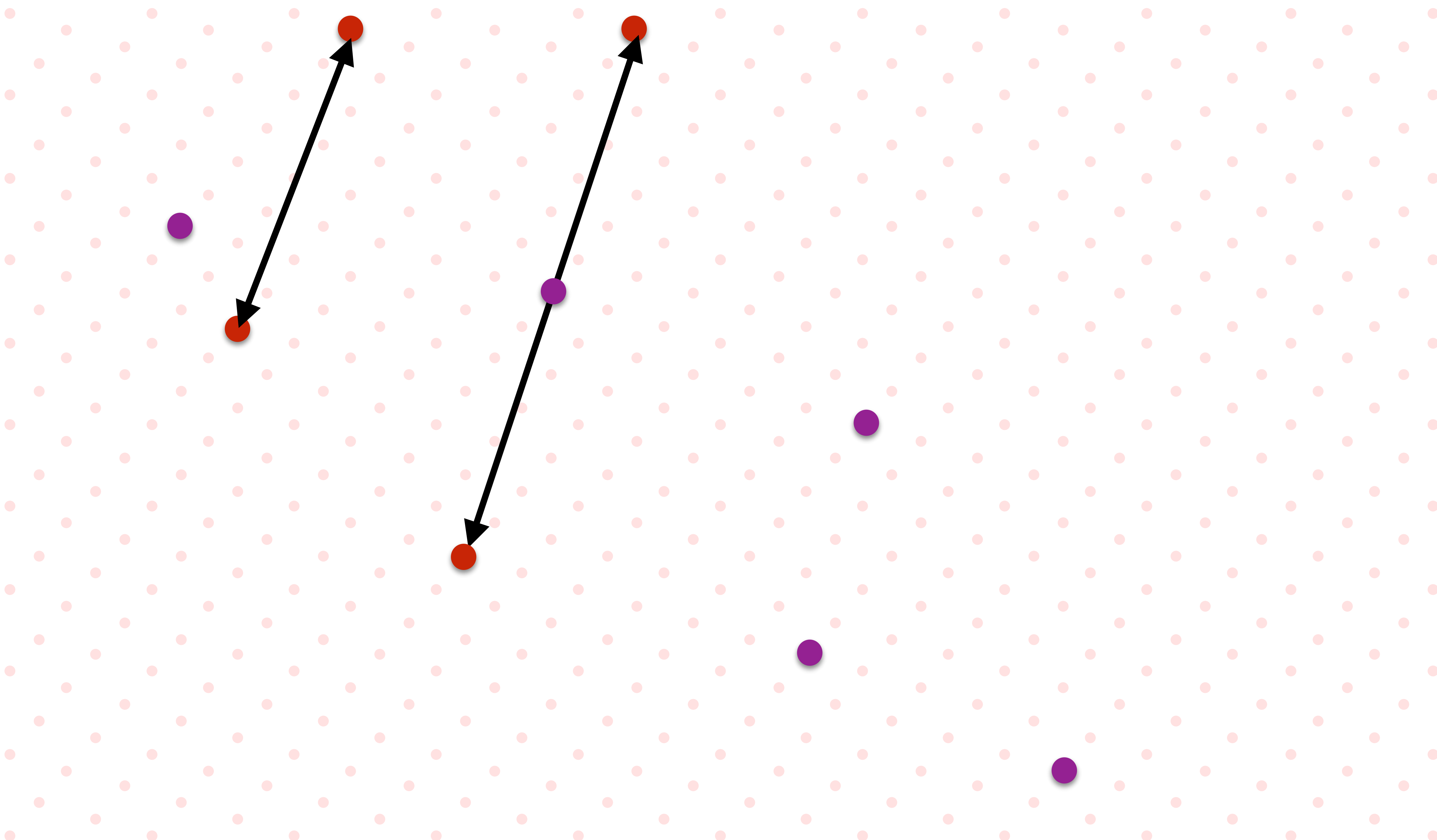

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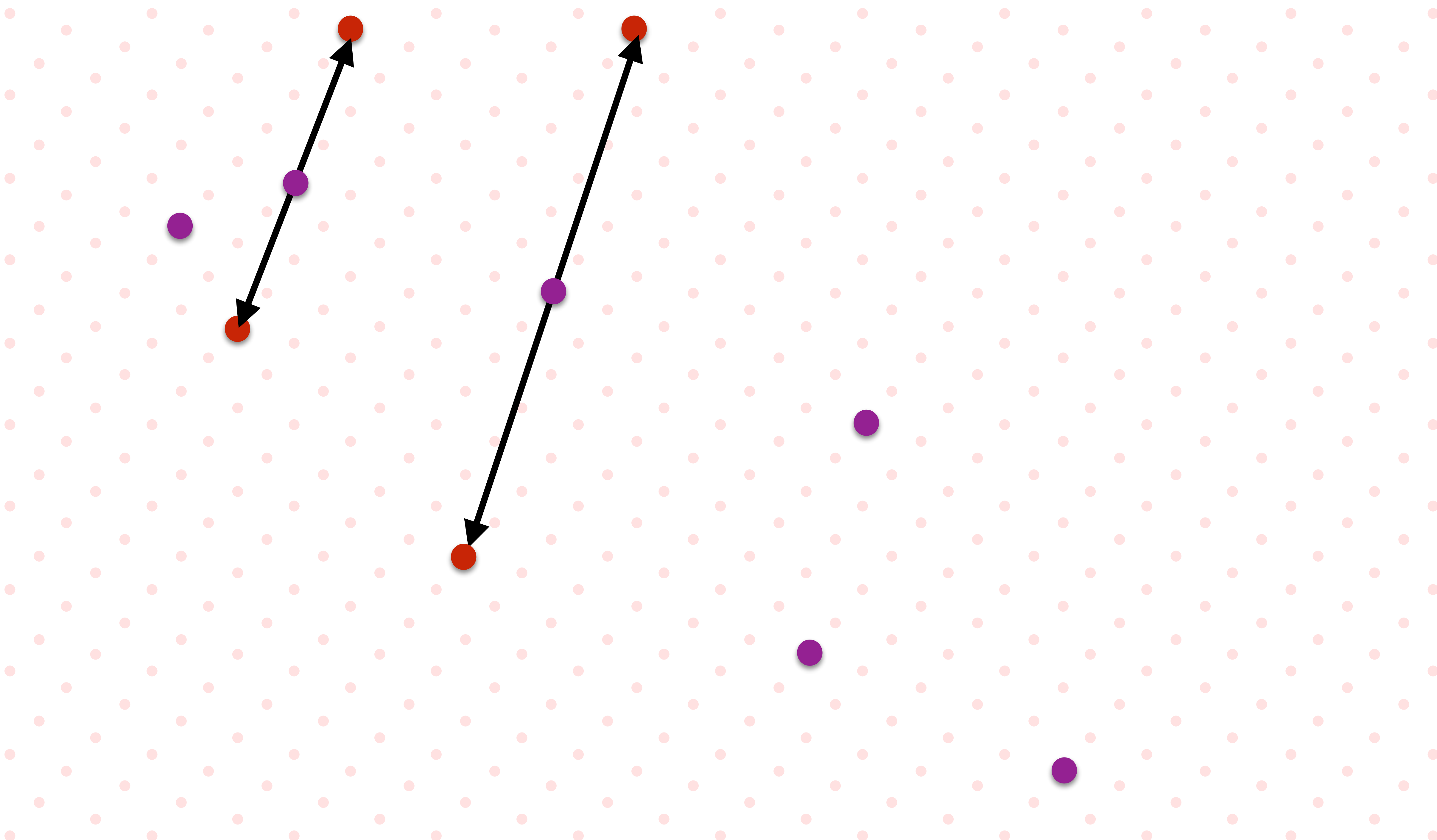

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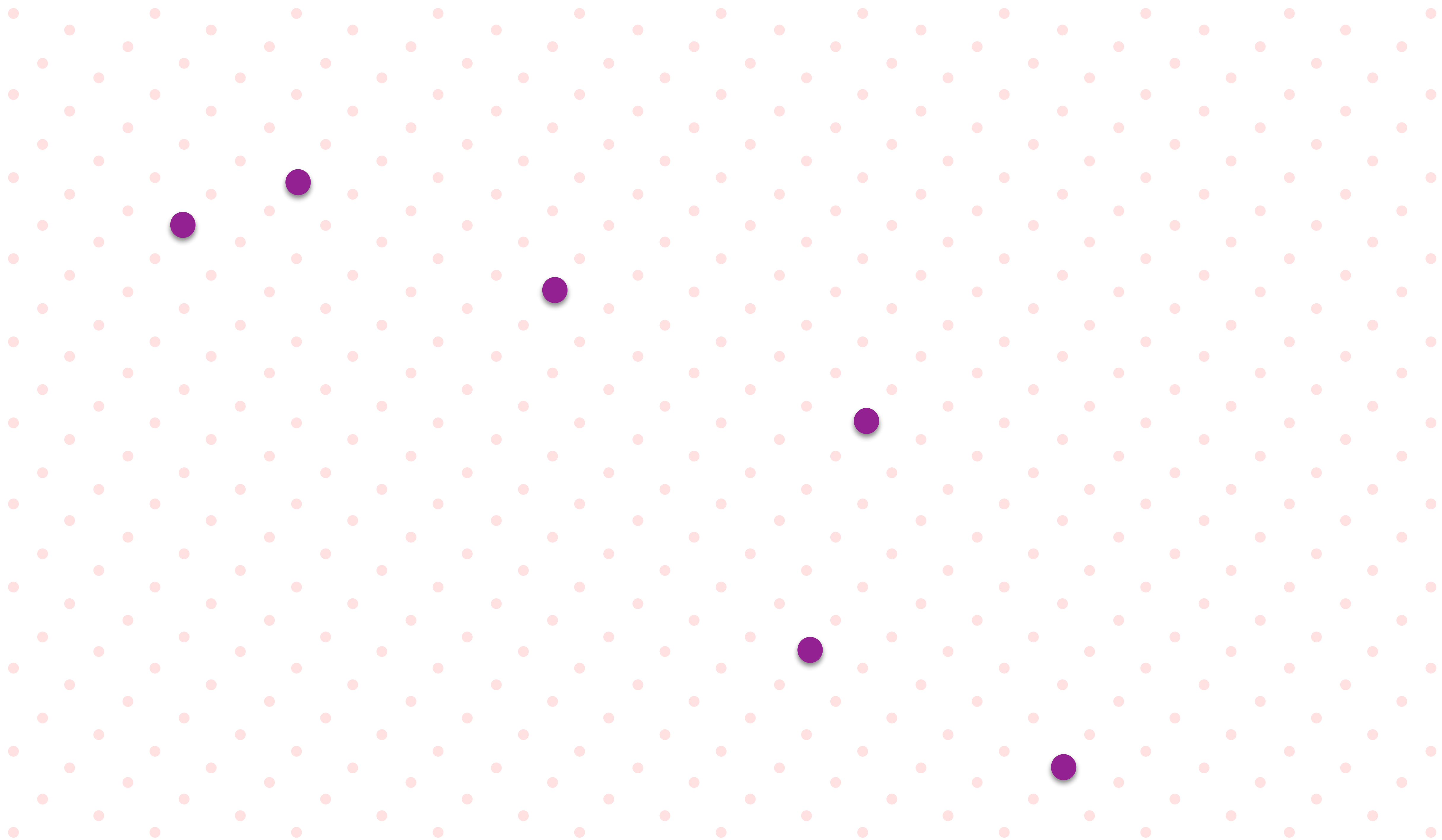

Sieving by Averages

1. Start with many vectors sampled from some nice distribution.
2. For each coset of $2\mathcal{L}$, group the vectors within the coset into disjoint pairs (randomly).
3. Take the average of each pair.
4. Repeat this procedure on the averages.

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Why Does This Work?

Cosets

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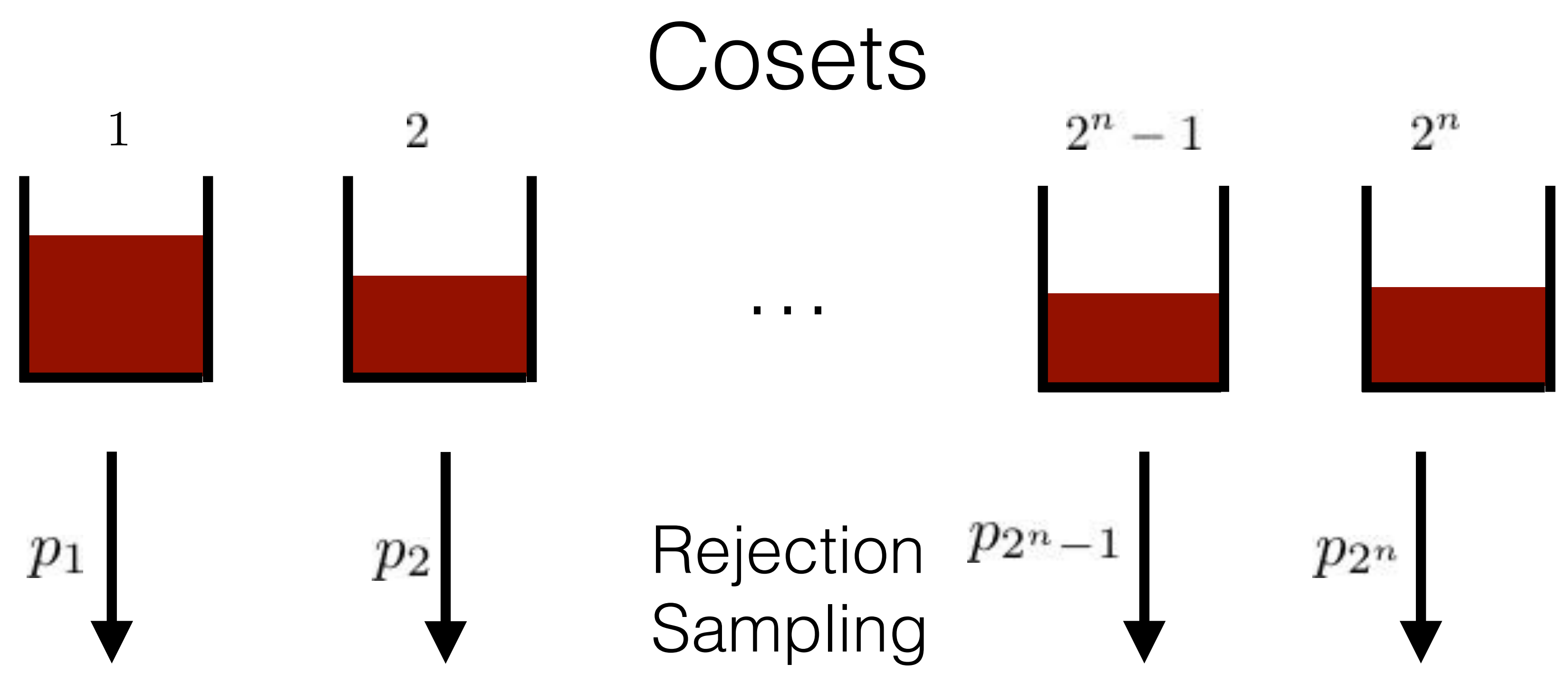
Rejection
Sampling

Why Does This Work?



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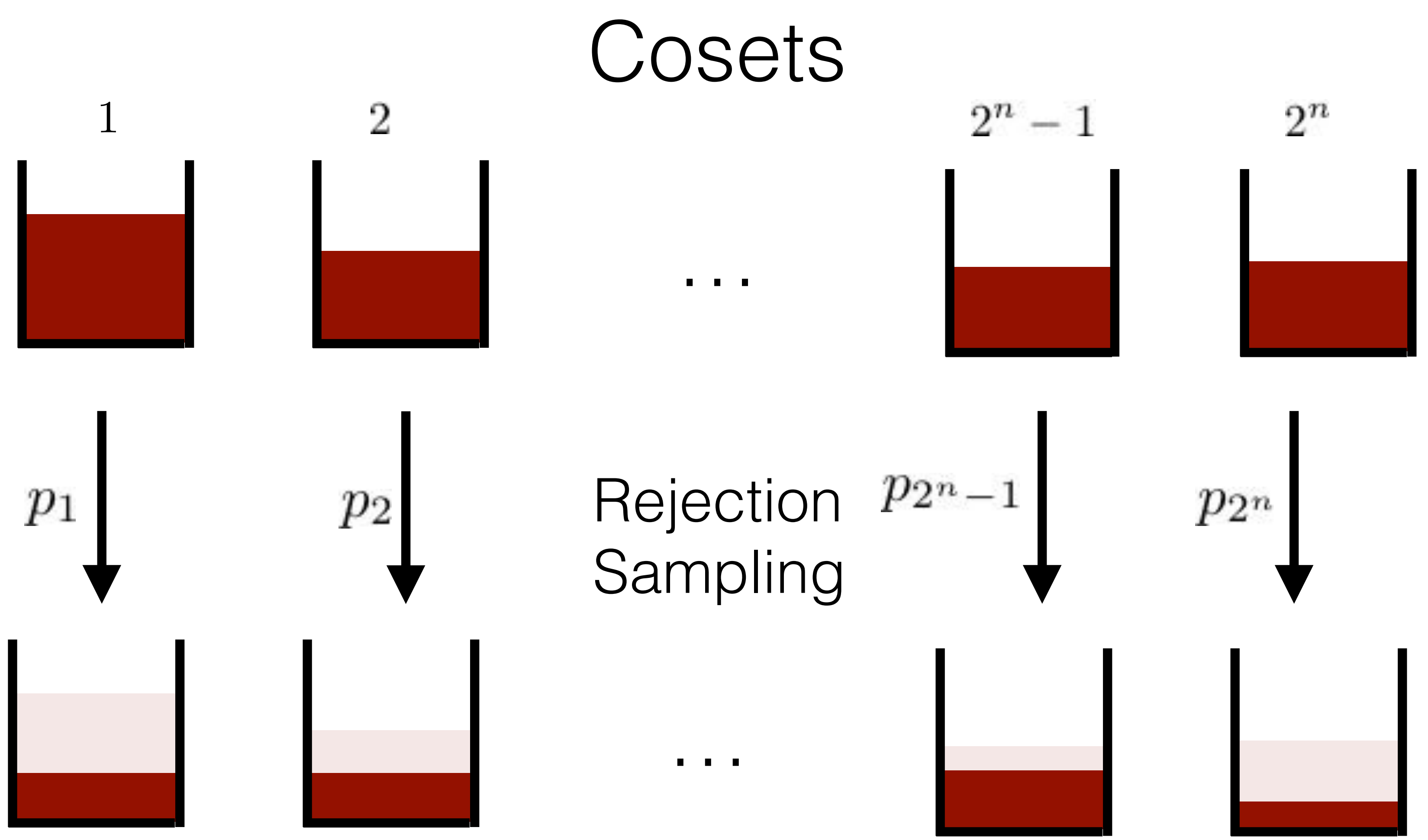

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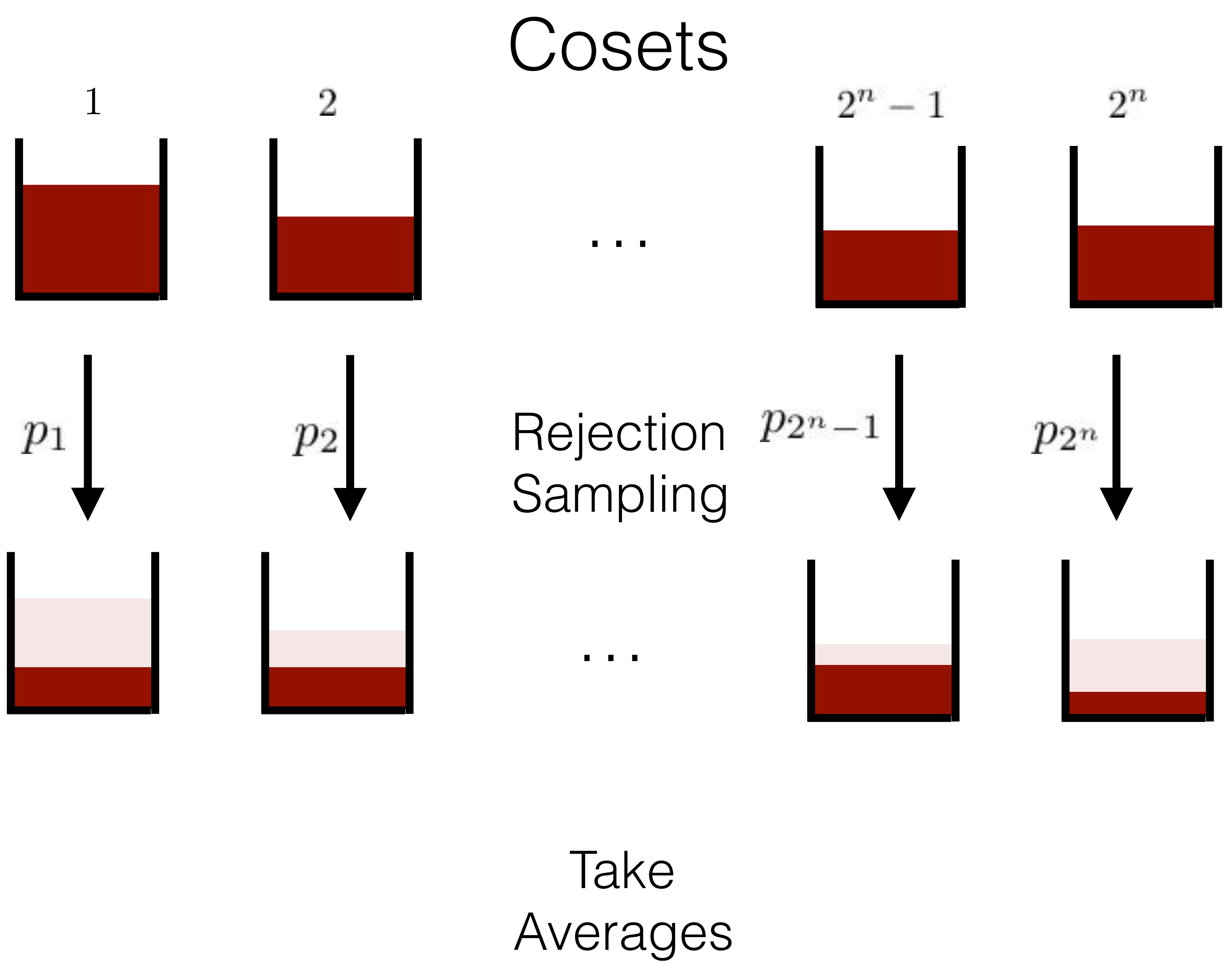

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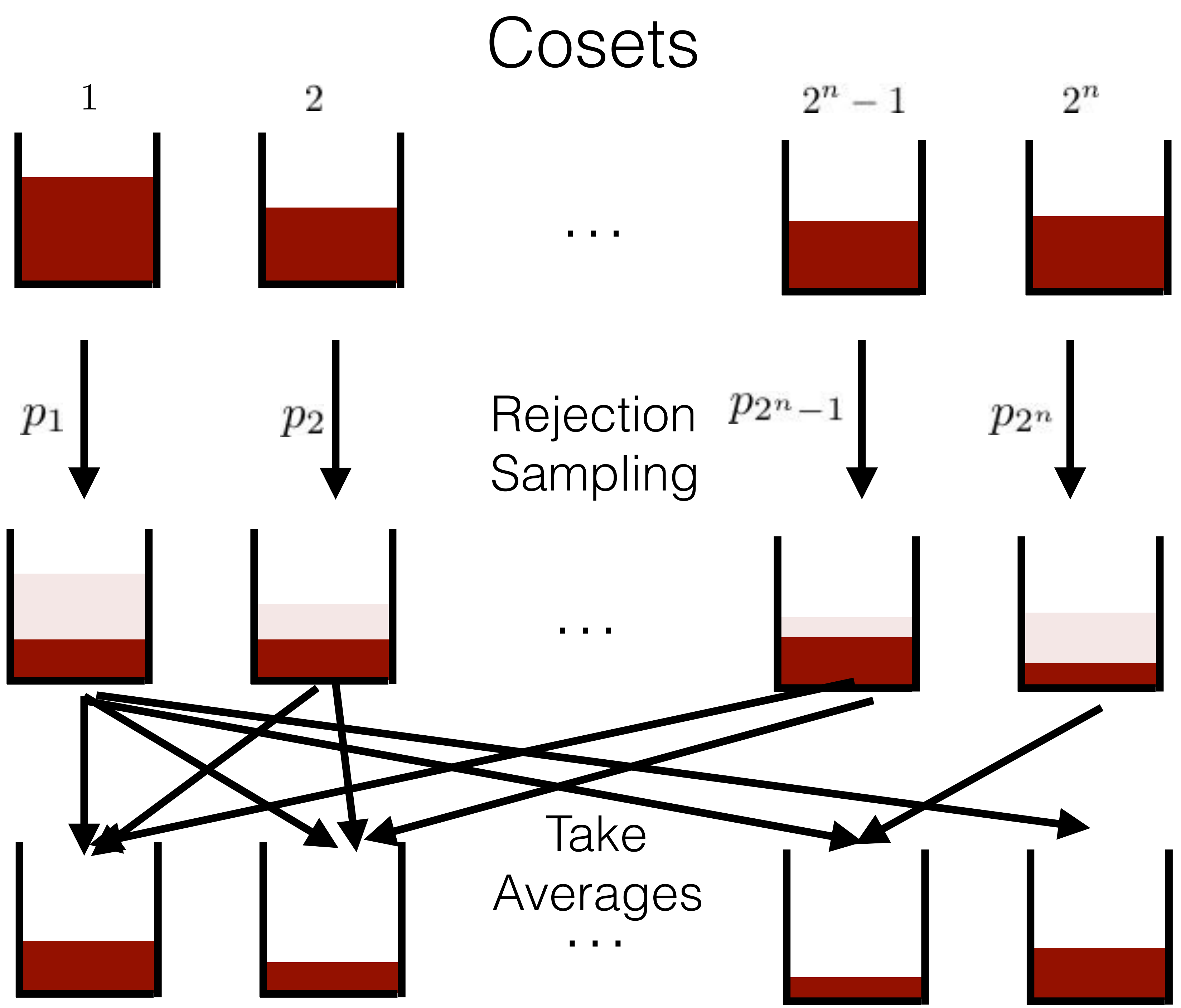

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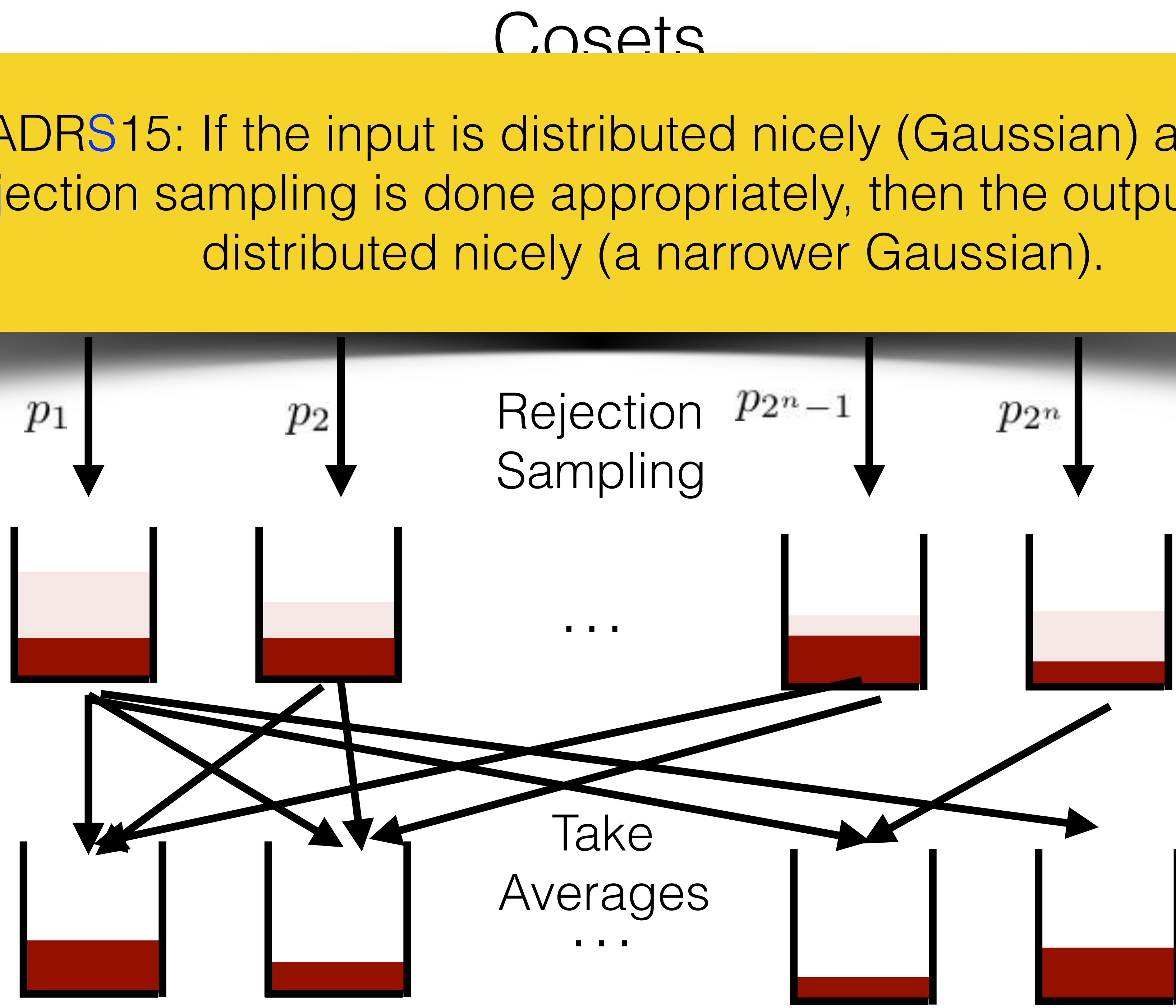
ADRS15: If the input is distributed nicely (Gaussian) and the rejection sampling is done appropriately, then the output will be distributed nicely (a narrower Gaussian).



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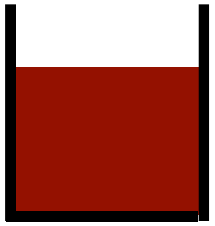

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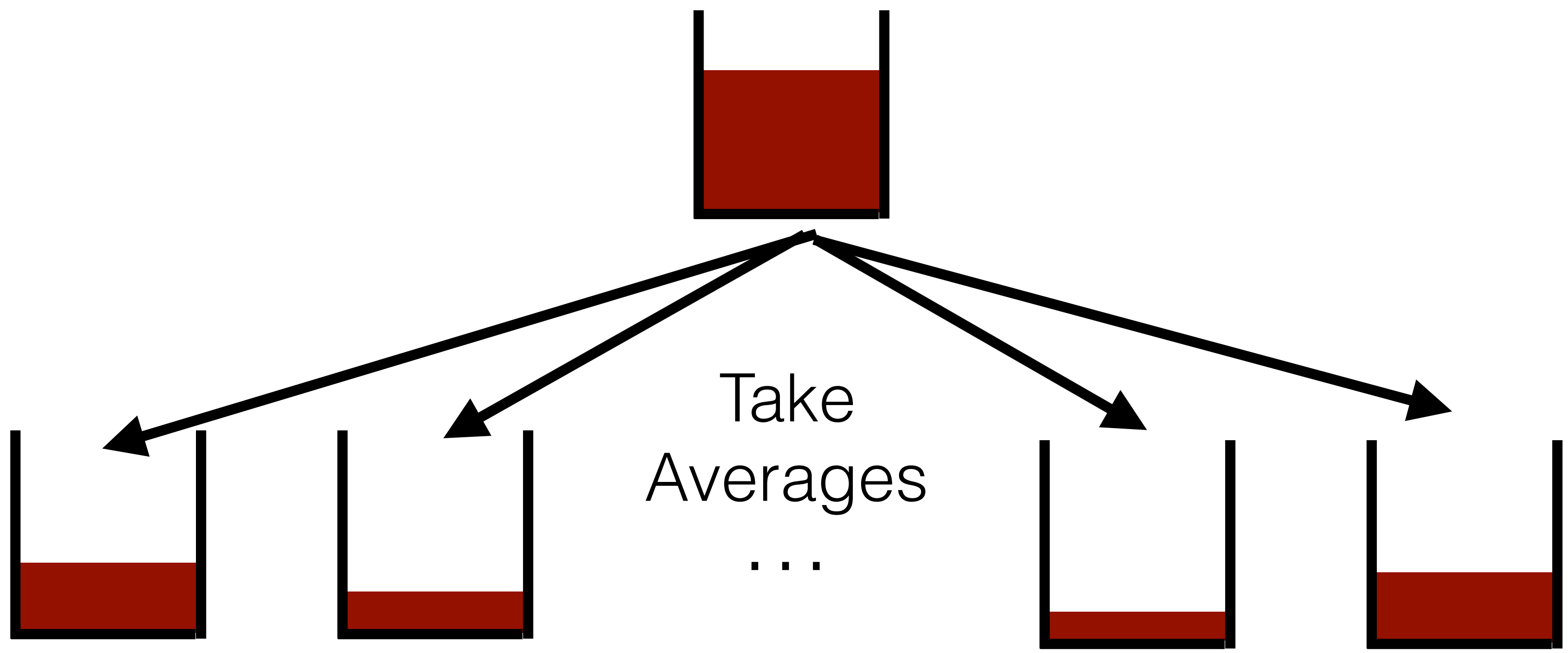

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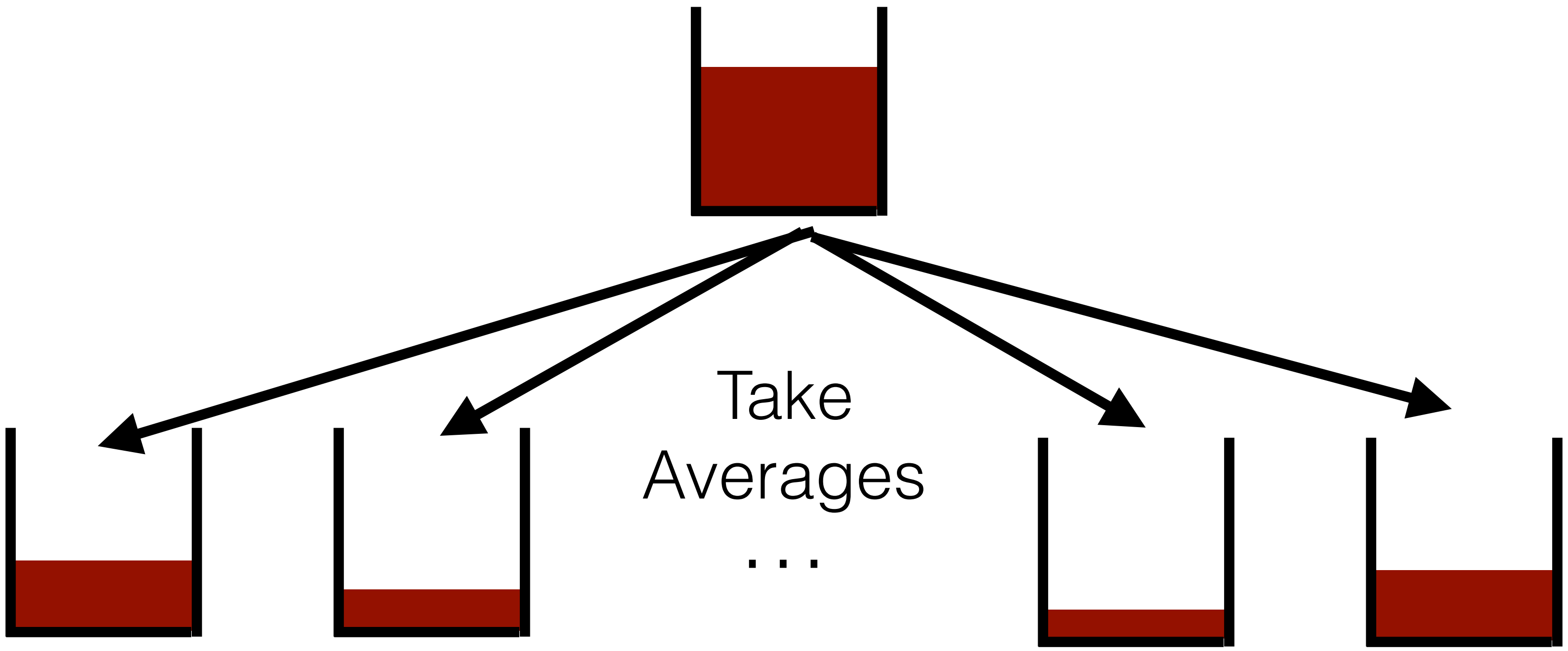

AS17: If the input is distributed nicely (Gaussian) *within* each coset, then the averages will be distributed nicely (a narrower Gaussian) *within* each coset.

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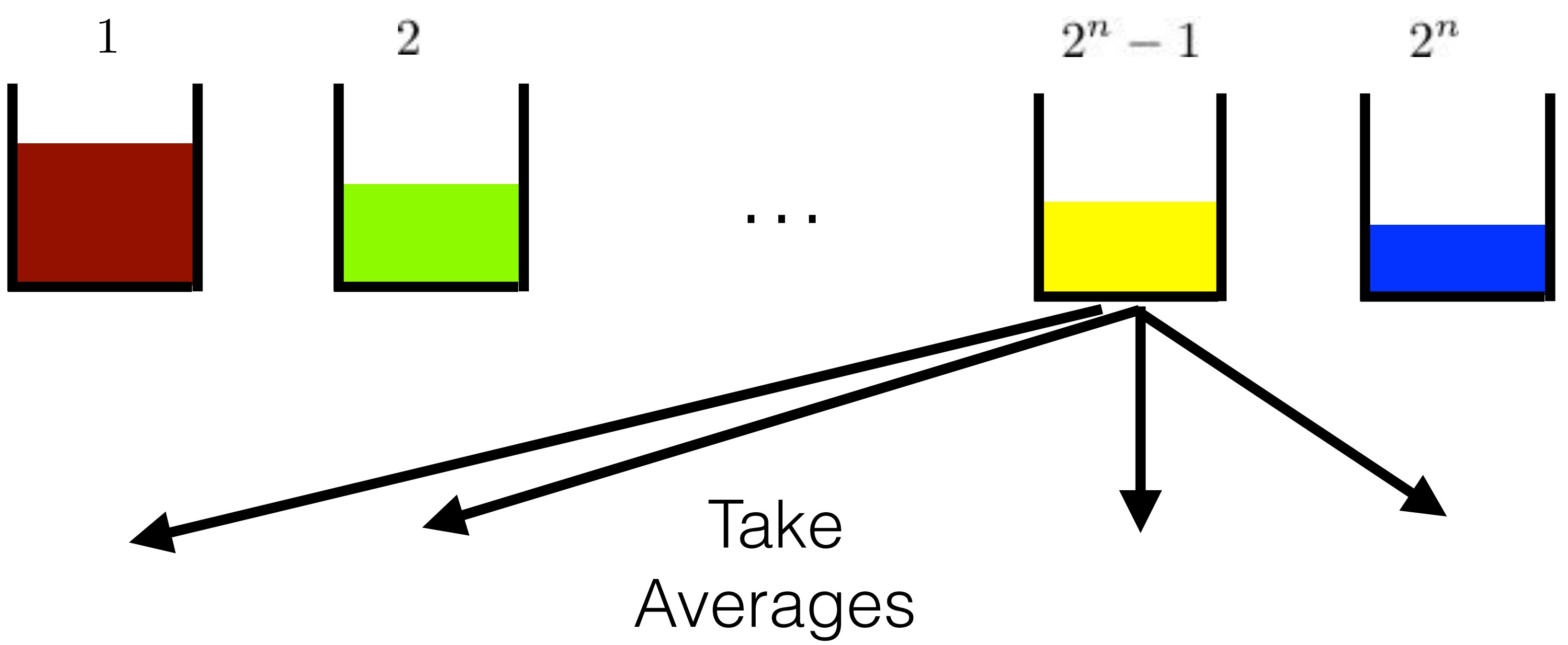

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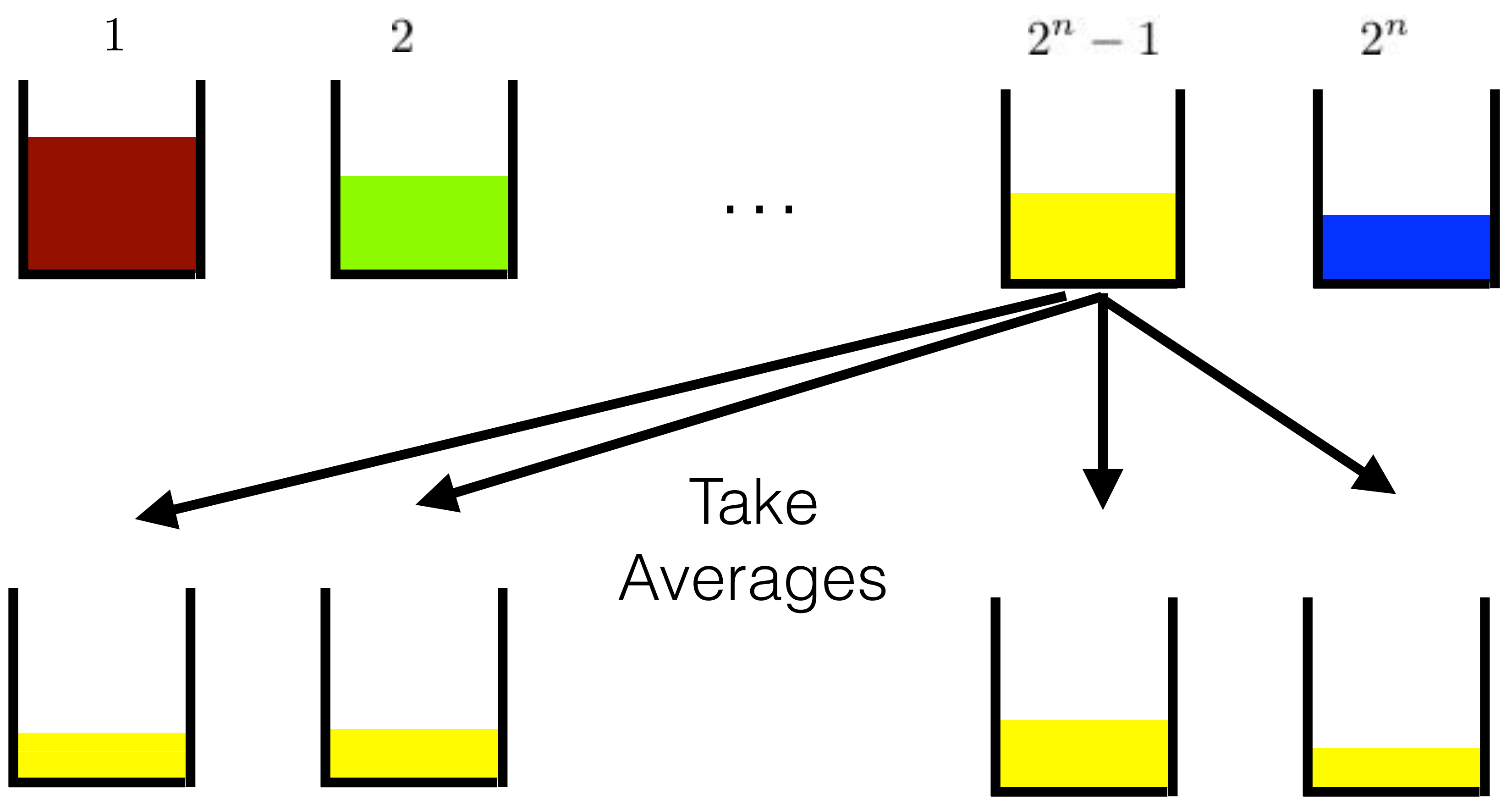

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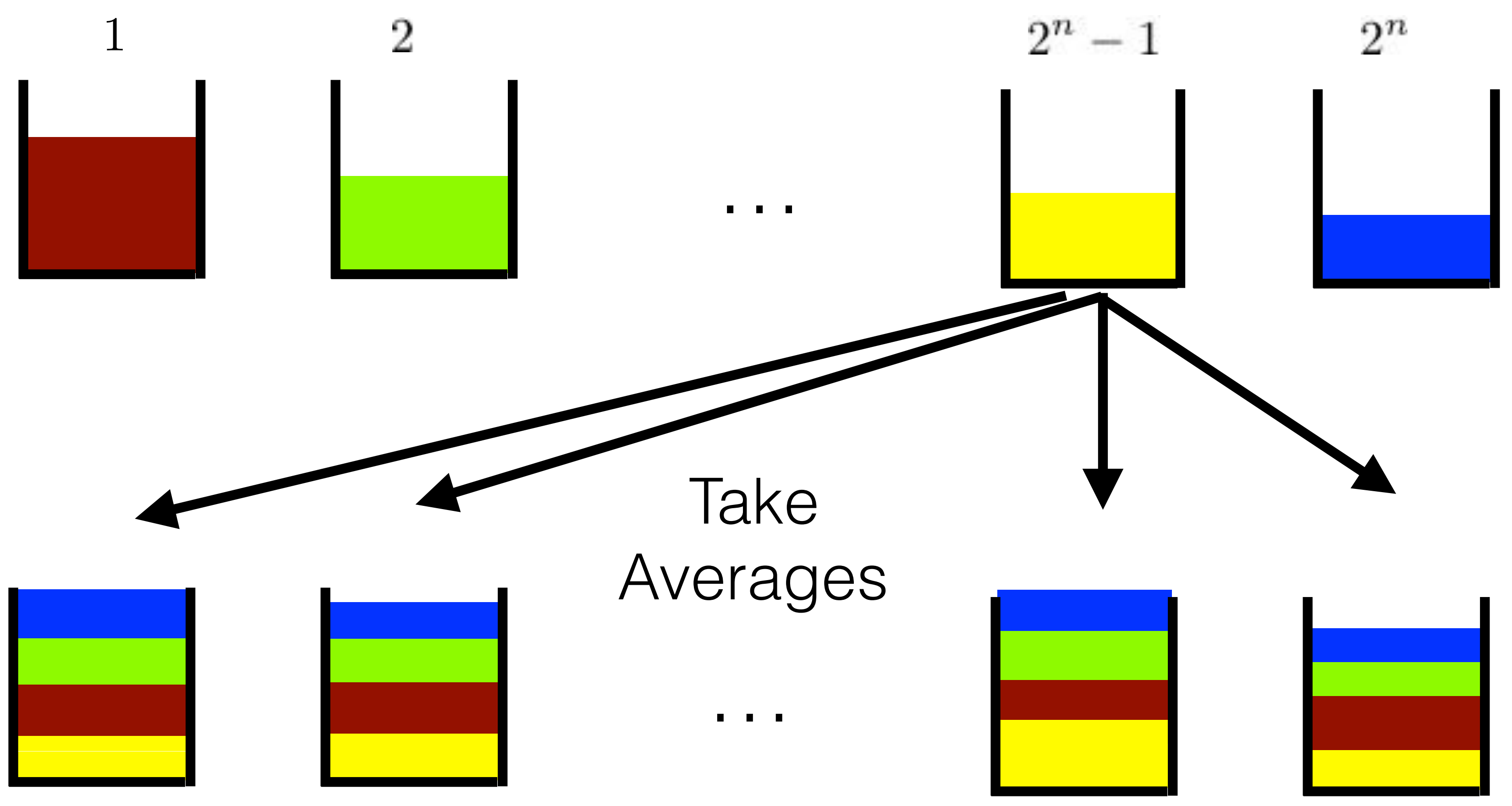

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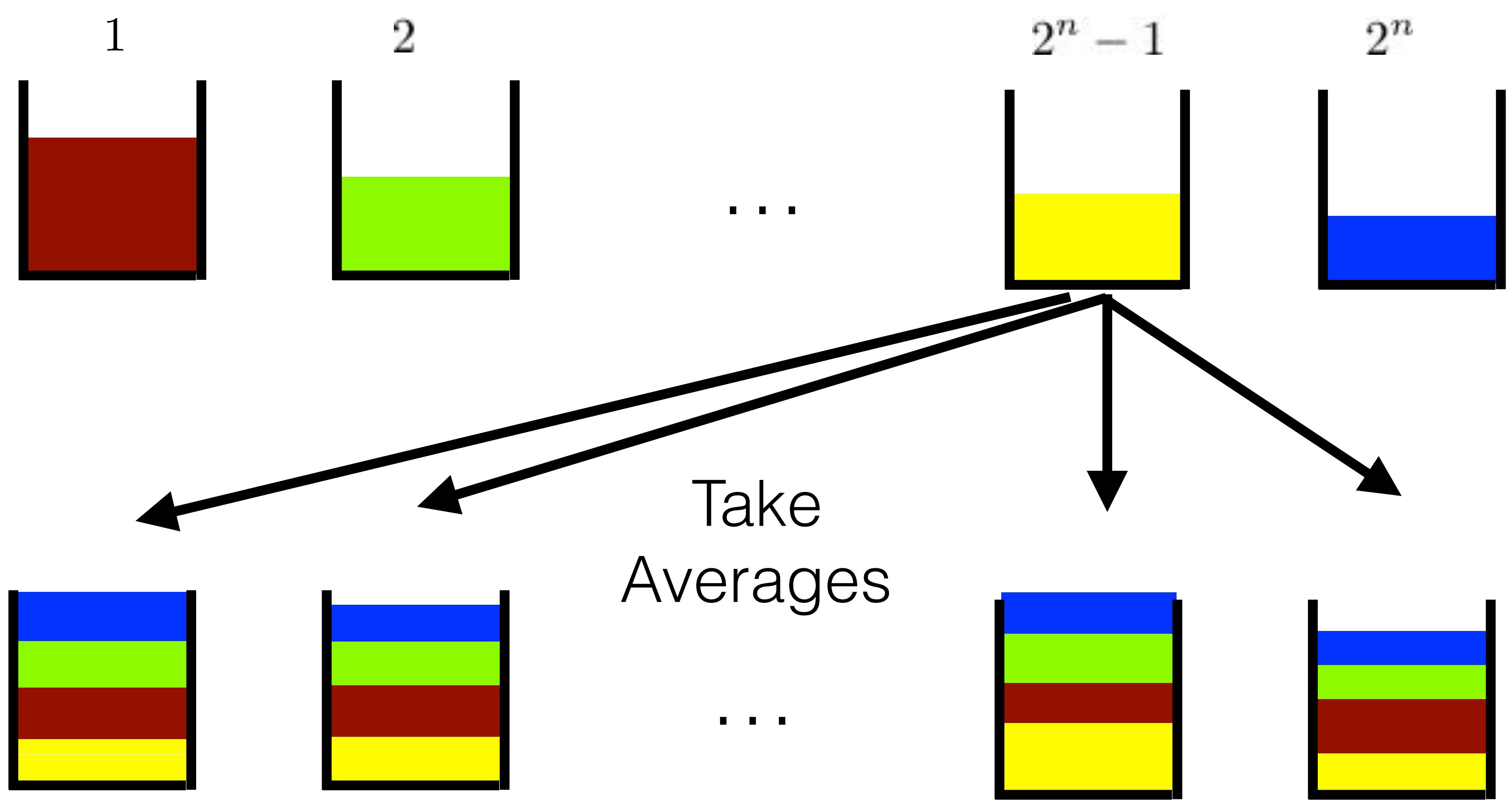

Had we done
rejection sampling:

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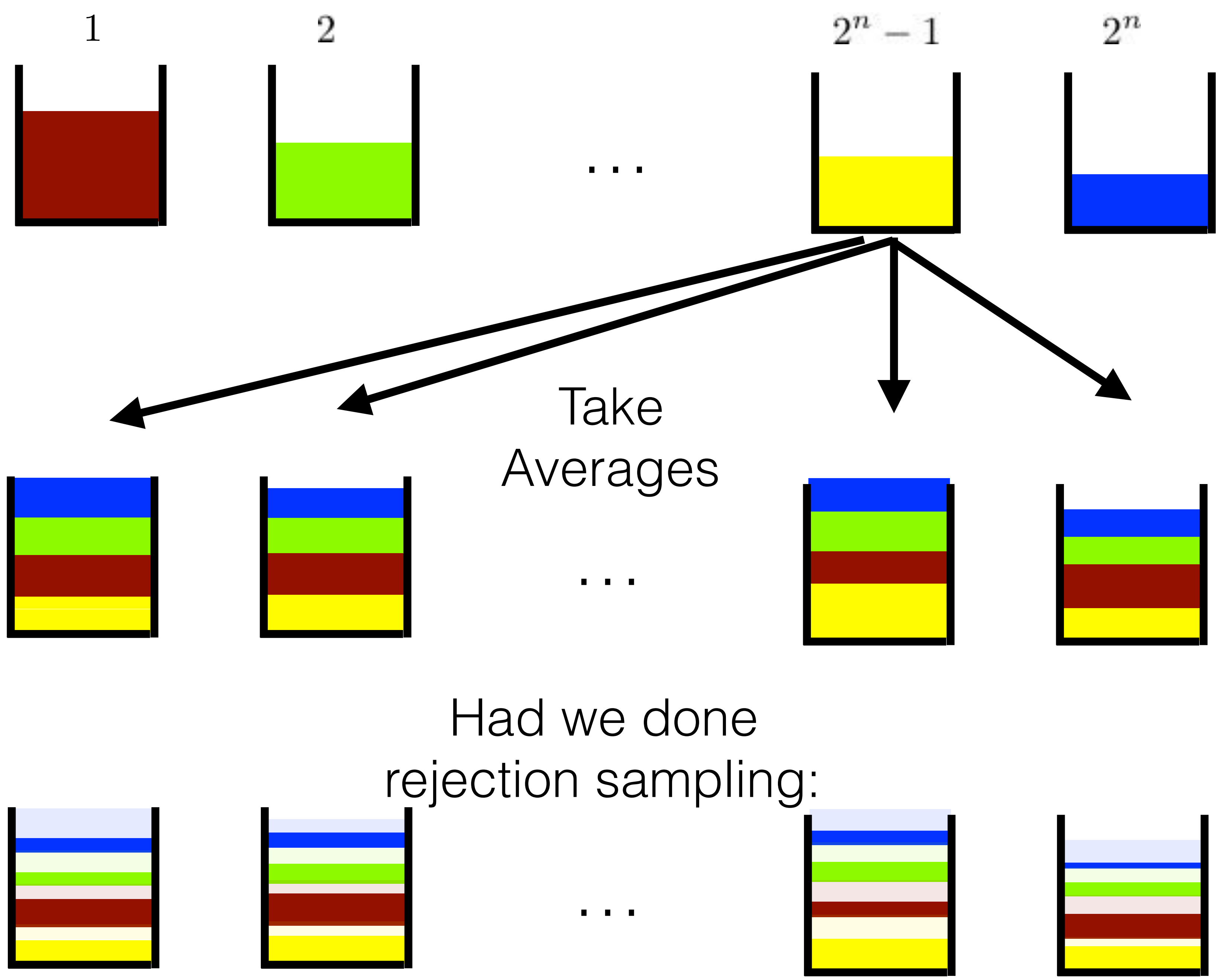

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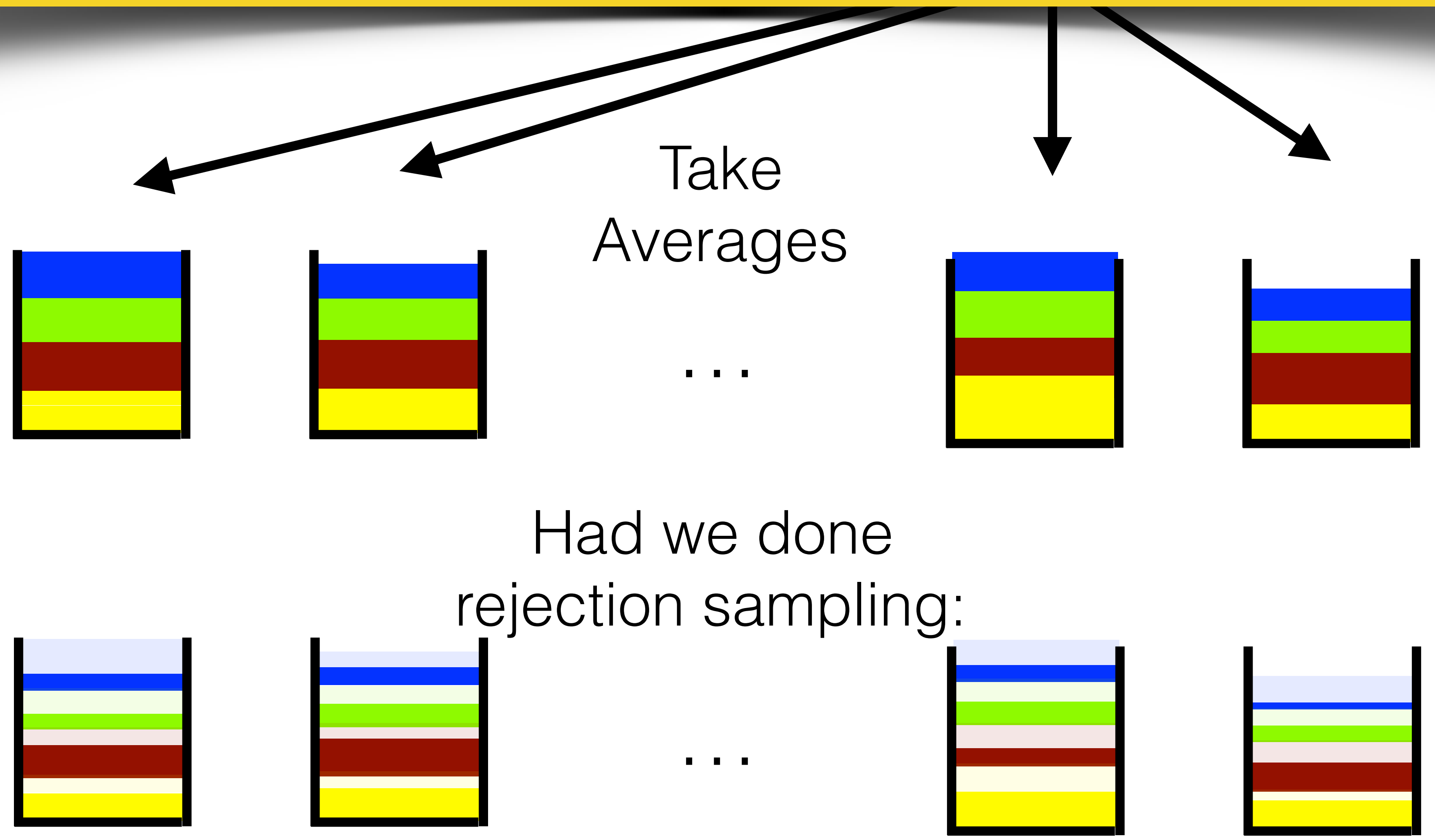
Rejection sampling only reweights the cosets in the output distribution. If we don't do rejection sampling, we just get more vectors from each coset.



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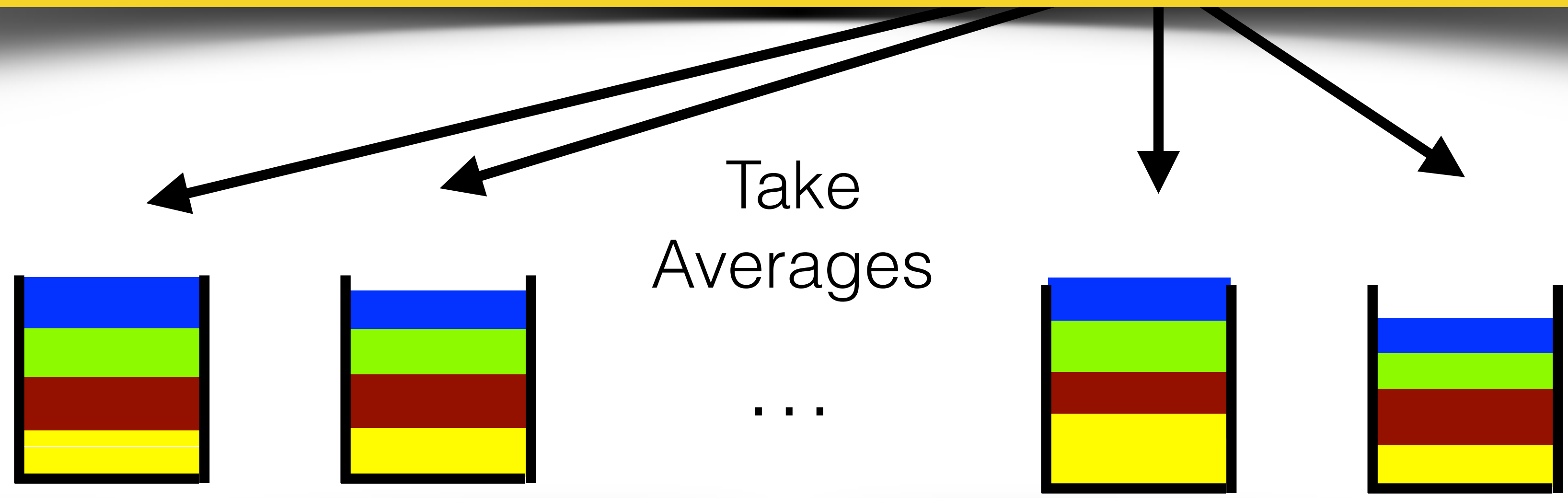
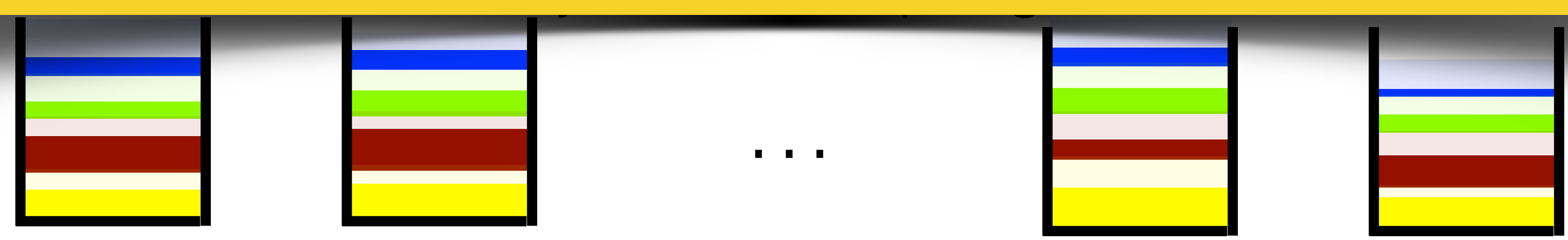


The rejection sampling procedure from ADRS15 was pointless!



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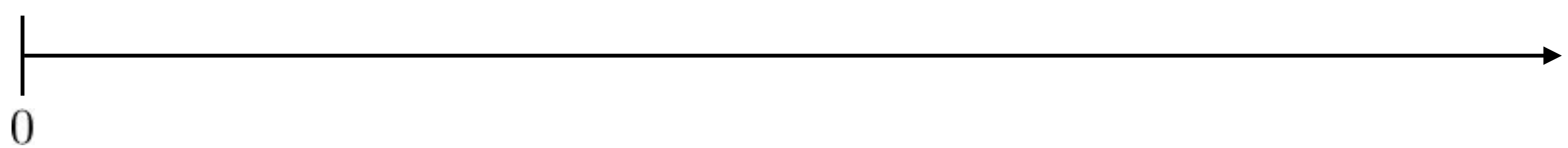

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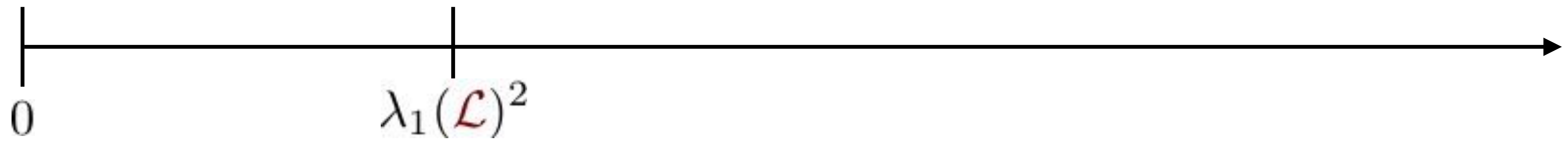

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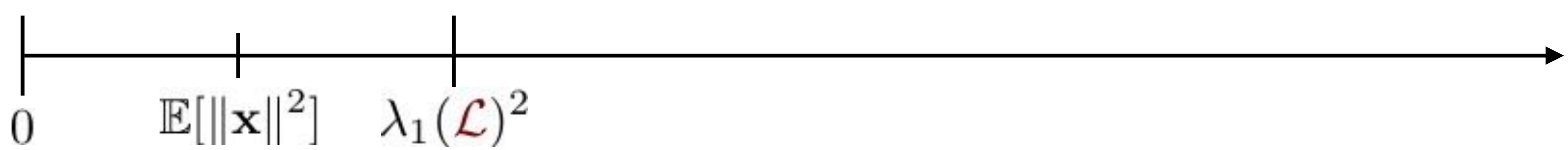

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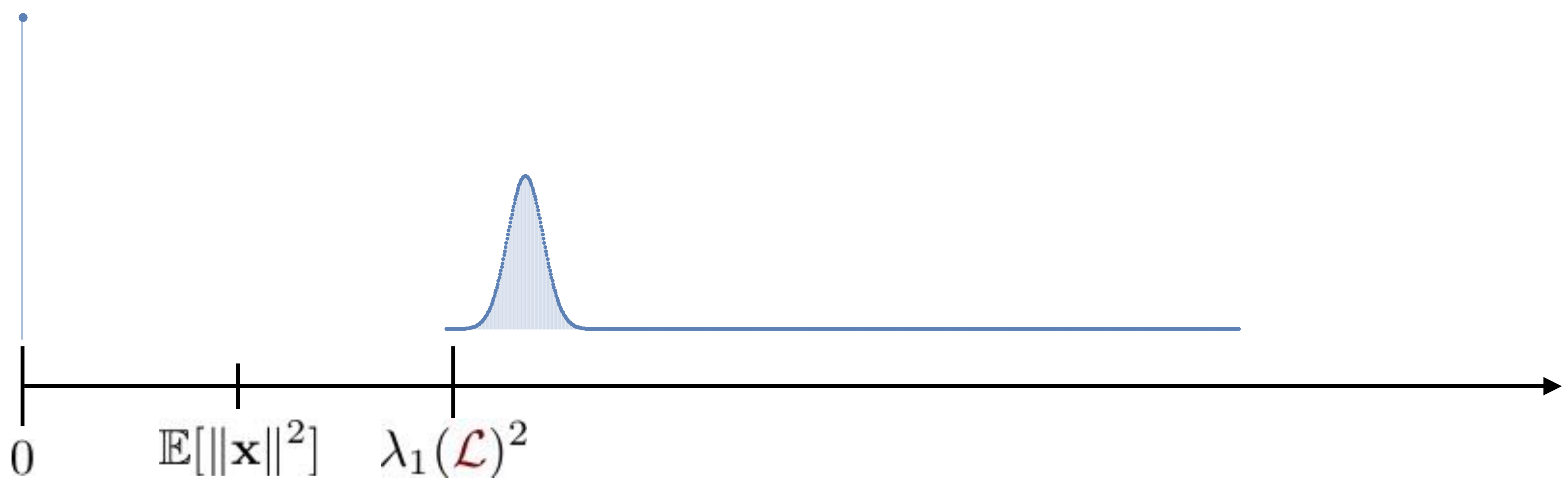

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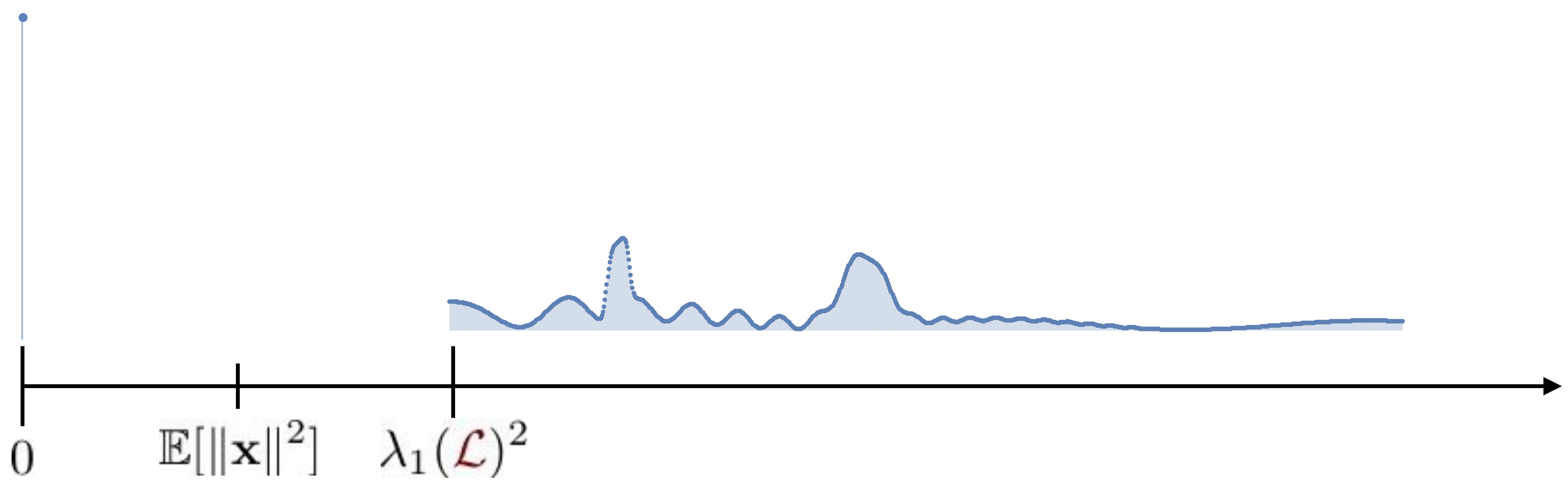

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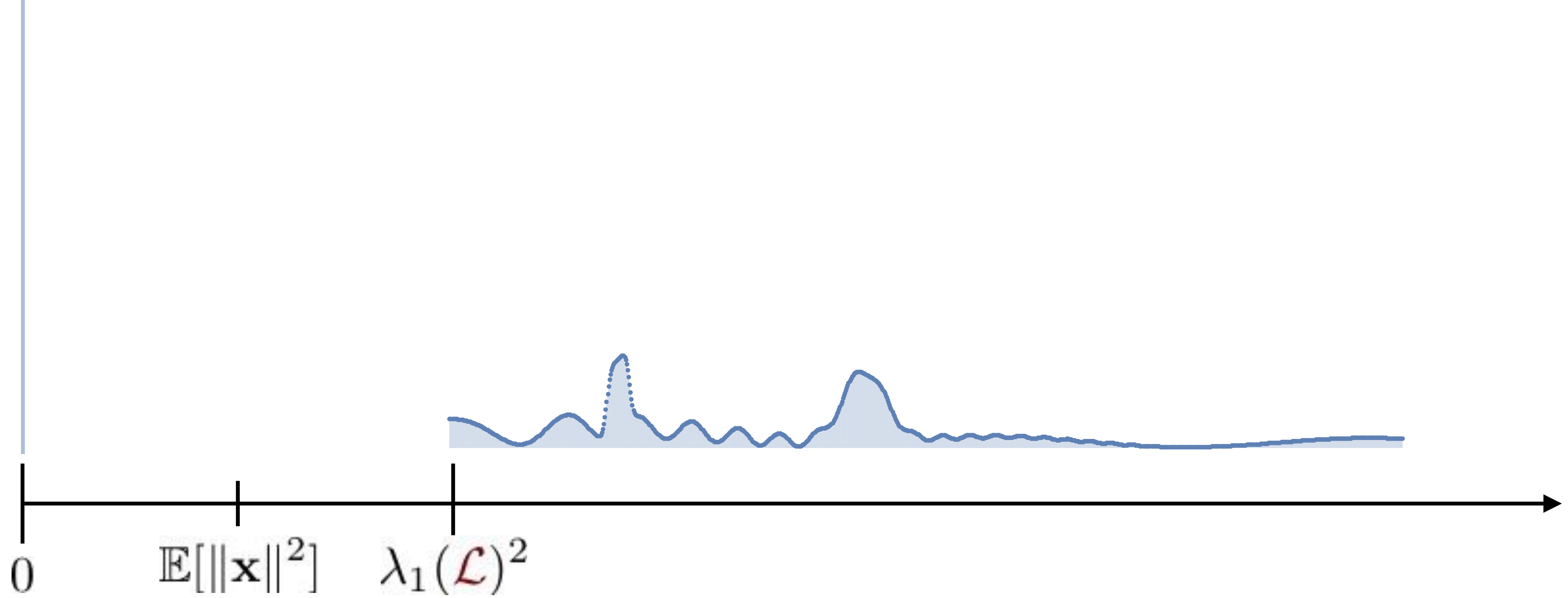

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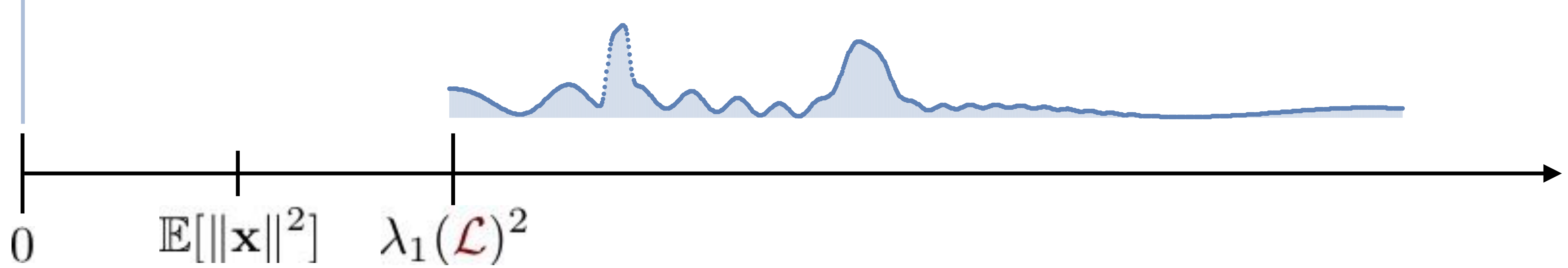

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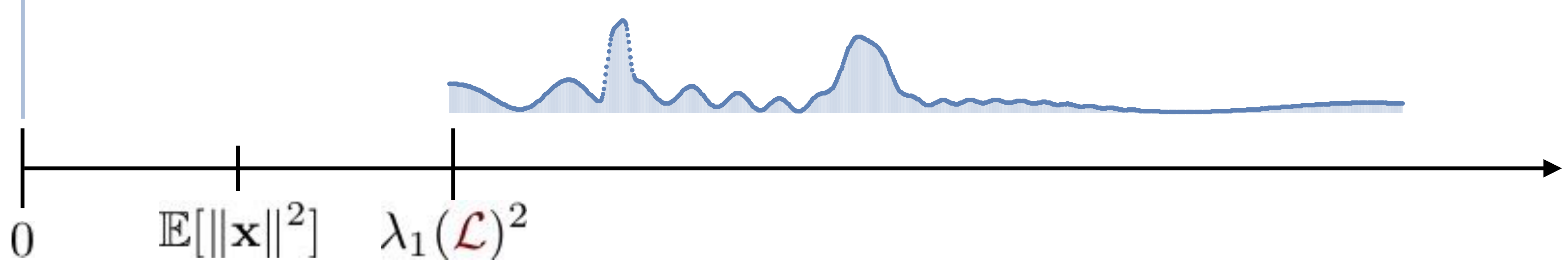

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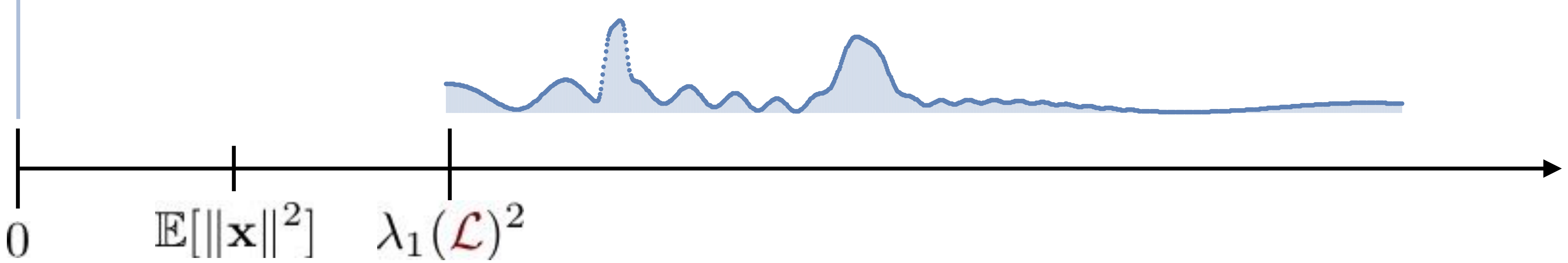
If we could show “anything non-trivial” about this distribution, then we could (probably) show that this algorithm solves (approximate) SVP.



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We know how to "remove the rejection sampling procedure" from the main [ADRS15] algorithm (the $2^{n+o(n)}$-time algorithm for SVP).

If we could do this for the $2^{n/2+o(n)}$-time algorithm, then it would provably solve SVP (at least approximately)!

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The running time will be roughly $|\mathcal{L}/2\mathcal{L}'| + |\mathcal{L}'/\mathcal{L}|$.

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We can take $|\mathcal{L}_i/2\mathcal{L}_{i+1}| = 2^{n/2+o(n)}$ to get an algorithm with this running time.

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Thanks!