CURVE

REPRESENTATION

Representation

Non-parametric form: $y = f(X)$

Implicit form: $f(x, y) = 0$

Explicit form: $y = mx + b$

Parametric form:
$x = x(t)$
$y = y(t)$

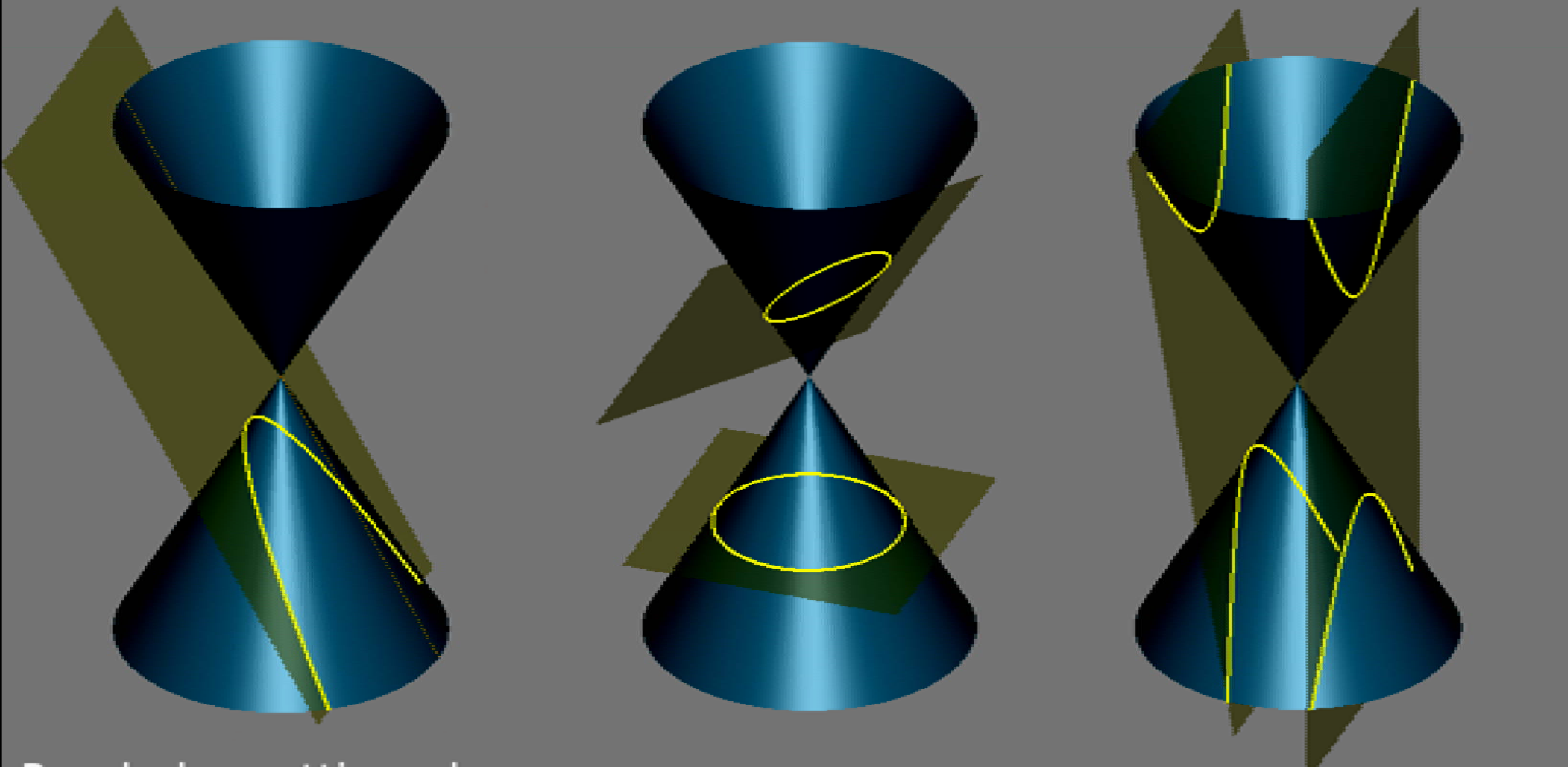
2nd degree implicit representation:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

Any guess, why the factor 2 is used ?

This form of the expression, with the coefficients, provide a wide variety of 2D curve forms called:

CONIC SECTIONS



Parabola- cutting plane parallel to side of cone.

Circle and Ellipse

Hyperbolas

CONIC SECTIONS

PARABOLA

$y^2 = 4ax; a > 0$

$Focus: (a, 0);$

$Directrix = -a.$

$\text{eccentricity}, e = 1$

$x = at^2; y = \pm 2at.$

or

$x = \tan^2(\phi);$

$y = \pm 2\sqrt{a\tan(\phi)}.$

HYPERBOLA

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$

$b^2 = a^2(e^2 - 1);$

$e > 1; Foci: (\pm ae, 0).$

$Directrices: x = \pm a / e;$

$x = a\sec(t),$

$y = b\tan(t);$

$-\pi/2 < t < \pi/2.$

Rectangular

Hyperbola :

$\mathbf{e} = \sqrt{2}; x = ct; y = c/t.$

ELLIPSE

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$

$a \geq b > 0.$

$b^2 = a^2(1 - e^2);$

$0 \leq e \leq 1.$

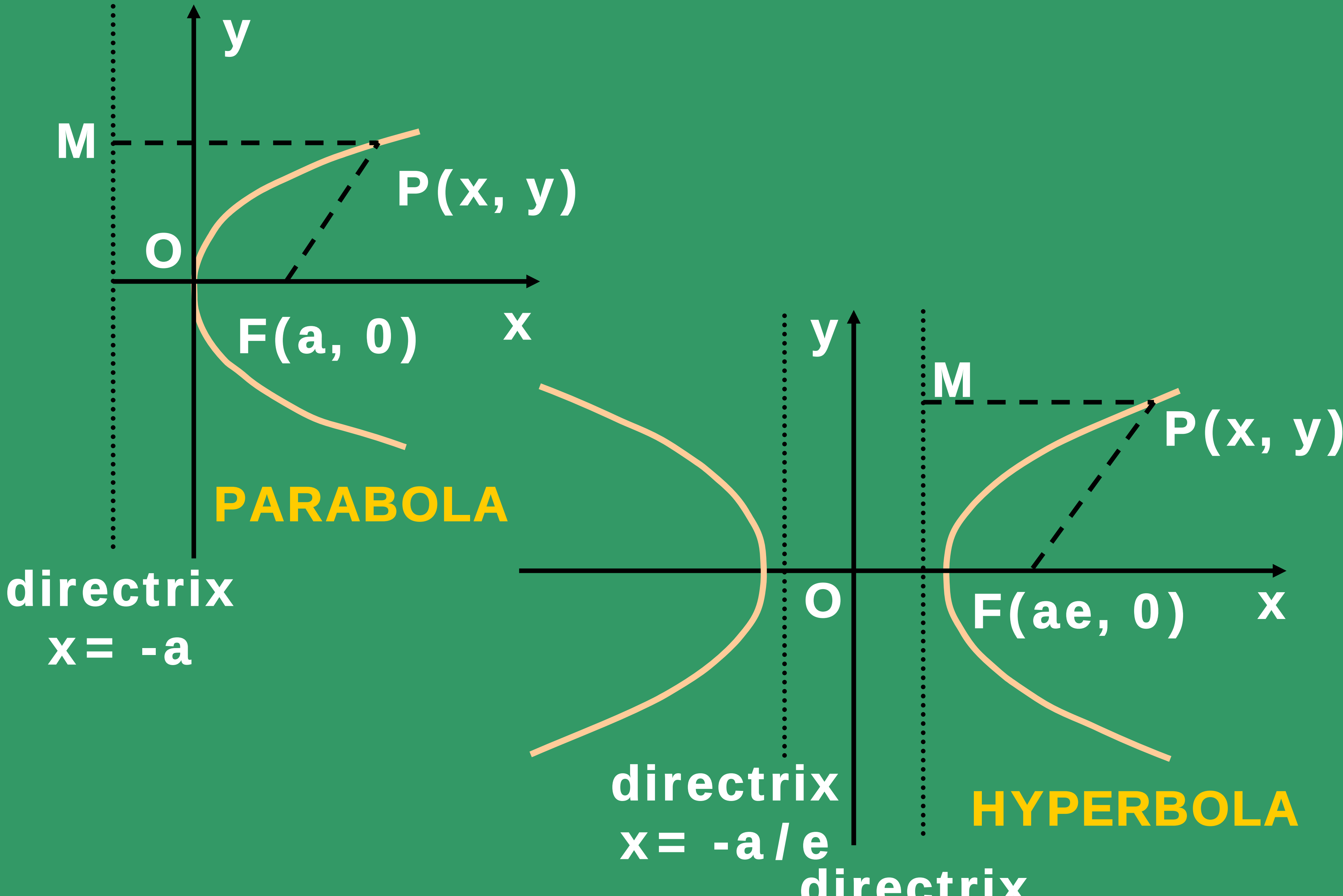
$Foci: (\pm ae, 0);$

$Directrices: x = \pm a / e.$

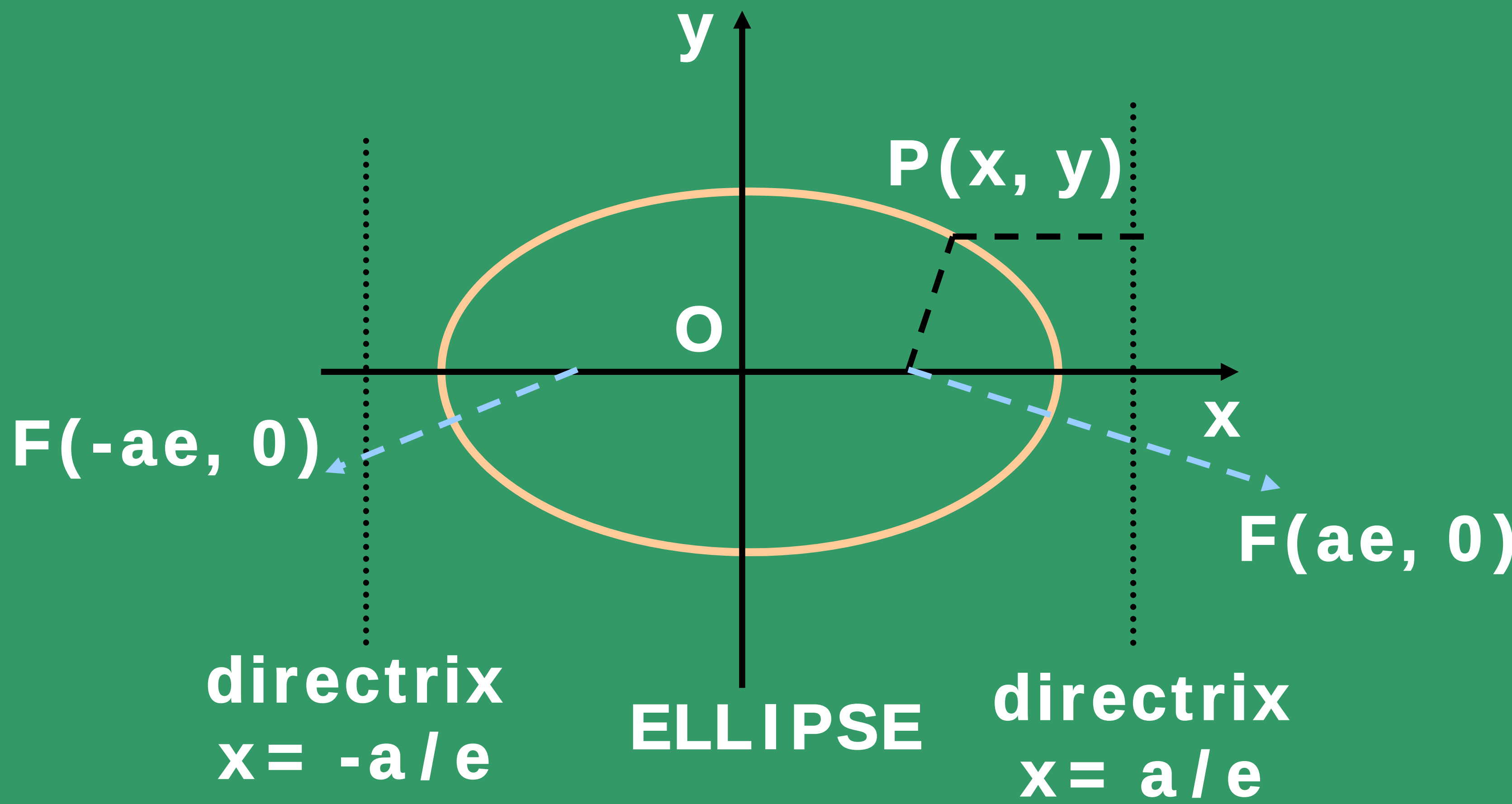
$x = a\cos(t),$

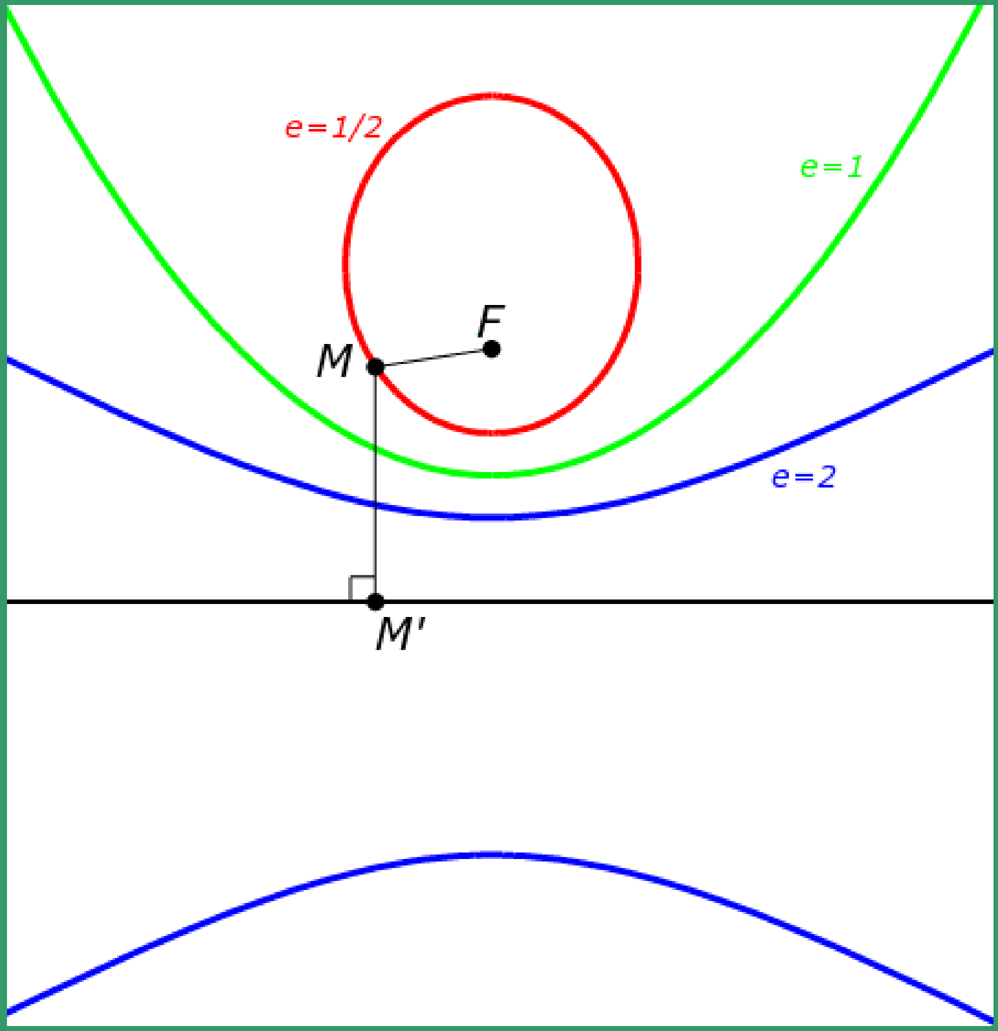
$y = b\sin(t);$

$t \in [-\pi, \pi].$



y
M
P(x, y)
O
F(a, 0)
x
PARABOLA
directrix
x= -a
y
M
P(x, y)
O
F(ae, 0)
x
directrix
x= -a/e
directrix
x= a/e
HYPERBOLA


y
P(x, y)
O
x
F(-ae, 0)
F(ae, 0)
directrix
x= -a/e
ELLIPSE
directrix
x= a/e



Ellipse (e=1/2), parabola (e=1) and hyperbola (e=2) with fixed focus F and directrix.

For circle, e = 0.

Polar Equation of a conic (***home assignment***):

$$r = \frac{e.L}{1 + e\cos(\theta)}, \quad \text{where, } L = dist(F, d)$$

F – Focal Point; d – Directrix;

e – Eccentricity.

Condns: Focal point at Origin;

e.L = l; is called the “semi-latus rectum”.

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

If the conic passes through the origin: f = 0.

Assuming, one of the parameters to be a constant, c = 1.0, f = 1.0

Remaining 5 Coeffs. may be obtained using 5 geometric conditions:

Say:

Boundary Conditions -

- **– two (2) end points**
- **- slope of the curves at two (2) end points. and**
- **- one (1) intermediate point**

Generalized CONIC

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

Re-organize:

as $\boldsymbol{XSX^T} = 0,$ **S is symmetric**

$$\Rightarrow [x \quad y \quad 1]\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or

$$\boldsymbol{XAX^T} + \boldsymbol{GX} + \boldsymbol{f} = 0$$

Special Conditions:

If $b^2 = ac$, the equation represents a PARABOLA;

If $b^2 < ac$, the equation represents an ELLIPSE;

If $b^2 > ac$, the equation represents a HYPERBOLA.

SPACE CURVE (3-D)

Explicit non-parametric representation:

x = x, y = f(x), z = g(x).

Non-parametric implicit representation:

f(x, y, z) = 0, g(x, y, z) = 0.

Intersection of the above two surfaces represents a curve.

Examples:

$$x = t^3, y = t^2, z = t.$$

A parametric space curve:

x = x(t), y = f(t), z = g(t).

Curve on the seam of a baseball:

$$x = \lambda[a.\cos(\theta + \pi / 4) - b.\cos 3(\theta + \pi / 4)],$$
$$y = \mu[a.\sin(\theta + \pi / 4) - b.\sin 3(\theta + \pi / 4)],$$
$$z = c.\sin(2\theta).$$

where,

$$\lambda = 1 + d.\sin(2\theta) = 1 + d(z / c),$$
$$\mu = 1 - d.\sin(2\theta) = 1 - d(z / c);$$
$$\theta = 2\pi t, 0 \le t \le 1.0.$$

HELIX:

$$x = r.\cos(t), y = r.\sin(t), z = bt;$$
$$b \ne 0, -\infty < t < \infty$$

PARAMETRIC CUBIC CURVES

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_{x},$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_{y},$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z.$$

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T.C,$$

$$where,\ T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} and\ C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

PARAMETRIC CUBIC Splines

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x,$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y,$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z.$$

Spline curve refers to any composite curve, formed with Polynomial sections, satisfying specific continuity conditions (1st and 2nd derivatives) at the boundary of the pieces.

$$P(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T \bullet CF,$$

$$\text{where, } T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \text{ and } CF = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

To solve for:

$$CF = T^{-1} P;$$

What do you need ??

$$P(t) = [x(t) \quad y(t) \quad z(t)] = T.CF,$$

$$\text{where, } T = [t^3 \quad t^2 \quad t \quad 1] \text{ and } CF = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

To solve for:

$$CF = T^{-1}P;$$

You need four (4) boundary conditions ??

$$P(t) = At^3 + Bt^2 + Ct + D; \; 0 \le t \le 1.$$

$$P(t) = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$P'(t) = [3t^2 \quad 2t \quad 1 \quad 0] \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

Hermite Boundary Conditions:

$$P(0) = P_0; P(1) = P_1;$$

$$P'(0) = DP_0; P'(1) = DP_1;$$

$$P(t) = At^3 + Bt^2 + Ct + D; \; 0 \le t \le 1.$$

$$P(0) = P_0; P(1) = P_1;$$

$$P'(0) = DP_0; P'(1) = DP_1;$$

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$P'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

Solve to get:

$$\begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$\begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = M_H G \quad (= CF);$$

In general:

$$Q(t) = [x(t) \quad y(t) \quad z(t)] = T.M.G,$$

$$\text{where, } \mathrm{T} = [t^3 \quad t^2 \quad t \quad 1],$$

$$\mathrm{M} = [m_{ij}]_{4x4} \text{ and } G = [g_1 \quad g_2 \quad g_3 \quad g_4]^T$$

M is a 4x4 ***basis matrix*** and G is a four element column vector of geometric constants, called the ***geometric vector***.

The curve is a weighted sum of the elements of the geometry matrix.

The weights are each cubic polynomials of t, and are called the ***blending functions***:

B = T.M.

CUBIC SPLINES

$$P(t) = \sum_{i=1}^{4} B_i t^{i-1}; t_i \le t \le t_2.$$

P(t) is the position vector of any point on the cubic spline segment.

P(t) = [x(t), y(t), z(t)]	***Cartesian***
or [r(t), θ(t), z(t)]	***Cylindrical***
or [r(t), θ(t), ϕ(t)]	***Spherical***

$$x(t) = \sum_{i=1}^{4} B_{ix} t^{i-1}$$

$$y(t) = \sum_{i=1}^{4} B_{iy} t^{i-1} \qquad t_1 \le t \le t_2.$$

$$z(t) = \sum_{i=1}^{4} B_{iz} t^{i-1}$$

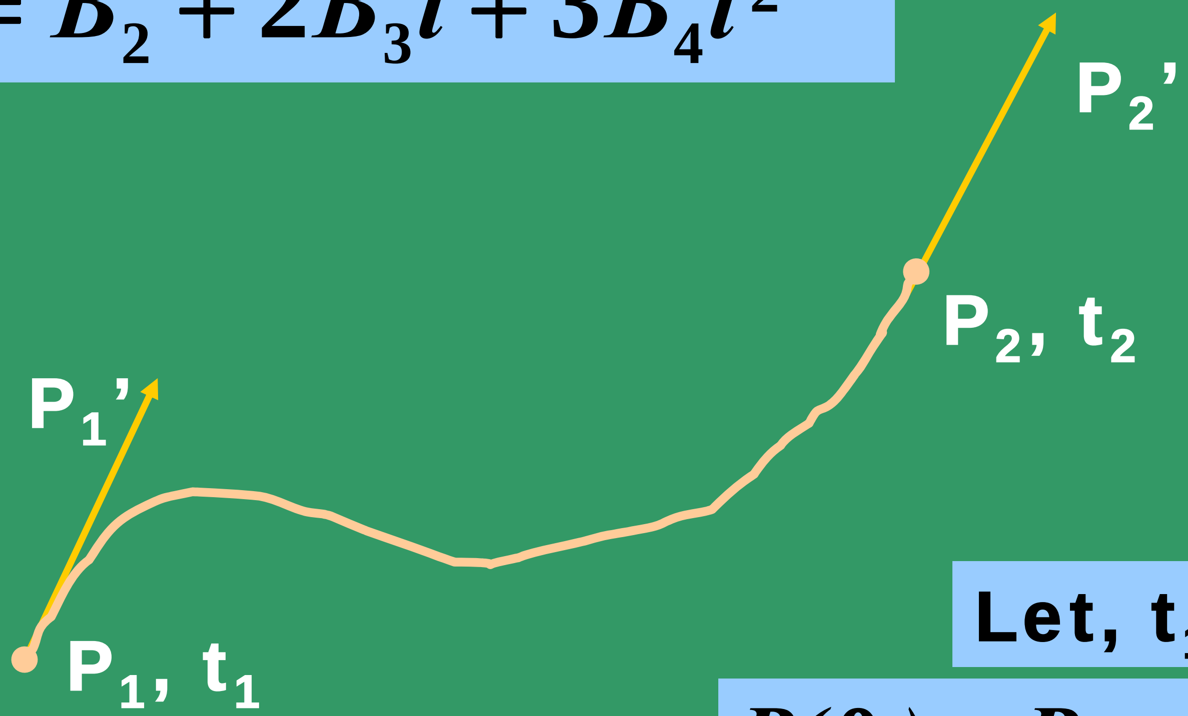
Use boundary conditions to evaluate the coeficients

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3,$$

$$t_1 \le t \le t_2$$

$$P'(t) = \sum_{i=1}^{4} (i-1) B_i t^{i-2}$$

$$= B_2 + 2B_3 t + 3B_4 t^2$$



Let, $t_1 = 0$:

$$P(0) = P_1; \quad P(t_2) = P_2.$$

$$P'(0) = P_1'; \quad P'(t_2) = P_1'.$$

Solutions:

$$B_1 = P_1; \; B_2 = P_1';$$

$$B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P(t_2);$$

$$B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P'(t_2);$$

$$B_3 =$$

$$B_4 =$$

Equation of a single cubic spline segment:

$$P(t) = P_1 + P_1' t + \left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right] t^2 + \left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right] t^3;$$

Rewrite as:

$$P(u) = \sum_{k=0}^{3} g_k H_k(u)$$

$$P(t) = P_1(2t^3 - 3t^2 + 1) + P_2(-2t^3 + 3t^2) + P_1'(t^3 - 2t^2 + t) + P_2'(t^3 - t^2)$$

Various other approaches used are:

- Normalized Cubic splines
- Blending
- Weighting functions.

Equation of a normalized cubic spline segment:

$$B = T.M;$$

$$P(t) = T.M.G =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}$$

Use, $t_2 = 1$;

Remember,
The derivation:

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = M_H G \quad (= CF);$$

Equation of a single cubic spline segment:

$$P(t) = P_1 + P_1' t + \left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right] t^2$$

$$+ \left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right] t^3;$$

$$P(t) = \left[x(t) \quad y(t) \quad z(t)\right] = T.M.G = B.G,$$

$$\text{where, } \mathrm{T} = \left[\mathrm{t}^3 \quad \mathrm{t}^2 \quad \mathrm{t} \quad 1\right], \ \mathrm{M} = \left[m_{ij}\right]_{4x4}$$

$$\text{and } G = \left[g_1 \quad g_2 \quad g_3 \quad g_4\right]^T;$$

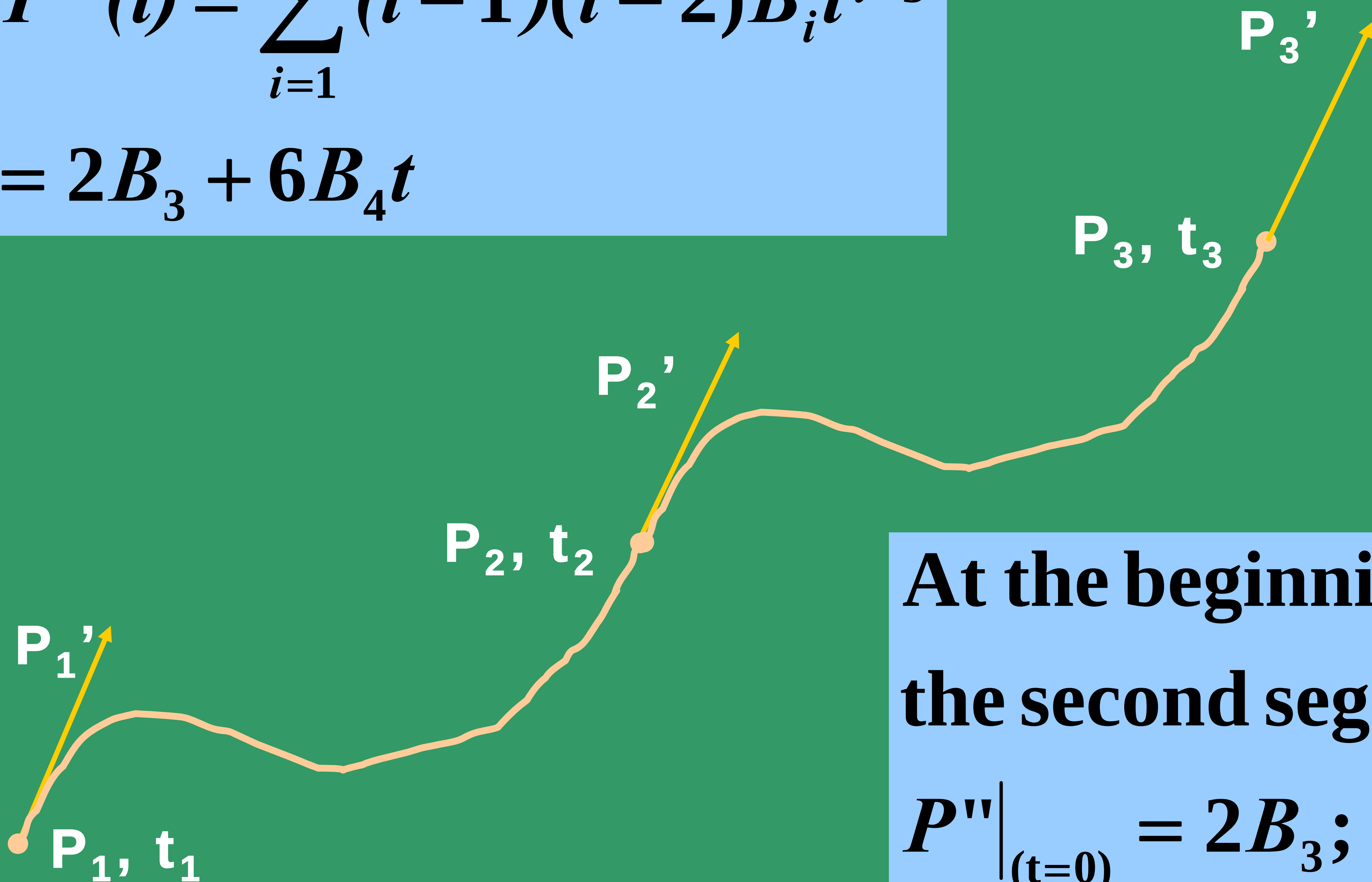
For piece-wise continuity for complex curves, two or more curve segments are joined together.

In that case, use second derivative $P_2''(t)$ at end-points (joints).

$$P''(t) = \sum_{i=1}^{4} (i-1)(i-2) B_i t^{i-3}$$

$$= 2B_3 + 6B_4 t$$

P₁’ and P₃’ known,
But what about P₂’ ?



At the beginning of the second segment :

$$P''\Big|_{(t=0)} = 2B_3;$$

$$P''(t_2) = 2B_3 + 6B_4 t_2 = P''(0) = 2\bar{B}_3$$

$$B_3 = \frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2};$$

$$B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2};$$

$$6t_2\left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right] + 2\left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right] = 2\left[\frac{3(P_3 - P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3}\right]$$

Multiplying both sides by $t_2 t_3$

Generalized equation for any two adjacent cubic spline segments, $P_k(t)$ and $P_{k+1}(t)$:

For first segment:

$$P_k(t) = P_k + P_k' t + \left[\frac{3(P_{k+1} - P_k)}{t_{k+1}^2} - \frac{2P_k'}{t_{k+1}} - \frac{P_{k+1}'}{t_{k+1}}\right]t^2 + \left[\frac{2(P_k - P_{k+1})}{t_{k+1}^3} + \frac{P_k'}{t_{k+1}^2} + \frac{P_{k+1}'}{t_{k+1}^2}\right]t^3;$$

For second segment:

$$P_{k+1}(t) = P_{k+1} + P_{k+1}' t + \left[\frac{3(P_{k+2} - P_{k+1})}{t_{k+2}^2} - \frac{2P_{k+1}'}{t_{k+2}} - \frac{P_{k+2}'}{t_{k+2}}\right]t^2 + \left[\frac{2(P_{k+1} - P_{k+2})}{t_{k+2}^3} + \frac{P_{k+1}'}{t_{k+2}^2} + \frac{P_{k+2}'}{t_{k+2}^2}\right]t^3;$$

Curvature Continuity ensured as:

$$t_{k+2}P_k' + 2(t_{k+1} + t_{k+2})P_{k+1}' + t_{k+1}P_{k+2}' = \frac{3}{t_{k+1}t_{k+2}}\left[t_{k+1}^2(P_{k+2} - P_{k+1}) + t_{k+2}^2(P_{k+1} - P_k)\right]$$

Equation of a normalized cubic spline segment:

$$F = T.N;$$

$$P(t) = T.N.G =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}$$

Use, $t_2 = 1$;

For curvature Continuity:

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

For curvature Continuity:

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

For three control points (knots) this works as:

$$P_2' = [3(P_3 - P_1) - P_1' - P_3']/4;$$

In general:

$$t_{k+2}P_k' + 2(t_{k+1} + t_{k+2})P_{k+1}' + t_{k+1}P_{k+2}' = \frac{3}{t_{k+1}t_{k+2}}\left[t_{k+1}^2(P_{k+2} - P_{k+1}) + t_{k+2}^2(P_{k+1} - P_k)\right]$$

For N points ??

For 3 points – 1 Eqn. (& 1 unknown)
For 4 points – 2 eqns. (& 2 unknowns)

.

.

.

For N points – (N-2) eqns. (& N-2 unknowns)

Write the eqn. set for N = 5; in matrix form.

$$
\begin{bmatrix}
b_1 & c_1 & & & 0 \\
a_2 & b_2 & c_2 & & \\
 & a_3 & b_3 & \cdot & \\
 & & \cdot & \cdot & c_{n-1} \\
0 & & & a_n & b_n
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}
=
\begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_n \end{bmatrix}.
$$

$$
= \begin{bmatrix}
P_1' \\
\frac{3}{t_2 t_3}\left[t_2^2(P_3 - P_2) + t_3^2(P_2 - P_1)\right] \\
\frac{3}{t_3 t_4}\left[t_3^2(P_4 - P_3) + t_4^2(P_3 - P_2)\right] \\
\cdot \\
\cdot \\
\cdot \\
\frac{3}{t_{n-1} t_n}\left[t_{n-1}^2(P_n - P_{n-1}) + t_n^2(P_{n-1} - P_{n-2})\right] \\
P_n'
\end{bmatrix}
$$

Solve using: Forward-backward substitution:

Thomas Algm.

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

Lets solve for N = 4;

$$P_1' + 4P_2' + P_3' = 3[P_3 - P_1];$$

$$P_2' + 4P_3' + P_4' = 3[P_4 - P_2]$$

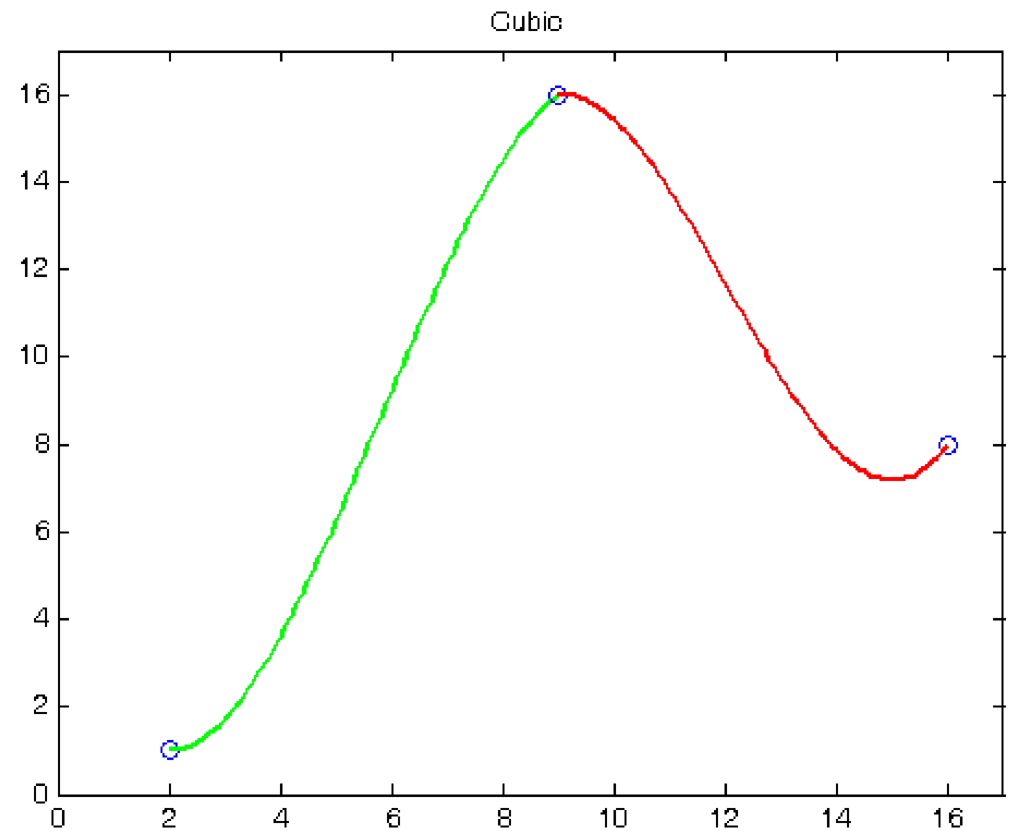
Re-arrange to get:

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} P_2' \\ P_3' \end{bmatrix} = \begin{bmatrix} 3(P_3 - P_1) - P_1' \\ 3(P_4 - P_2) - P_4' \end{bmatrix};$$

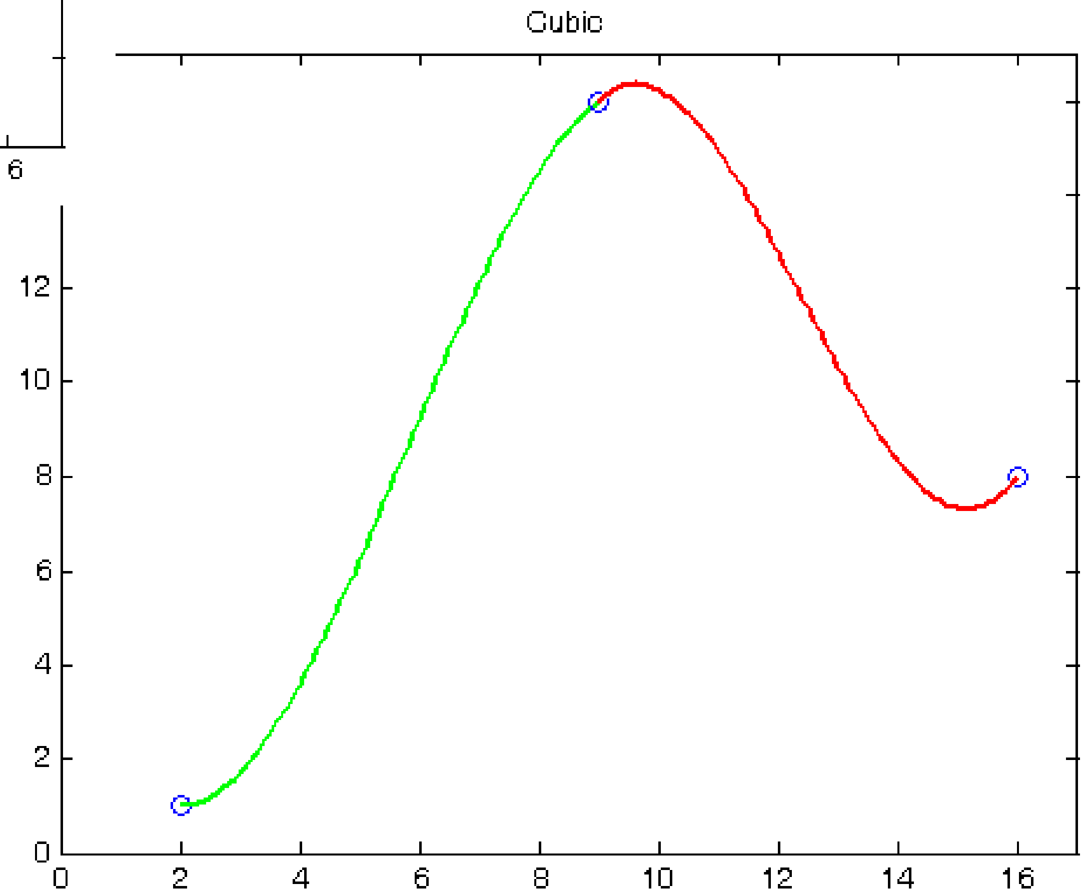
$$\begin{bmatrix} P_2' \\ P_3' \end{bmatrix} = (1/15) \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3(P_3 - P_1) - P_1' \\ 3(P_4 - P_2) - P_4' \end{bmatrix}$$

Problem: The position vectors of a normalized cubic spline are given as (0 0), (1 1), (2 -1) and (3 0). The tangent vectors at the ends are both (1 1).

Soln: The 2 internal tangent vectors are calculated, and both are equal to (1 –0.8).



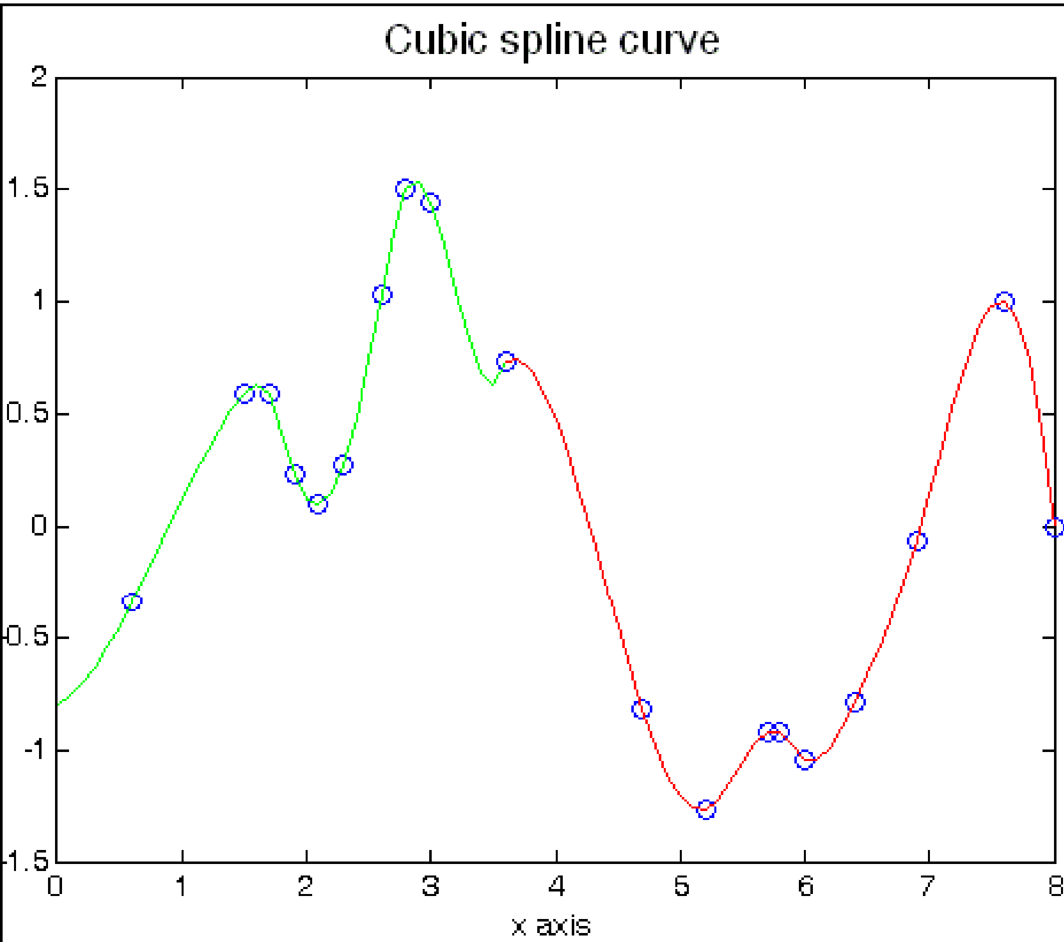
No use of 2^{nd} derivative smoothing



Two piecewise cubic spline segments

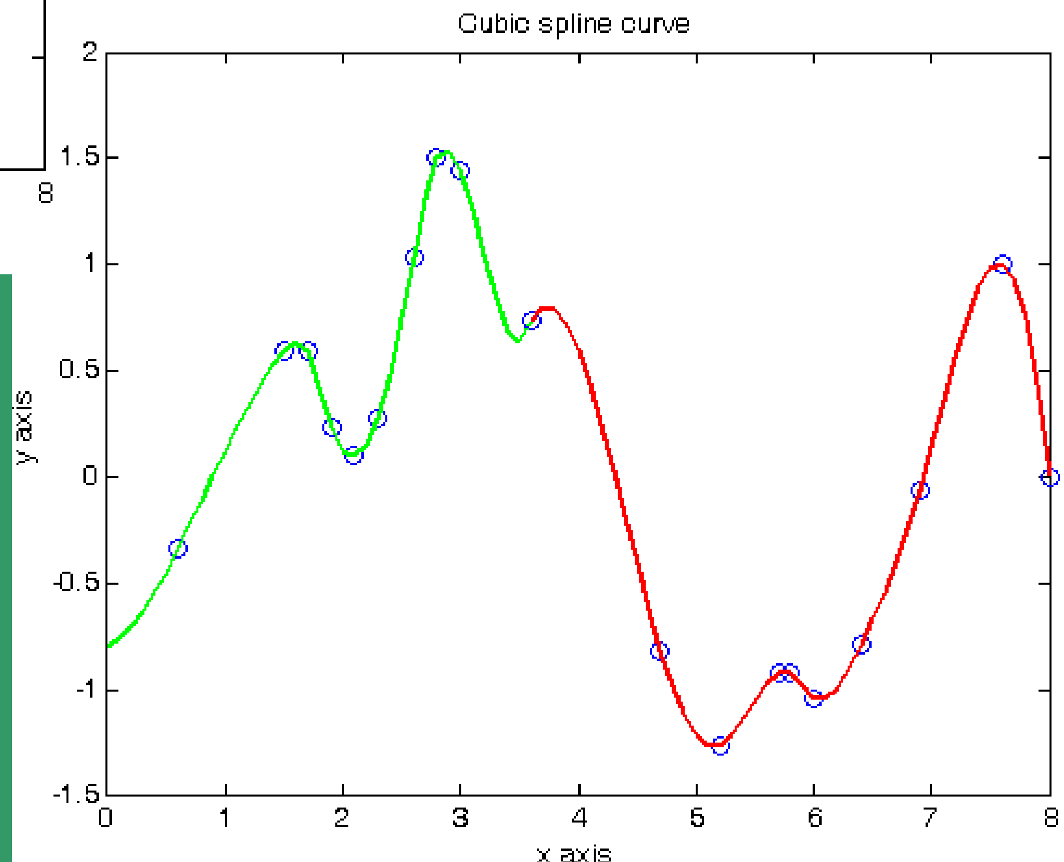
Using 2^{nd} derivative smoothing

Examples of spline interpolation



No use of 2nd derivative smoothing

Using 2nd derivative smoothing

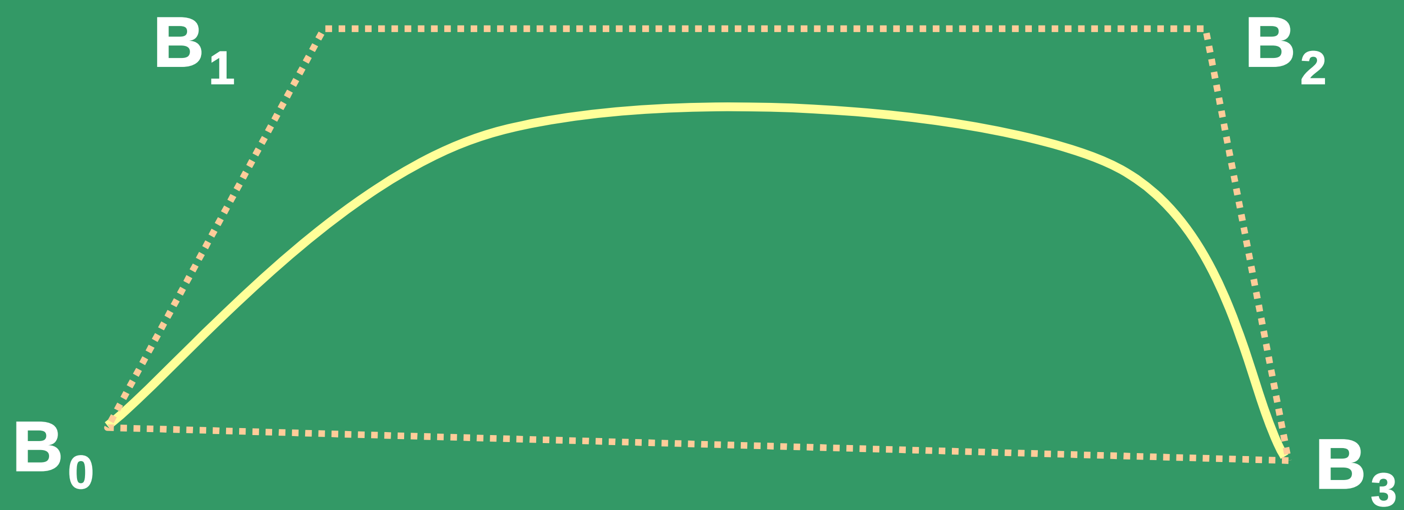


BEZIER CURVES

- **Basis functions are real**
- **Degree of polynomial is one less than the number of points**
- **Curve generally follows the shape of the defining polygon**
- **First and last points on the curve are coincident with the first and last points of the polygon**
- **Tangent vectors at the ends of the curve have the same directions as the respective spans**
- **The curve is contained within the convex hull of the defining polygon**
- **Curve is invariant under any affine transformation.**

A few typical examples of cubic polynomials for Bezier

BEZIER CURVES



Equation of a parametric Bezier curve:

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t); \;\; 0 \le t \le 1$$

B_i's are called the *control points*;

where the **Bezier or Bernstein basis** or **blending function** is:

Binomial Coefficients:
(*i*th, *n*th-order Bernstein basis function)

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$J_{n,i}(t)$ is the ith, nth order Bernstein basis function.

n is the degree of the defining Bernstein basis function (polynomial curve segment).

This is one less than the number of points used in defining Bezier polygons.

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t); \quad 0 \le t \le 1$$

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Limits for $i = 0:$

$0^0 = 1$; 0! = 1

$$J_{n,0}(0) = \frac{n!}{0!n!} 0^0 (1-0)^{n-0} = 1;$$

For $i \neq 0:$ $J_{n,i}(0) = \frac{n!}{i!(n-i)!} 0^i (1-0)^{n-i} = 0;$

Also:

$$J_{n,n}(1) = 1, i = n;$$

$$J_{n,i}(1) = 0, i \neq n.$$

Thus:

$$P(0) = B_0 J_{n,0}(0) = B_0.$$

$$P(1) = B_n J_{n,n}(1) = B_n.$$

For any t:

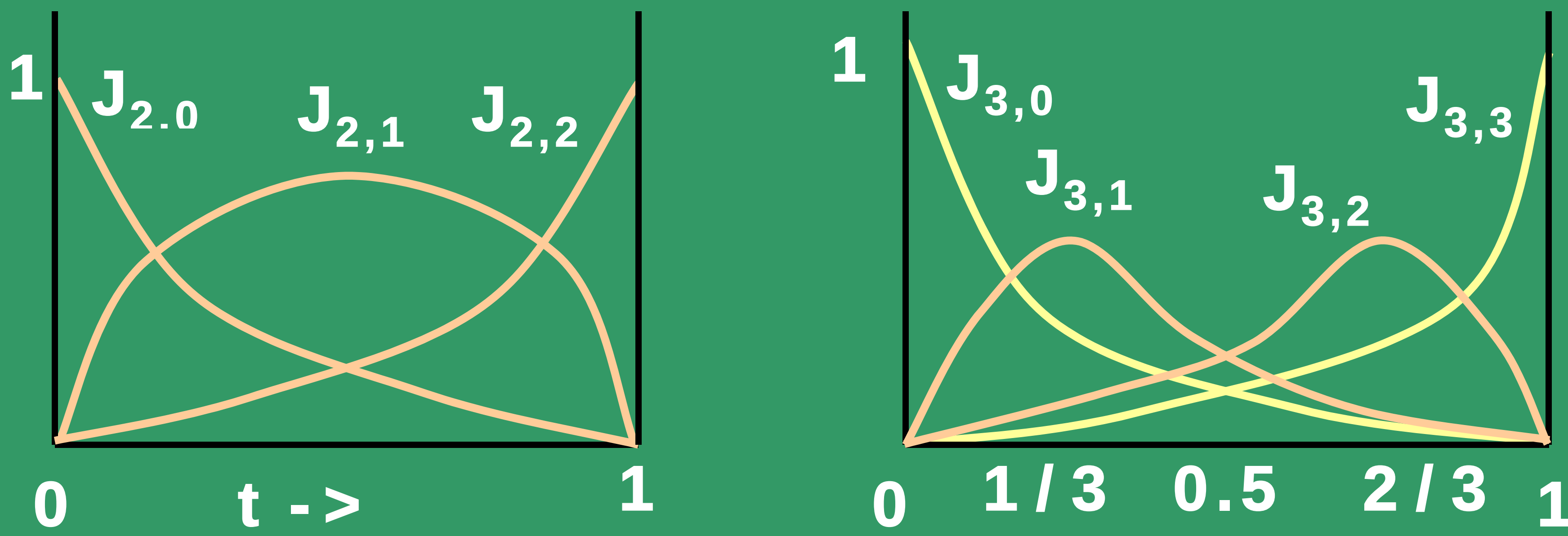
$$\sum_{i=0}^{n} J_{n,i}(t) = 1$$

Also Verify:

$$J_{n,i}(t) =$$

$$(1-t).J_{(n-1),i}(t) + t.J_{(n-1),(i-1)}(t); \; n > i \geq 1$$

Below are some examples of BBF (Bezier / Bernstein blending functions:



n = 2

n = 3 (cubic)

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}; \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Take n = 3:

$$\binom{n}{i} = \binom{3}{i} = \frac{6}{i!(3-i)!}$$

$$J_{3,0}(t) = 1.t^0(1-t)^3 = (1-t)^3;$$

$$J_{3,1}(t) = 3.t.(1-t)^2;$$

$$J_{3,2}(t) = 3.t^2.(1-t);$$

$$J_{3,3}(t) = t^3.$$

Thus, for Cubic Bezier:

$$P(t) = (1-t)^3 B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t) B_2 + t^3 B_3$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}; n = 3.$$

For Cubic-splines:

$$P(t) = T.N.G =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}^T$$

For n = 4:

$$P(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

$$= T.N.G = F.G;$$

where:

$$F = [J_{n,o}(t) \;\; J_{n,1}(t) \;\; .... \; J_{n,n}(t)]$$

$$N = \left\lfloor \lambda_{ij} \right\rfloor_{nxn}$$

where:

$$\lambda_{ij} = \begin{cases} \binom{n}{j}\binom{n-j}{n-i-j}(-1)^{n-i-j} & 0 \le (\mathrm{i}+\mathrm{j}) \le \mathrm{n} \\ 0 & \text{otherwise} \end{cases}$$

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

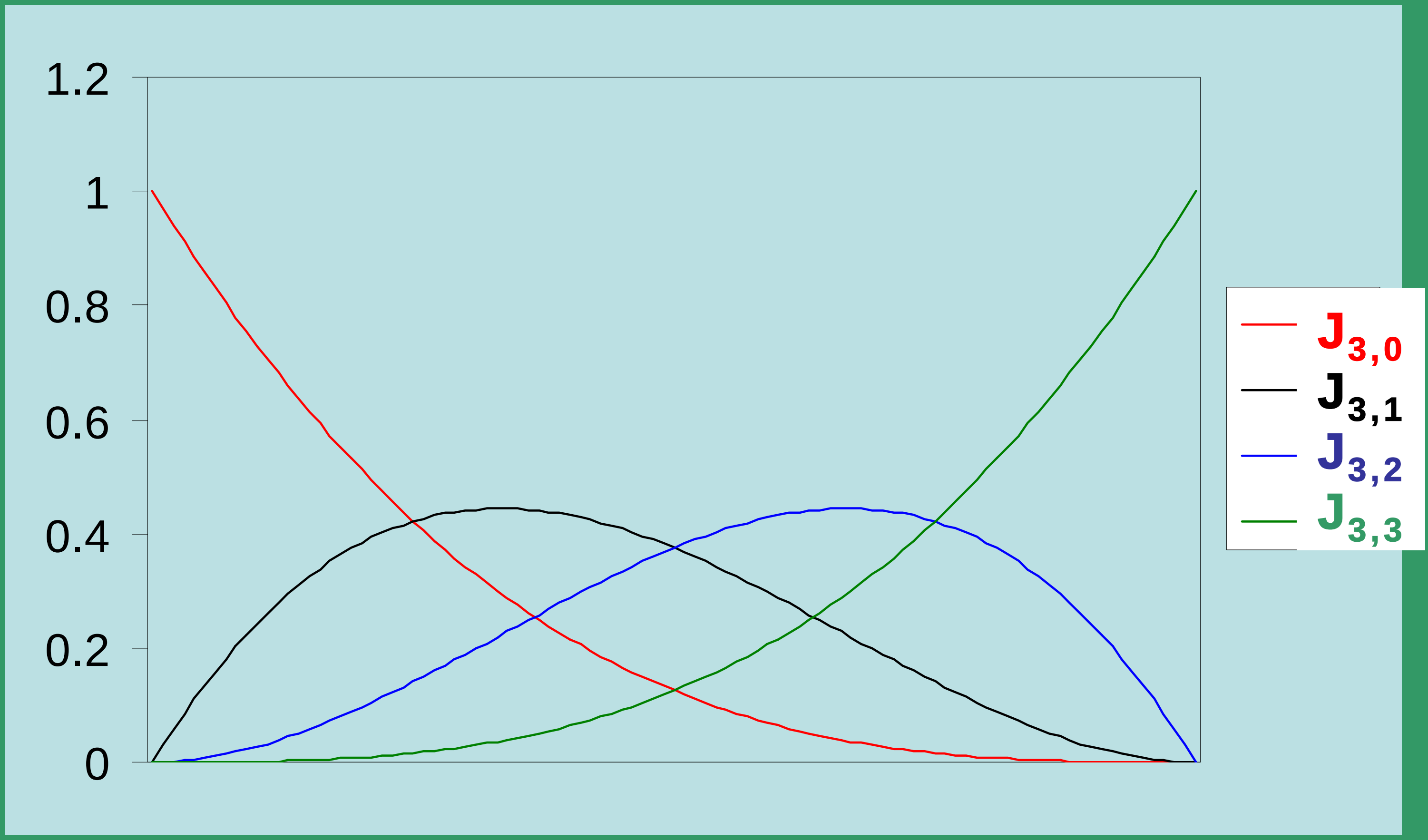
Computation of successive binomial coefficients:

$$\binom{n}{i} = \left(\text{[redacted]}\right)\binom{n}{i-1}$$

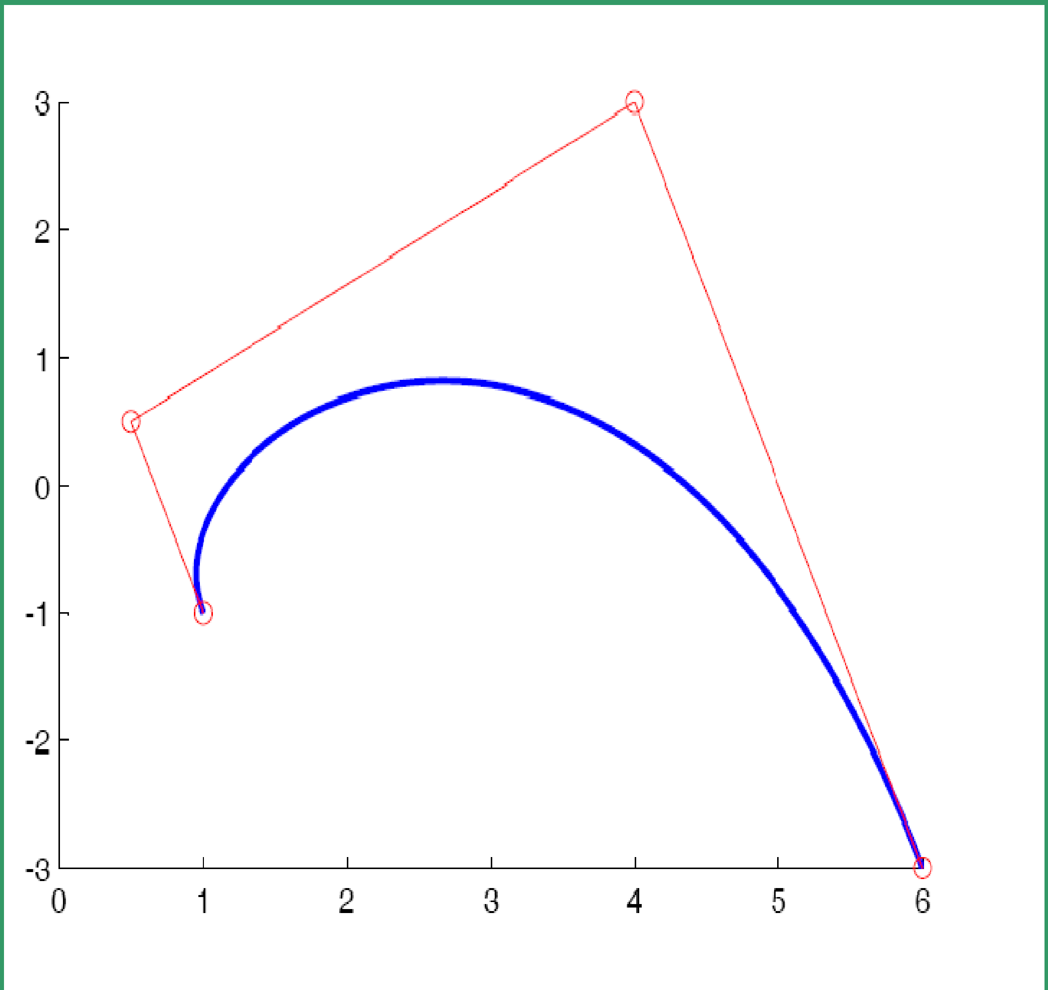
Home Assignment:

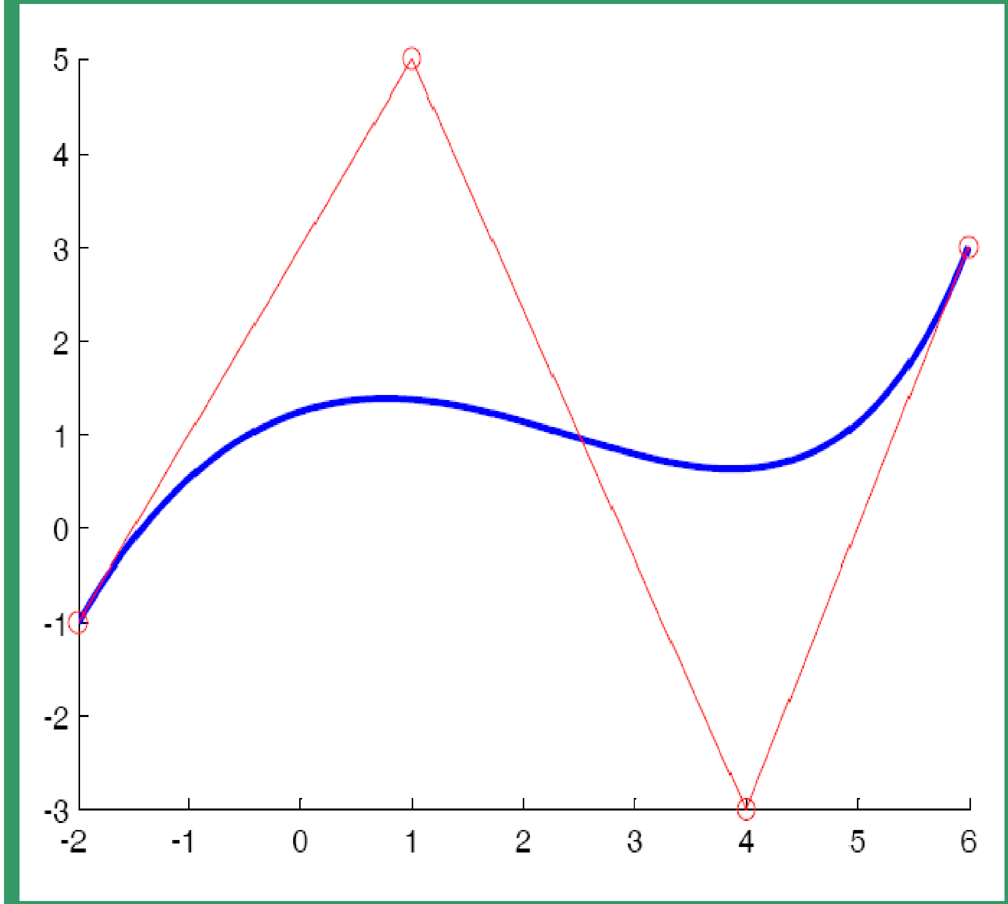
Get the expressions of $J_{2,i}$ and $J_{4,i}$

Bezier Basis Functions

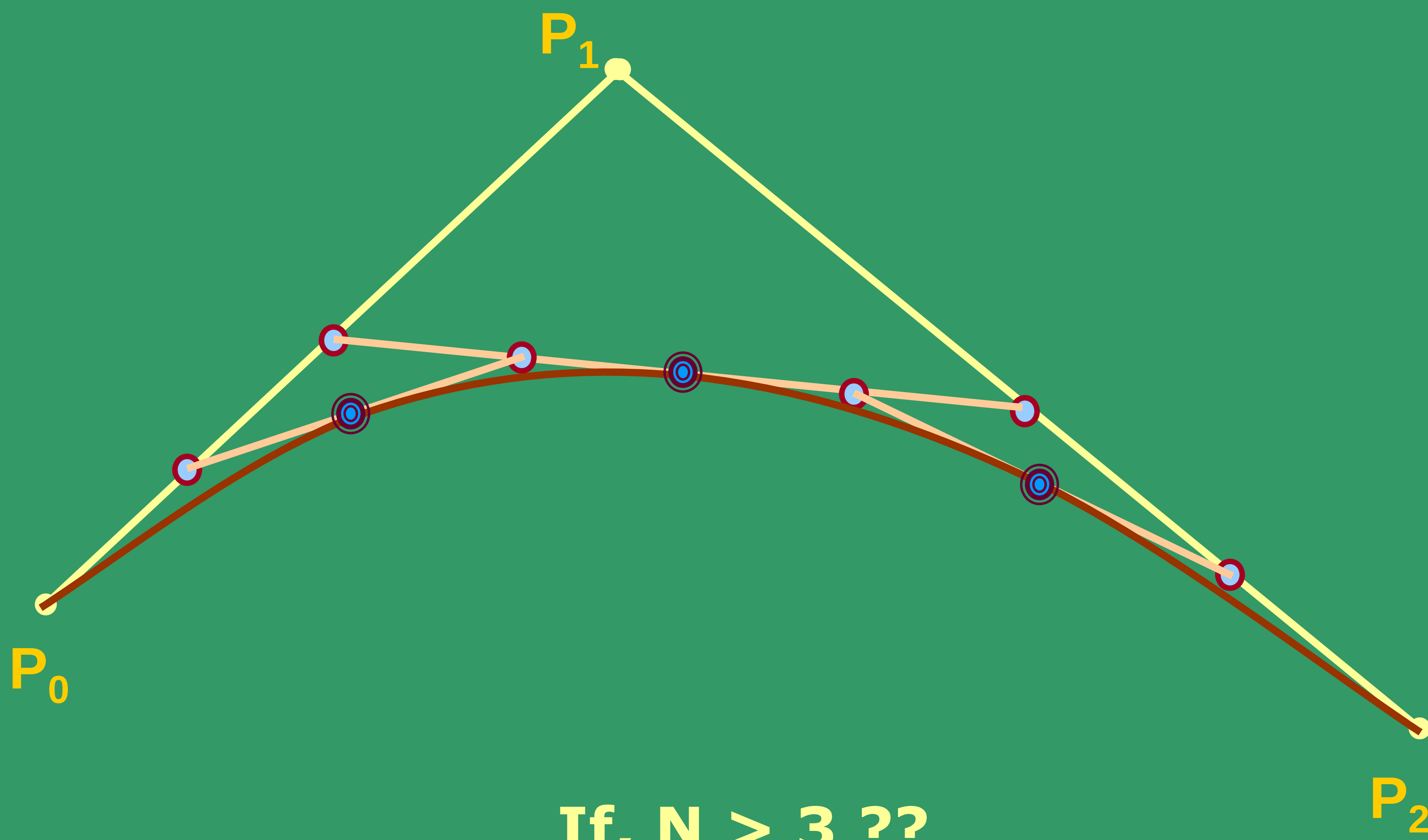


Bezier Curve Examples



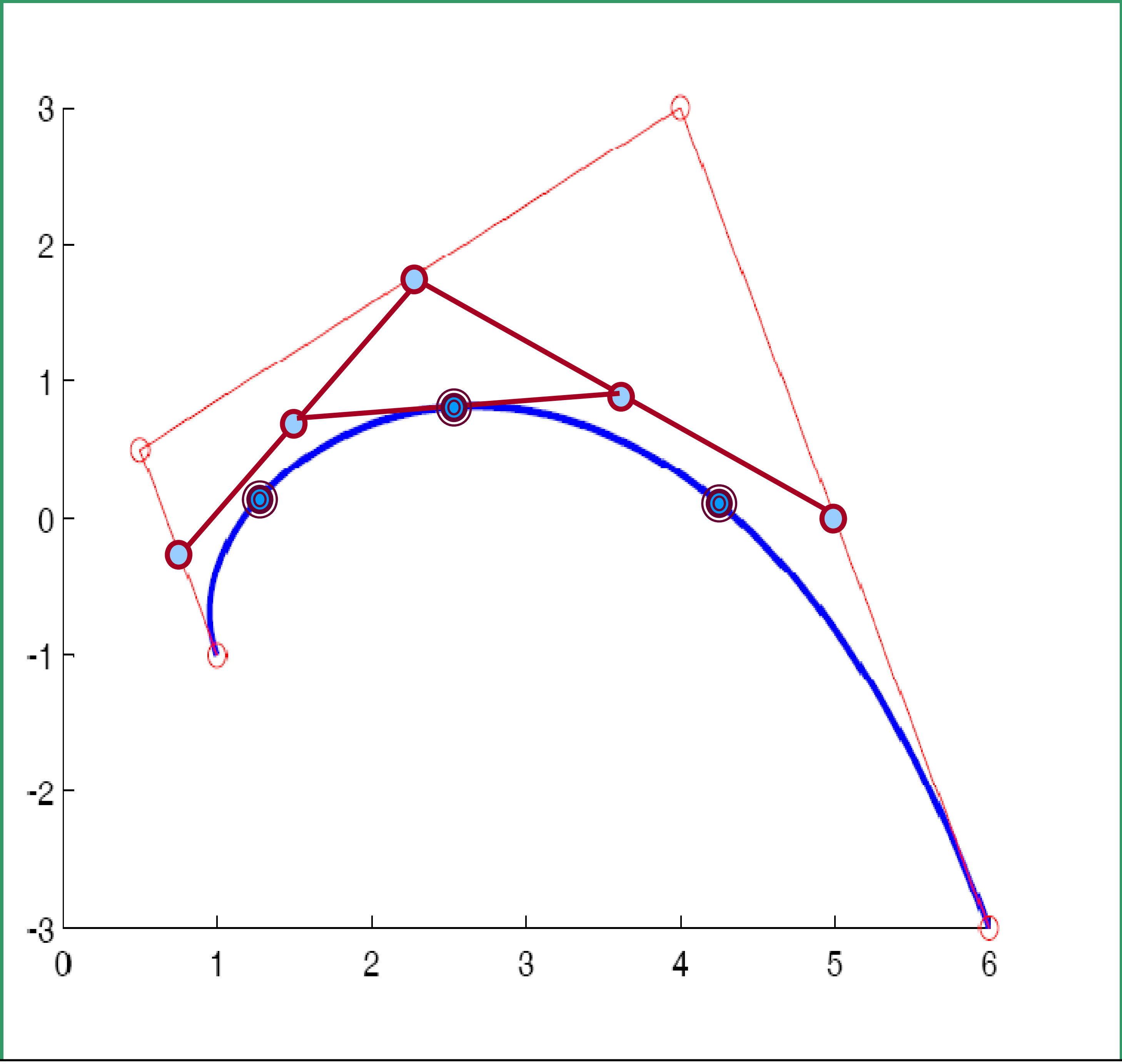


Recursive geometric definition of BEZIER CURVES

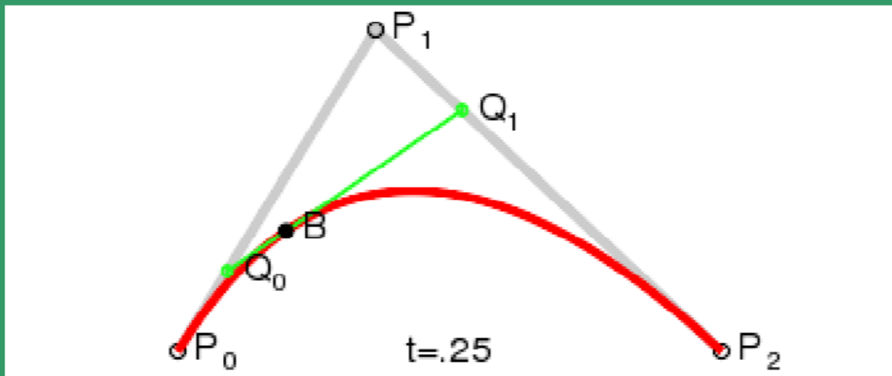


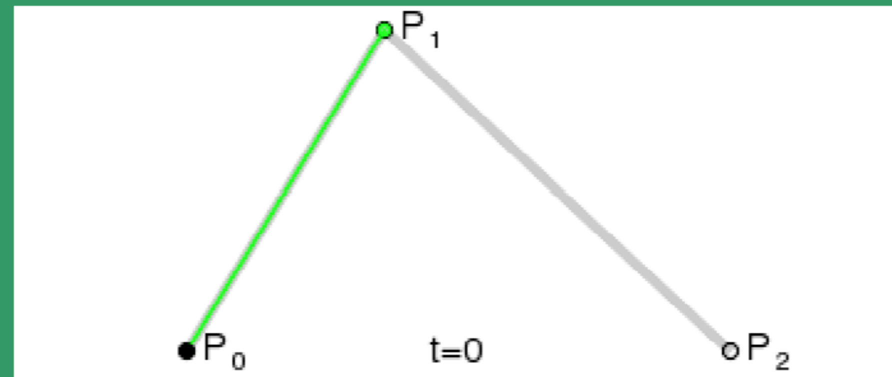
If, N > 3 ??

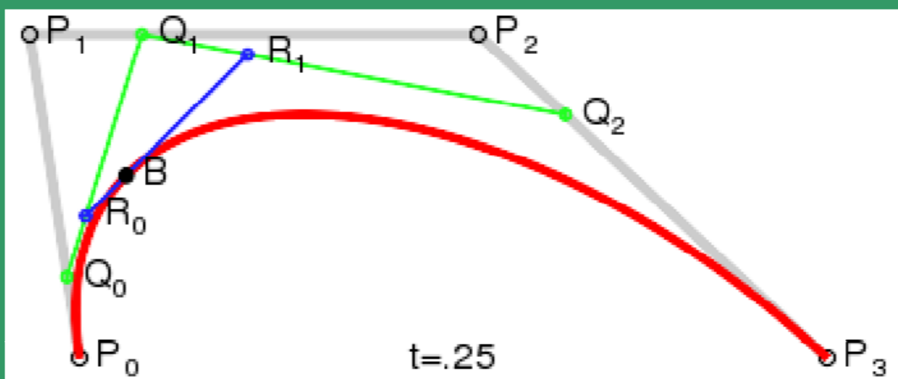
Recursive Bezier Curve Example

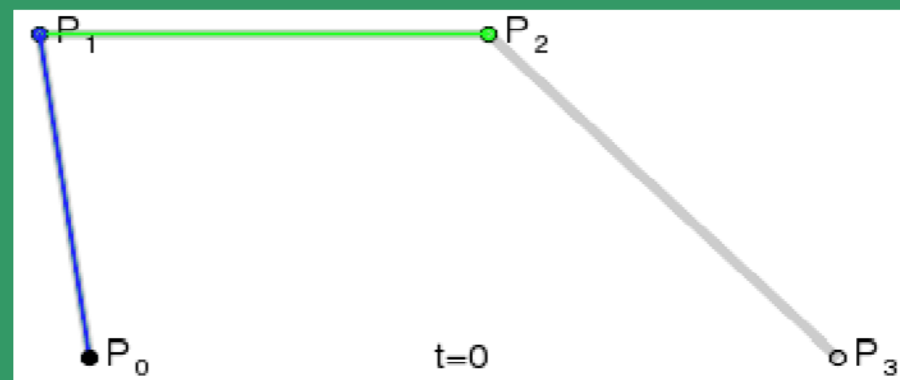


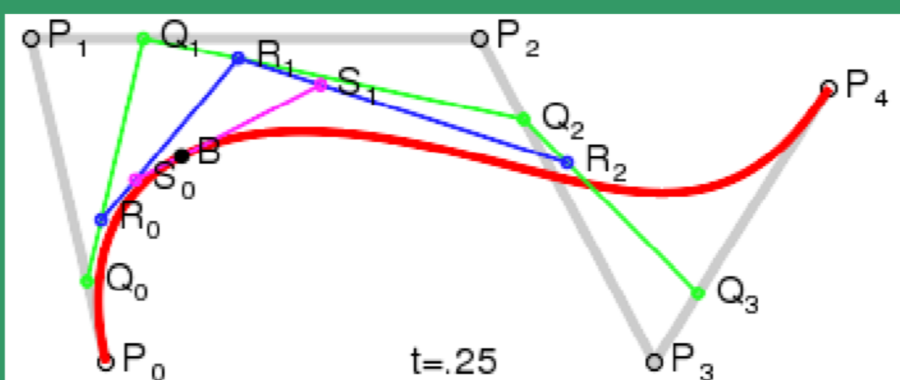
Iterative Bezier Curve Animation

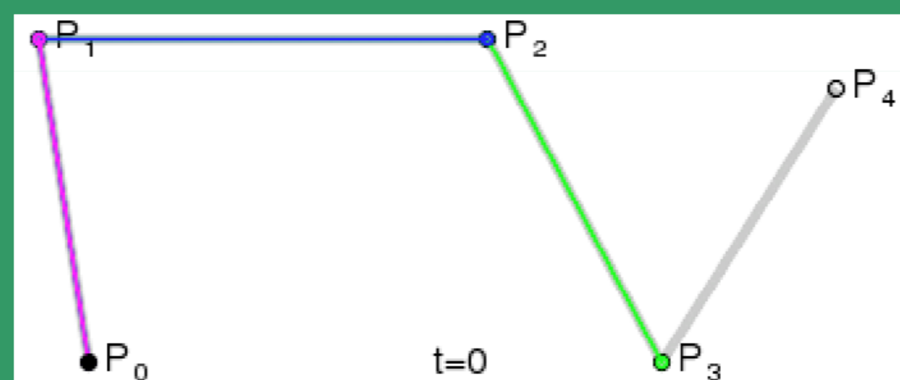




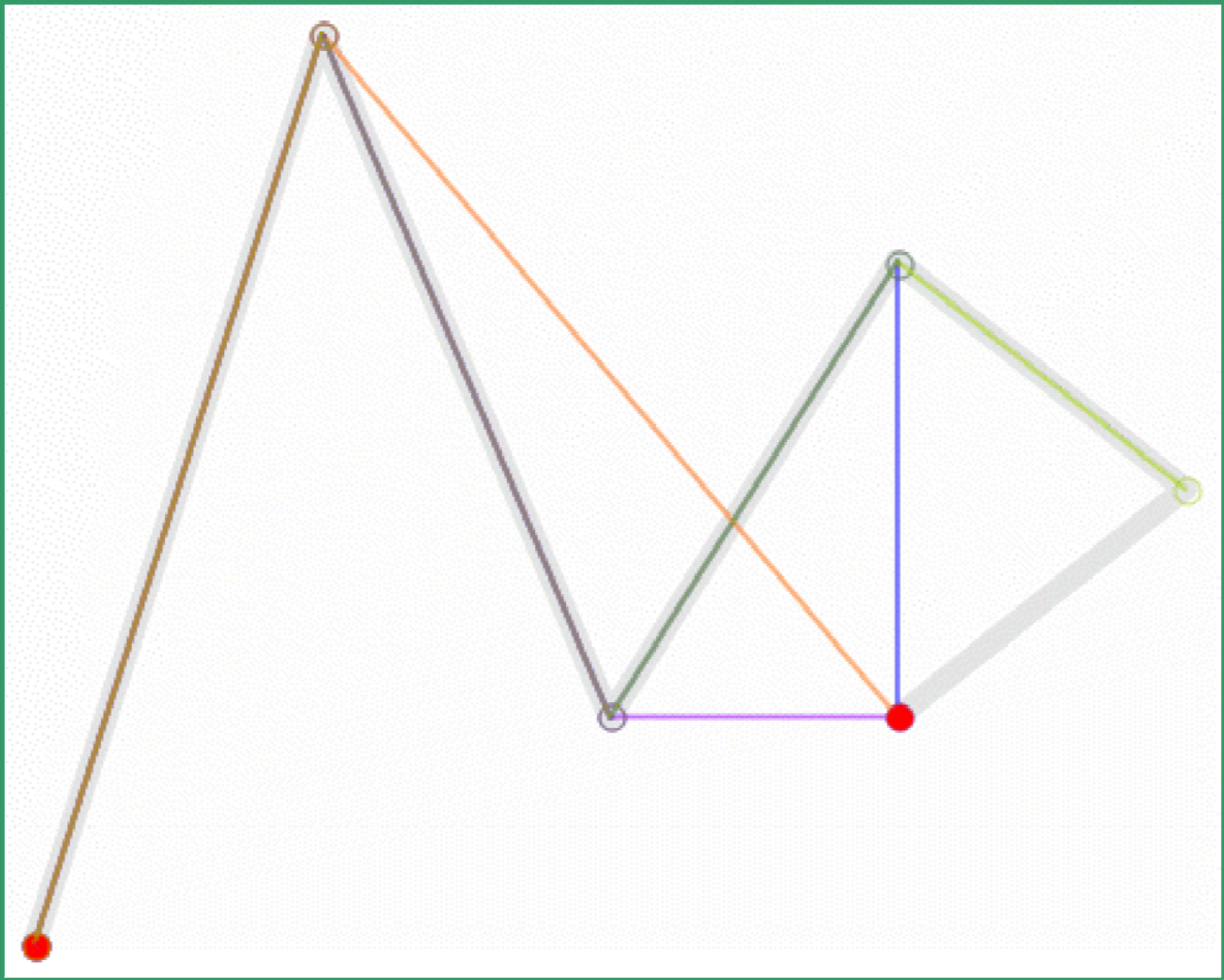








Iterative Higher-order Bezier Curve Animation



Read about:

- **B-splines represented as blending functions**
- **Conversion between one format to another.**
- **Knots and control points.**
- **When B-spline becomes a Bezier?**

QUADRICS – 3-D analogue of conics:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Jz + K = 0$$

Basis Splines (B-splines):

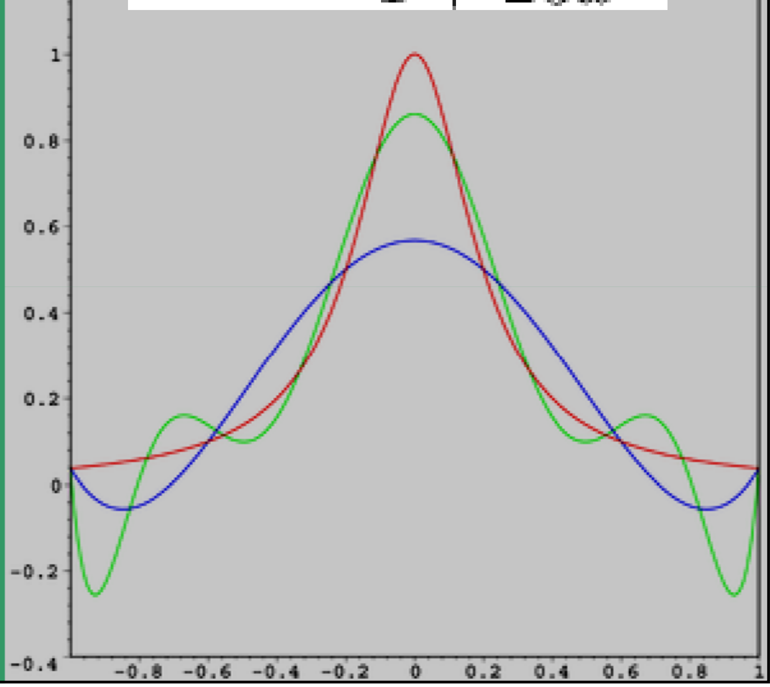
- a generalisation of a Bézier curve, avoids the Runge phenomenon without increasing the degree of the B-spline

The red curve is the Runge (The Cauchy–Lorentz distribution or Breit–Wigner distribution) function.

The blue curve is a 5th-order interpolating polynomial (using six equally-spaced interpolating points).

The green curve is a 9th-order interpolating polynomial (using ten equally-spaced interpolating points).

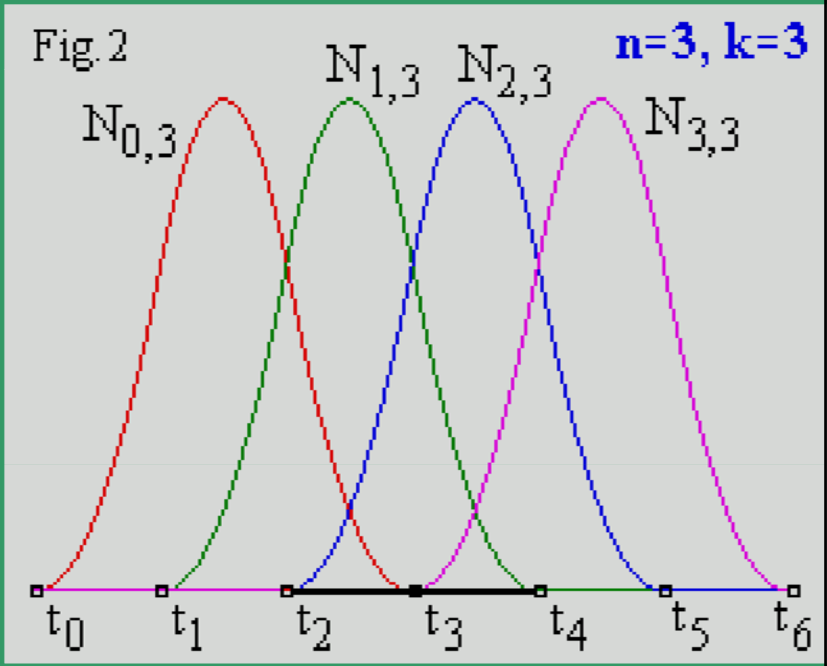
$$f(x) = \frac{1}{1 + 25x^2}.$$



At the interpolating points, the error between the function and the interpolating polynomial is (by definition) zero.

Between the interpolating points (especially in the region close to the endpoints 1 and −1), the error between the function and the interpolating polynomial gets worse for higher-order polynomials.

In mathematics, a spline is a special function defined piece-wise by polynomials.



Spline interpolation is often preferred to polynomial interpolation because it yields similar results, even when using low-degree polynomials, while avoiding Runge's phenomenon for higher degrees.

$N_{i,k}$ (i-th B-spline blending function, of order k) is a polynomial of order *k* (degree *k-1*) on each interval:

$$t_i < t < t_{i+1}.$$

k must be at least 2 (linear) and can be not more, than *n+1* (the number of control points).

A <u>knot vector</u> $(t_0 , t_1 , \ldots , t_{n+k})$ must be specified. Across the knots basis, functions are C^{k-2} continuous.

Basis Splines (B-splines):

- **Degree is independent of the No. of control Points**
- **Local Control over Shape**
- **More complex than Bezier**

Given *m* values $t_i \in [0,1]$, called *knots*, with $t_0 \leq t_1 \leq \cdots \leq t_{m-1}$

a B-spline of degree *n* is a parametric curve $\mathbf{S} : [t_0, t_{m-1}] \to \mathbb{R}^2$

composed of linear combination of basis B-splines of degree *n*

$$\mathbf{S}(t) = \sum_{i=0}^{m-n-2} \mathbf{P}_i b_{i,n}(t)\ ,\ t \in [t_n, t_{m-n-1}]$$

The P_i are called control points or de Boor points (there are *m-n-1* control points). A polygon can be constructed by connecting the de Boor points with lines, starting with P_0 and finishing with P_{m-n-2}. This polygon is called the de Boor polygon

The *m-n-1* basis B-splines of degree *n* for n = 0,1,...,m-2, can be defined using the Cox-de Boor recursion formula:

$$b_{j,0}(t) := \begin{cases} 1 & \text{if} \quad t_j \le t < t_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

j = 0,1,...,m-2

$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t).$$

j = 0,1,...,m-n-2

(j+n+1) can not exceed m-1, which limits both j and n.

The above recursion formula specifies how to construct nth-order function from two B-spline function of order (n-1).

No. of Control Points: (m – n - 1);

Degree of Spline: n;

(m-n-1=4; n=3) - B-spline will have [0 0 0 0 1 1 1 1] knot vector.

No. of Knots: m (= No. of Control Points + degree + 1);

When the knots are equidistant we say the B-spline is uniform otherwise we call it non-uniform.

NURBS: Non-uniform Regularized B-Splines

Uniform B-spline

When the B-spline is uniform, the basis B-splines for a given degree *n* are just shifted copies of each other. An alternative non-recursive definition for the *m−n-1* basis B-splines is:

$$b_{j,n}(t) = b_n(t - t_j), \qquad j = 0, \ldots, m-n-2$$

with

$$b_n(t) := \frac{n+1}{n} \sum_{i=0}^{n+1} \omega_{i,n}(t - t_i)_+^n$$

and

$$\omega_{i,n} := \prod_{j=0, j\neq i}^{n+1} \frac{1}{t_j - t_i}$$

where

$$(t - t_i)_+^n := \begin{cases} (t - t_i)^n & \text{if } t \geq t_i \\ 0 & \text{if } t < t_i \end{cases}$$

is the truncated power function.

When the number of knots is the same as the degree, the B-Spline degenerates into a Bézier curve. The shape of the basis functions is determined by the position of the knots. Scaling or translating the knot vector does not alter the basis functions

The "**Standard Knot Vector**" for a B-spline of order (n + 1) begins and end with a knot of "multiplicity" (n+1) and uses unit spacing for the remaining knots.

Let, No. of control points: m-n-1 = 8;
and for a cubic (n=3) B-spline: n + 1 = 4;

So, m = 12; The "**Standard Knot Vector**" is"

[0 0 0 0 1 2 3 4 5 5 5 5]

Periodic,
Cubic B-spline
Blending functions :

$$B_{0,3}(t) = (1-t)^3 / 6;$$

$$B_{1,3}(t) = (3.t^3 - 6t^2 + 4) / 6;$$

$$B_{2,3}(t) = (-3.t^3 + 3t^2 + 3t + 1) / 6;$$

$$B_{3,3}(t) = t^3 / 6.$$

Linear B-spline:

$$b_{j,1}(t) = \begin{cases} \frac{t-t_j}{t_{j+1}-t_j} & \text{if} \quad t_j \le t < t_{j+1} \\ \frac{t_{j+2}-t}{t_{j+2}-t_{j+1}} & \text{if} \quad t_{j+1} \le t < t_{j+2} \\ 0 & \text{otherwise} \end{cases}$$

Uniform quadratic B-spline (uniform knot vector):

$$b_{j,2}(t) = \begin{cases} \frac{1}{2}t^2 \\ -t^2 + t + \frac{1}{2} \\ \frac{1}{2}(1-t)^2 \end{cases}$$

V = [1, 2, 3, 4, 5, 6];

$$\mathbf{S}_i(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$

$$t \in [0, 1], i = 1, 2 \ldots m-2$$

For Bezier:

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}; n = 3.$$

For Cubic-splines:

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}^T$$

For Cubic B-splines, with uniform Knot vector:

$$\mathbf{S}_i(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \end{bmatrix}$$

$$\boldsymbol{P(t)} = \sum_{i=1}^{4} \boldsymbol{B}_i \boldsymbol{t}^{i-1}; \boldsymbol{t}_i \le \boldsymbol{t} \le \boldsymbol{t}_2.$$

$$P(u) = \sum_{k=0}^{3} g_k H_k(u)$$

$$P(t) = P_1(2t^3 - 3t^2 + 1) + P_2(-2t^3 + 3t^2)$$

$$+ P_1'(t^3 - 2t^2 + t) + P_2'(t^3 - t^2)$$

CUBIC SPLINES

$$\boldsymbol{P(t)} = \sum_{i=0}^{n} \boldsymbol{B}_i \boldsymbol{J}_{n,i}\boldsymbol{(t)}; \quad 0 \le \boldsymbol{t} \le 1$$

BEZIER CURVES

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}; \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\mathbf{S}(t) = \sum_{i=0}^{m-n-1} \mathbf{P}_i b_{i,n}(t), \; t \in [t_n, t_{m-n}]$$

B-splines

$$b_{j,0}(t) := \begin{cases} 1 & \text{if} \quad t_j \le t < t_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

j = 0,1,...,m-2

j = 0,1,...,m-n-2

$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t).$$

The recursion for integer knots

$$(m-1)B_{jm}(t) =$$
$$(t-j)B_{j,m-1}(t) + (m+j-t)B_{j+1,m-1}(t)$$

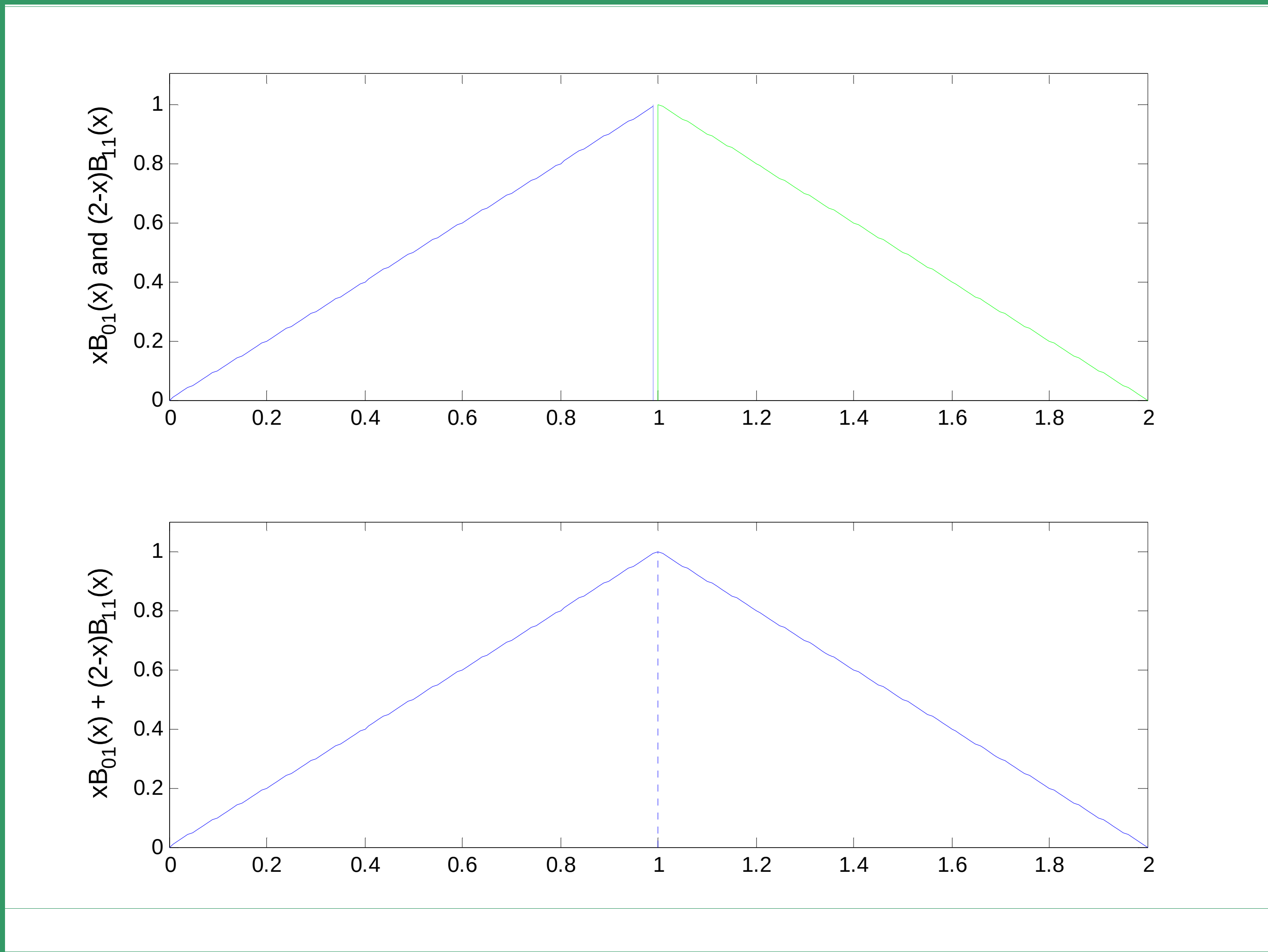
$$b_{j,n}(t) := \frac{t-t_j}{t_{j+n}-t_j} b_{j,n-1}(t) + \frac{t_{j+n+1}-t}{t_{j+n+1}-t_{j+1}} b_{j+1,n-1}(t).$$

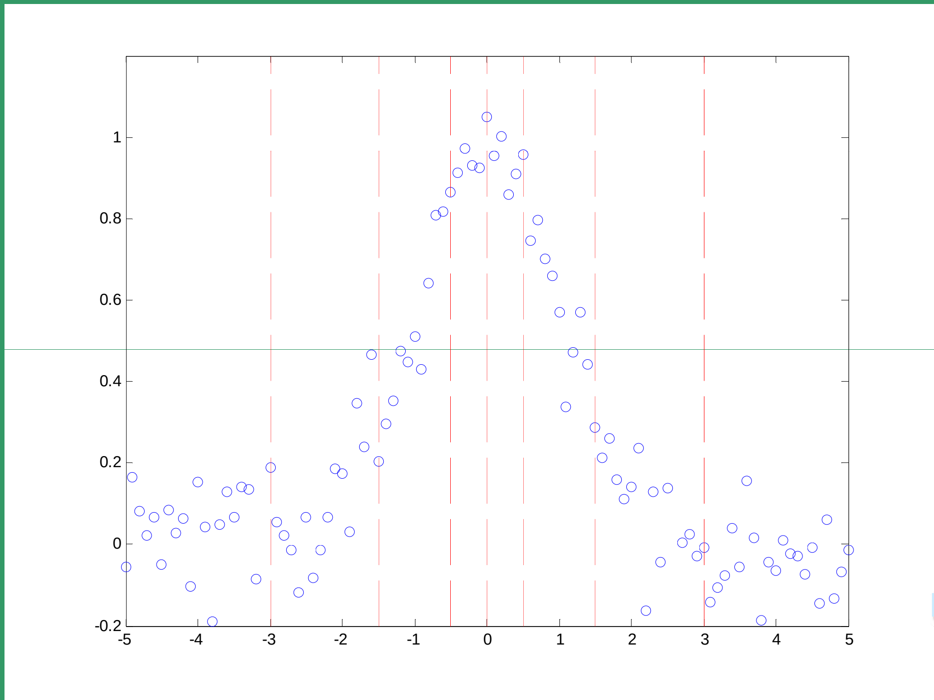
For the B-spline function of order 2 beginning at 0, the recursion is:

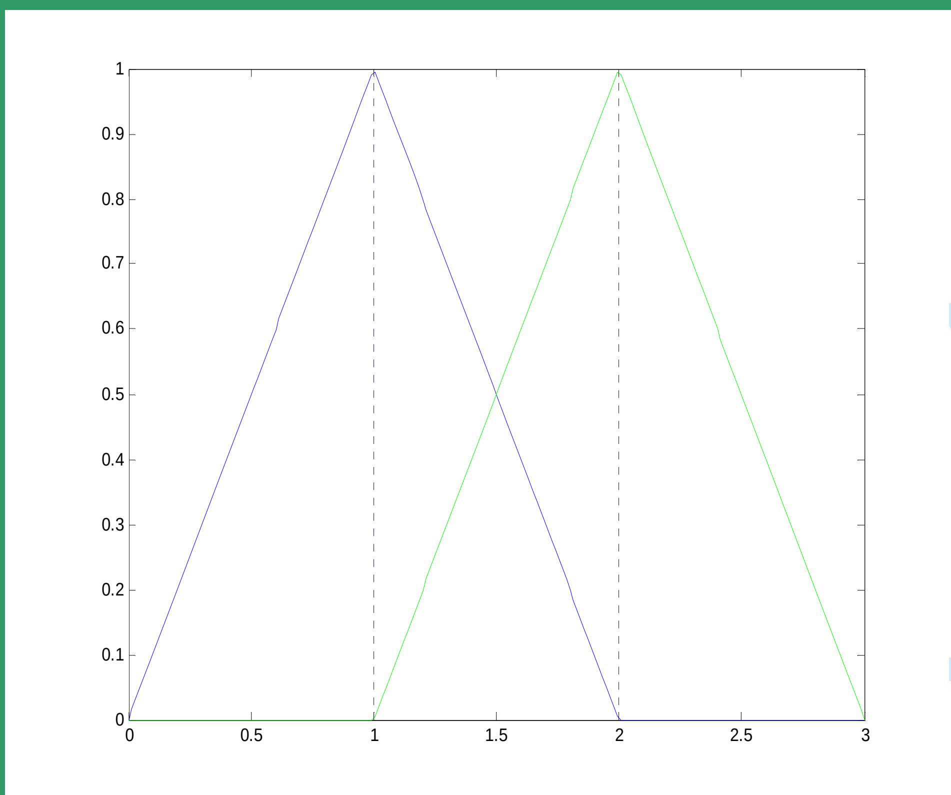
$$B_{02}(t) = tB_{01}(t) + (2-t)B_{01}(t)$$

Degree is "n" and order is "m" = n + 1.

Tent $B_{02}(t)$ from Two Boxes $B_{01}(t)$ and $B_{11}(t)$







knots: $\xi_0, \xi_1, \ldots, \xi_L$.

- B-splines of order 2 are tent functions, starting at a knot, rising linearly to 1 at the next knot, and decaying linearly to 0 two knots over.
- They ($B_{0,2}$ & $B_{1,2}$) are continuous. Order 2 implies a continuous derivative of order 0.
- Order 2 knots are piecewise linear

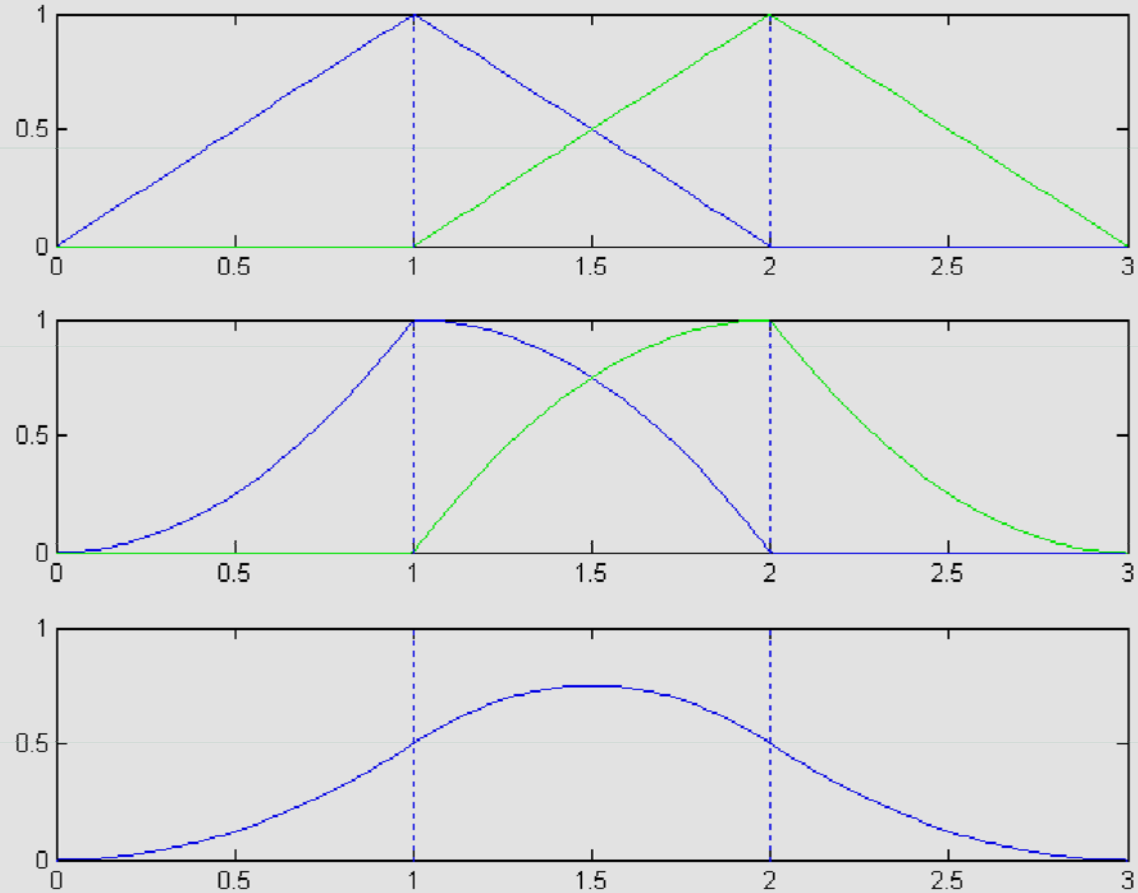
Order 3 $B_{03}(t)$ from Two Tent Functions

$$B_{0,3}(t) = (t/2) B_{0,2}(t) + \frac{3-t}{2} B_{1,2}(t)$$

$t.(2-t)$

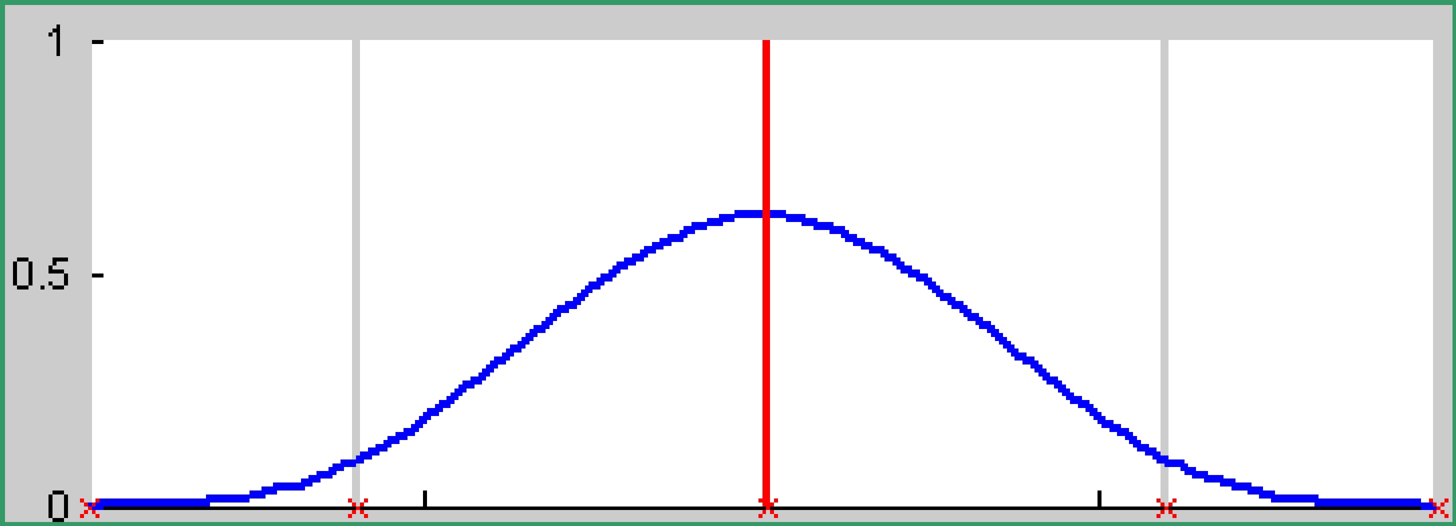
$(3-t)(t-1)$

$(3-t)^2$

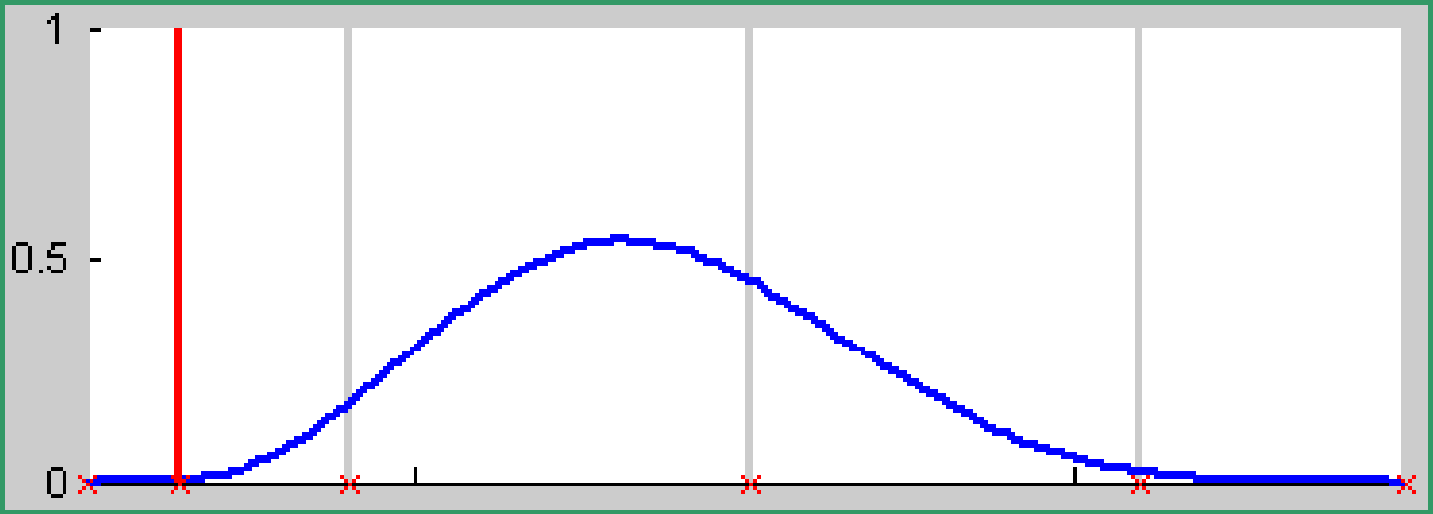


$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t).$$

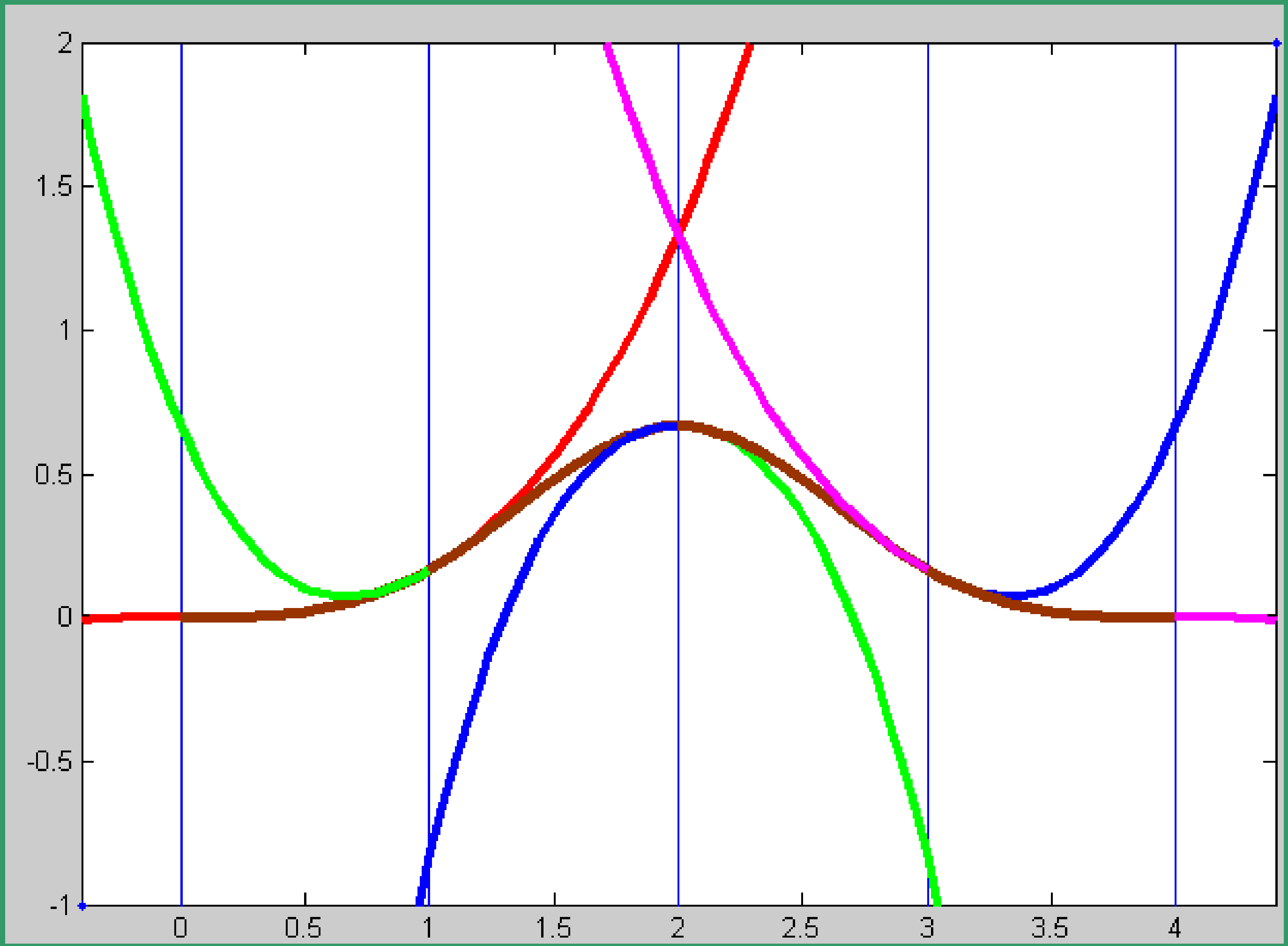
B-Spline Examples



Order 4, Degree 3, Knots = 5, Poly pieces = 4.



Order 5, Degree 4, Knots = 6, Poly pieces = 5.



A B-Spline of Order 4, and the Four Cubic Polynomials from which It Is Made

Knot Sequence: [0 1 2 3 4]

A B-Spline of Order 4, and the Four Cubic Polynomials from which It Is Made

Knot Sequence: [0 1.5 2.3 4 5]

QUADRIC SURFACES

Some trivial examples:

SPHERE

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2;$$

$$x=r.\cos\phi.\cos\theta,\ -\pi/2\le\phi\le\pi/2$$

$$y=r.\cos\phi.\sin\theta,\ \ -\pi\le\phi\le\pi$$

$$z=r.\sin\phi.$$

ELLIPSOID

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1;$$

$$x = a.\cos\phi.\cos\theta, \ -\pi/2 \le \phi \le \pi/2$$

$$y = b.\cos\phi.\sin\theta, \quad -\pi \le \phi \le \pi$$

$$z = c.\sin\phi.$$

TORUS

$$\left[r - \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}\right]^2 + \left(\frac{z}{c}\right)^2 = 1;$$

$$x = a.(r + \cos\phi).\cos\theta, \ -\pi \le \phi \le \pi$$

$$y = b.(r + \cos\phi).\sin\theta, \quad -\pi \le \phi \le \pi$$

$$z = c.\sin\phi.$$

SUPERELLIPSOID

$$[(\frac{x}{a})^{2/s_2} + (\frac{y}{b})^{2/s_2}]^{s_2/s_1} + (\frac{z}{c})^{2/s_1} = 1;$$

$$x = a.\cos^{s_1}\phi.\cos^{s_2}\theta, \ -\pi/2 \le \phi \le \pi/2$$

$$y = b.\cos^{s_1}\phi.\sin^{s_1}\theta, \ \ -\pi \le \phi \le \pi$$

$$z = c.\sin s_1\phi.$$

SUPERQUADRICS:

$$(\alpha x)^n + (\beta y)^n + (\gamma z)^n = k$$

General expression of a Quadric Surface

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz$$
$$+ Gx + Hy + Jz + K = 0.$$

The above is a generalization of the general conic equation in 3-D. In matrix form, it is:

$$XSX^T = 0,$$

$$\Rightarrow \begin{bmatrix} x & y & z & 1 \end{bmatrix} (1/2) \begin{bmatrix} 2A & D & F & G \\ D & 2B & E & H \\ F & E & 2C & J \\ G & H & J & 2K \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

Parametric forms of the quadric surfaces, are often used in computer graphics

***Ellipsoid* :**

$$x = a\cos(\theta).\sin(\phi);\ \ 0 \le \theta \le 2\pi;$$
$$y = b\sin(\theta).\sin(\phi);\ \ 0 \le \phi \le 2\pi;$$
$$z = c\cos(\phi);$$

***Elliptic Cone* :**

$$x = a\phi\cos(\theta);\ \ 0 \le \theta \le 2\pi$$
$$y = b\phi\sin(\theta);\ \ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = c\phi$$

***Hyperbolic Paraboloid* :**

$$x = a\phi\cosh(\theta);\ \ -\pi \le \theta \le \pi$$
$$y = b\phi\sinh(\theta);\ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = \phi^2$$

***Elliptic Paraboloid* :**

$$x = a\phi\cos(\theta);\ \ 0 \le \theta \le 2\pi$$
$$y = b\phi\sin(\theta);\ \ 0 \le \phi \le \phi_{\max}$$
$$z = \phi^2$$

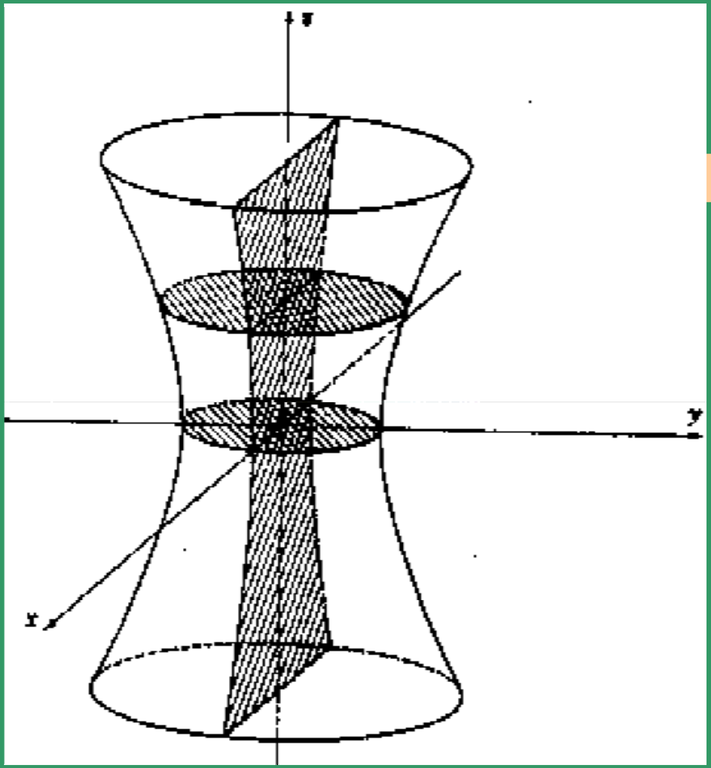
***Hyperboloid*:**

$$x = a\cos(\theta)\cosh(\phi);\ \ 0 \le \theta \le 2\pi$$
$$y = b\sin(\theta)\sinh(\phi);\ \ -\pi \le \phi \le \pi$$
$$z = \sinh(\phi)$$

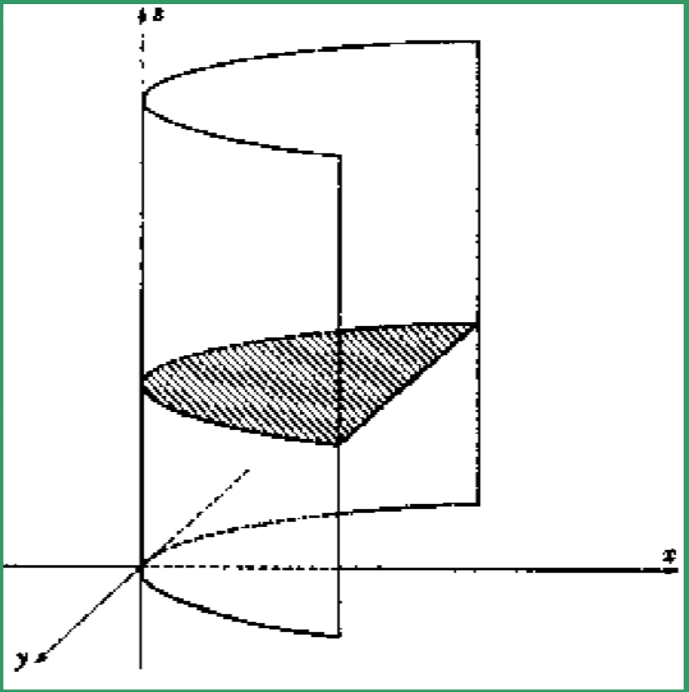
***Parabolic Cylinder* :**

$$x = a\theta^2;\ \ 0 \le \theta \le \theta_{\max}$$
$$y = 2a\theta;\ \ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = \phi$$

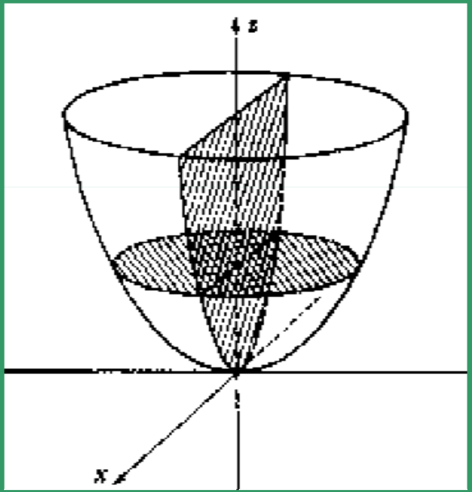
Some examples of Quadric Surfaces



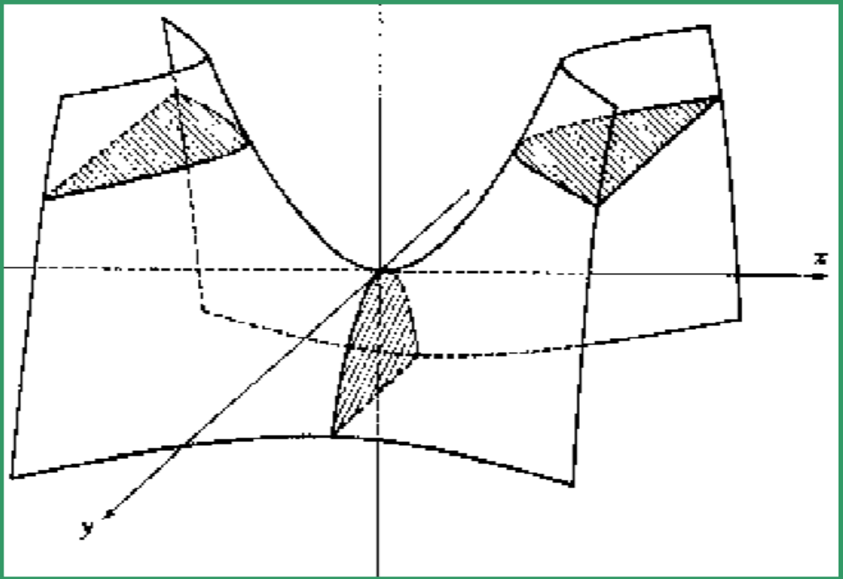
Hyperboloid



Parabolic Cylinder



Elliptic Paraboloid



Hyperbolic Paraboloid

BEZIER Surfaces

- Degree of the surface in each parametric direction is one less than the number of defining polygon vertices in that direction
- Surface generally follows the shape of the defining polygon net
- Continuity of the surface in each parametric direction is two less than the number of defining polygon net
- Only the corner points of the defining polygon net and the surface are coincident
- The surface is contained within the convex hull of the defining polygon
- Surface is invariant under any affine transformation.

Equation of a parametric Bezier surface:

$$Q(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{i,j} J_{n,i}(u) K_{m,j}(w);$$

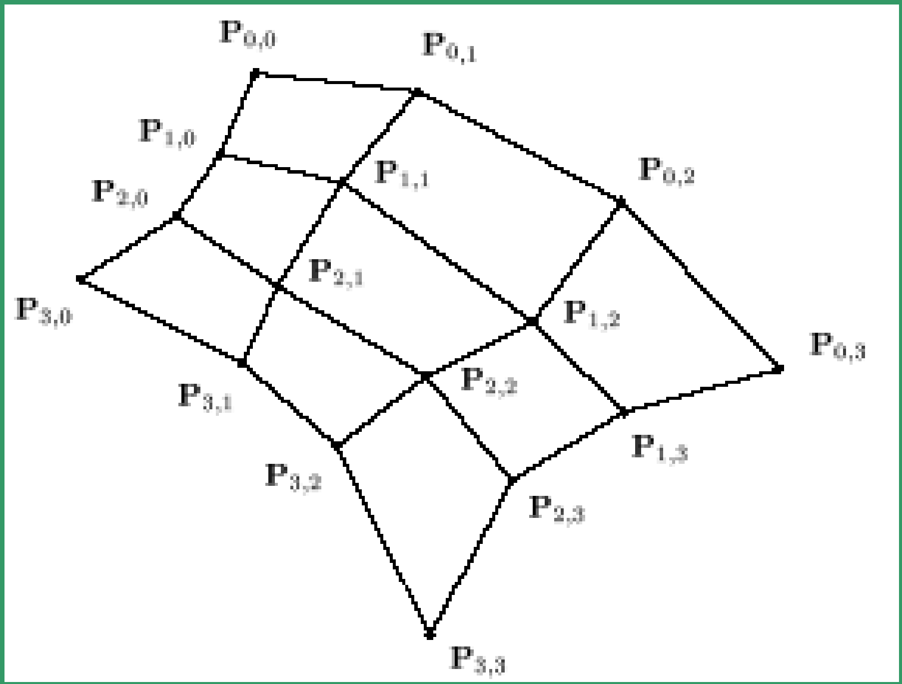
$$J_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i};$$

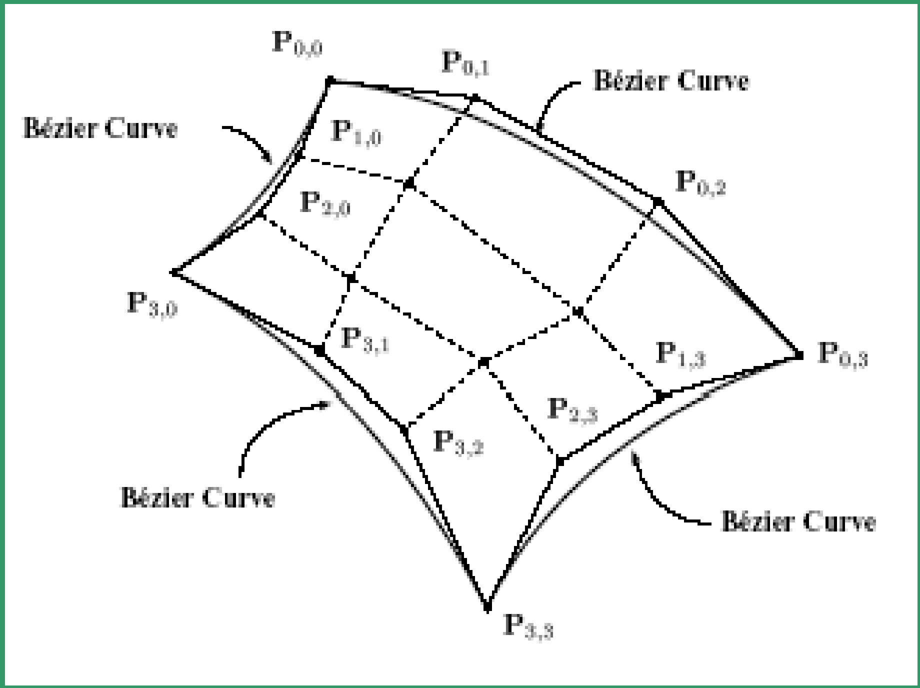
$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$K_{m,j}(w) = \binom{m}{j} w^j (1-w)^{m-j};$$

$$\binom{m}{j} = \frac{m!}{j!(m-j)!}$$

BEZIER Surfaces





$$Q(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{i,j} J_{n,i}(u) K_{m,j}(w)$$

$$= \sum_{i=0}^{n} [\sum_{j=0}^{m} P_{i,j} J_{n,i}(u)] K_{m,j}(w);$$

BEZIER Surface in matrix form:

$$Q(u,w) = U.N.B.M^T W;$$

where,

$$U = [u^n \quad u^{n-1} \quad . \quad . \quad 1],$$

$$W = [w^m \quad w^{m-1} \quad . \quad . \quad 1]^T,$$

$$B = \begin{bmatrix} B_{0,0} & . & . & B_{0,m} \\ . & . & . & . \\ . & . & . & . \\ B_{n,0} & . & . & B_{n,m} \end{bmatrix}$$

4x4 bicubic BEZIER Surface in matrix form:

$$Q(u,w) =$$

$$\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3} \\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,2} \\ B_{2,0} & B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,0} & B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix}$$

$$X \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix};$$

Non-square 4x4 bicubic BEZIER Surface in matrix form:

$$Q(u,w)=$$

$$\begin{bmatrix} u^4 & u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & -12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} \\ B_{1,0} & B_{1,1} & B_{1,2} \\ B_{2,0} & B_{2,1} & B_{2,2} \\ B_{3,0} & B_{3,1} & B_{3,2} \\ B_{4,0} & B_{4,1} & B_{4,2} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w^2 \\ w \\ 1 \end{bmatrix};$$

NURBS

$$Q(u,v) = \frac{\sum_{i=0}^{M}\sum_{k=0}^{L} w_{i,j} P_{i,k} B_{i,m}(u) B_{k,n}(v)}{\sum_{i=0}^{M}\sum_{k=0}^{L} w_{i,j} B_{i,m}(u) B_{k,n}(v)}$$

End of Lectures on

CURVES and SURFACE REPRESENTATION