Computer Vision - Histogram Processing

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HISTOGRAM

- In a gray level image the probabilities assigned to each gray level can be given by the relation:

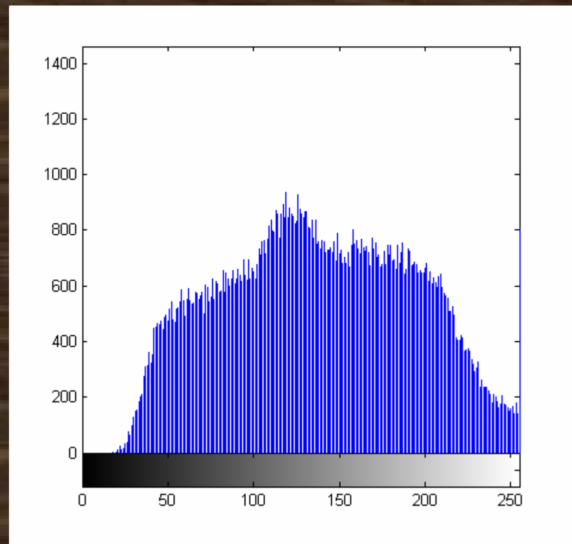
$$p_r(r_k) = \frac{n_k}{N} \qquad 0 \le \mathrm{r_k} \le 1,\ \mathrm{k} = 0,1,2\ldots \mathrm{L}-1$$

r_k - The normalized intensity value

L - No. of gray levels in the image

n_k - No. of pixels with gray level $\mathrm{r_k}$

N - Total number of pixels

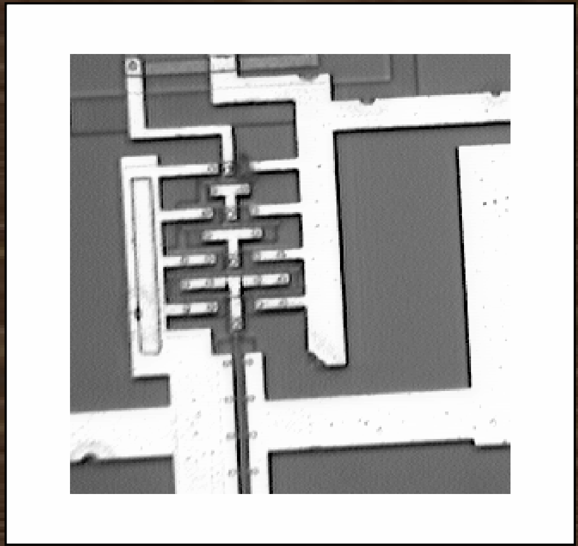


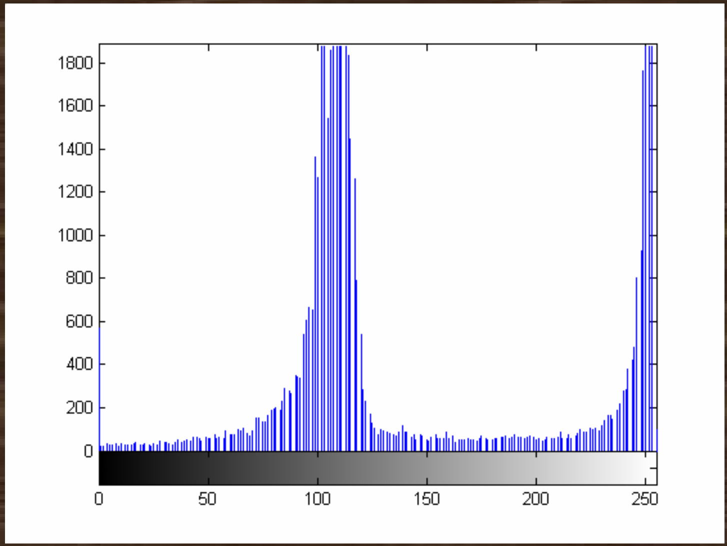
histogram

The plot of $p_r(r_k)$ with respect to r_k is called *HISTOGRAM* of the image

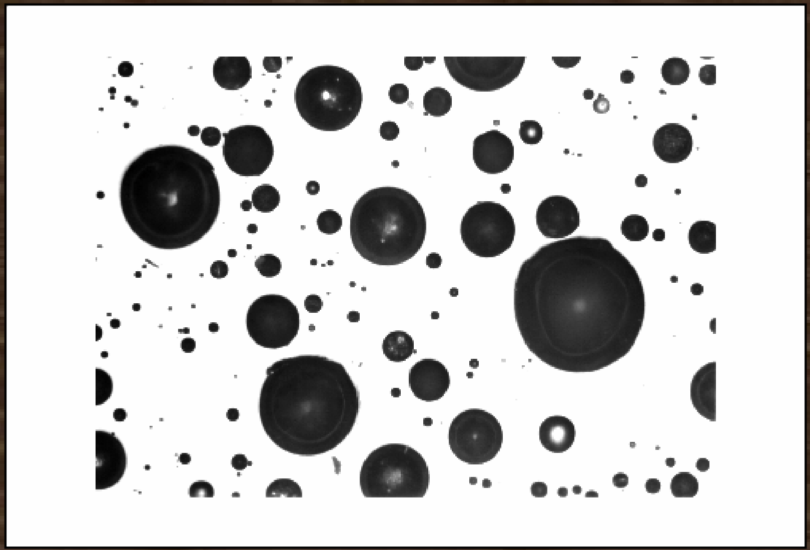
Some *images* and their *histograms*

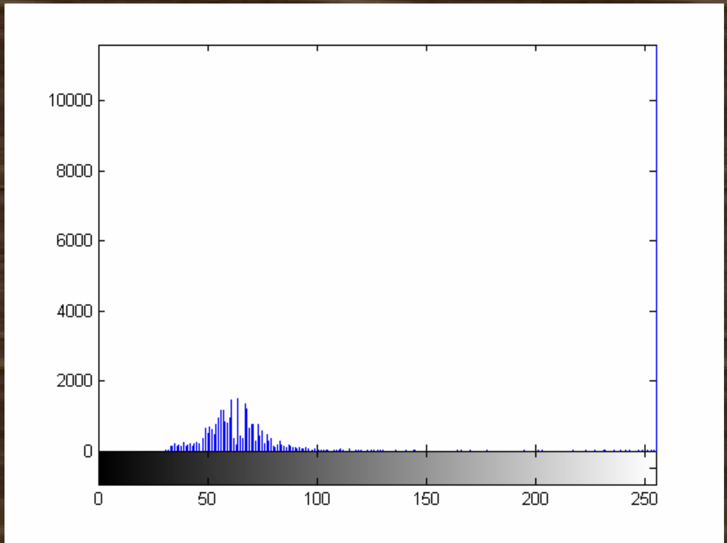
Image with
2 prominent
intensities





Foreground
white





Some images and their histograms

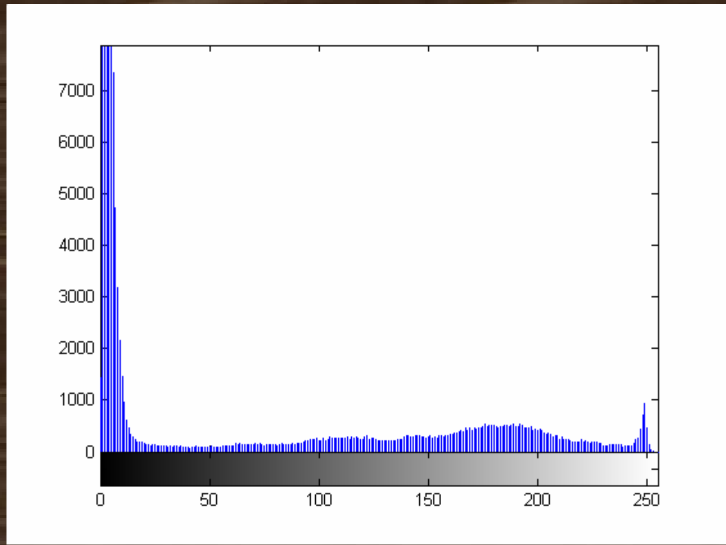


Image with more prominent dark background

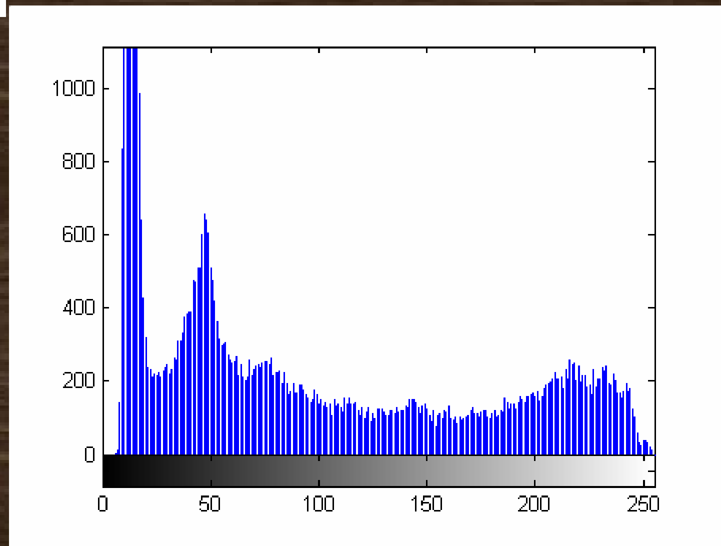


Image with more or less uniform coloring except the background

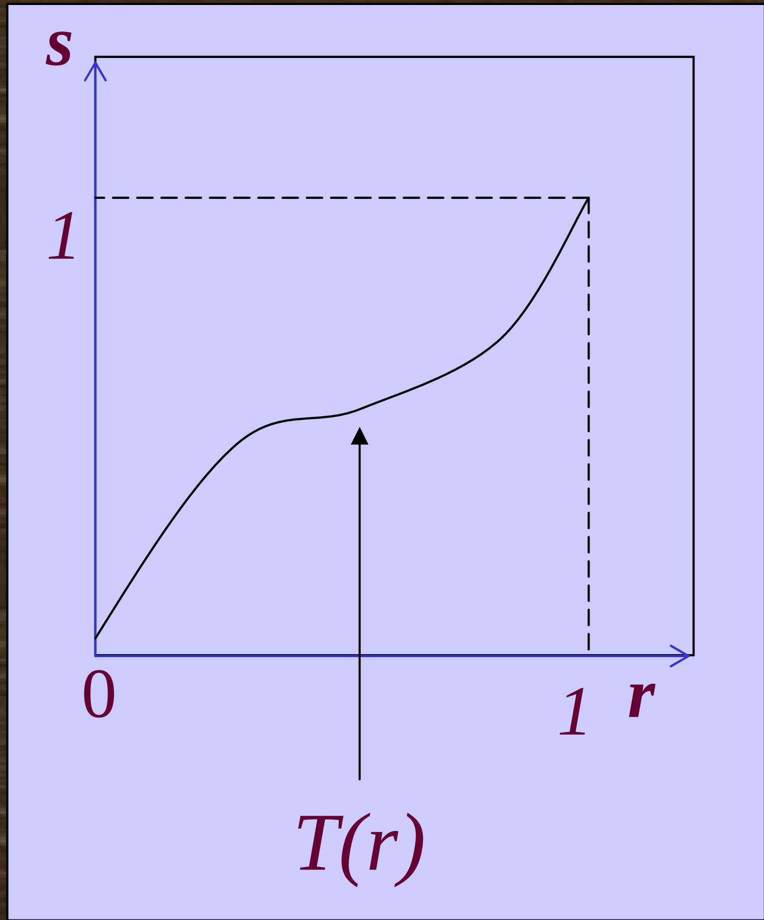
Use of histogram

- Histograms are
  - ✓ simple to calculate
  - ✓ Give information about the kind (global appearance) of image and its properties.
- Used for Image enhancement
- Used for Image compression
- Used for Image segmentation
- Can be used for real time processing

We shall now have a look at Histogram Equalization:
Here, the goal is to obtain a uniform histogram for the output image.

Some basics

- ***r*** represents the gray levels of image to be enhanced. It is normalized in the range [0,1]
  - ***r = 0*** represents black
  - ***r = 1*** represents white
- ***s = T(r)*** is transformation that produces a level s for every pixel r in the original image.



HISTOGRAM EQUALIZATION

Constraint on *T(r)*

- ***T*** should satisfies the following conditions
  - ***T(r)*** is single valued and monotonically increasing where r is in the range [0,1]
  - ***T(r)*** also varies in the range [0,1]

1. The first requirement is to ensure that T is invertible, and monotonicity ensures the order of increasing intensities (one to one)
2. The second requirement is to ensure that resulting gray levels are in the same range as input levels (onto)

- The inverse transformation from *s* back to *r* is denoted by $r = T^{-1}(s)$

HISTOGRAM EQUALIZATION (Continuous case)

- The gray levels in an image can be viewed as random variables in the interval [0, 1] and their pdf calculated
- If p_r and p_s are two different probability distributions on r and s (of input and transformed image) respectively, then the probability of a gray level value r in the range dr should be same in the transformed image at gray level s and range ds.

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

- So the pdf of s depends on pdf of r and the transformation function.

HISTOGRAM EQUALIZATION

Continuous case

- Consider the CDF to be the transformation function. i.e.

$$s = T(r) = \int_0^r p_r(w)\,dw$$

- This T(r) is single valued and monotonically increasing also the integration of a pdf is a pdf in the same range. So both constraints satisfied.

$$\frac{ds}{dr} = \frac{dT(r)}{dr} = \frac{d}{dr}\left[\int_0^r p_r(w)\,dw\right] = p_r(r)$$

(applying the leibniz rule)

- Substituting this into the first equation we get

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| = p_r(r)\left|\frac{1}{p_r(r)}\right| = 1, \quad 0 \le s \le 1$$

HISTOGRAM EQUALIZATION

Continuous case

- $p_s(s)$ is a pdf that is 0 outside the interval [0,1] and 1 in the interval [0,1] :- a uniform density.
- Thus the transformation $T(r)$ yields a random variable s characterized by a uniform pdf.
- $T(r)$ depends on $p_r(r)$ but always $p_s(s)$ is always a uniform pdf.
- In discrete case r takes discrete values

r_k , $k=0,1\ldots L\text{-}1$ and

probability of occurrence of a gray level r_k in an image is approximated by:

$$p_r(r_k) = \frac{n_k}{N} \quad \mathrm{k} = 0,1 \ldots \mathrm{L}-1$$

- **Mathematically, the discrete form of the transformation function for histogram equalization is given by**

$$s_k = T(r_k) = \sum_{j=0}^{k} n_j / N = \sum_{j=0}^{k} P_r(r_j)$$

$$0 \le r_k \le 1, \qquad k = 0,1,2,\ldots, L-1$$

Where

  - **n_j is the number of times the j^th gray level appears in the image**
  - **L is the number of gray levels**
  - **$P_r(r_j)$ is the probability of the j^{th} gray level and**
  - **N is the total number of pixels in the image**

- **Unlike the continuous part the discrete transformation may not produce the discrete equivalent of a uniform pdf. Nevertheless, it spreads the histogram to span a larger range.**

Discussion on Histogram equalization

- **This method is completely automatic.**
- **Inverse transformation $r_k = T^{-1}(s_k)$ satisfies both the conditions (onto and one to one) and hence can be used in the continuous case**
- **In case of discrete histogram, it is possible if the input image has all the gray levels.**

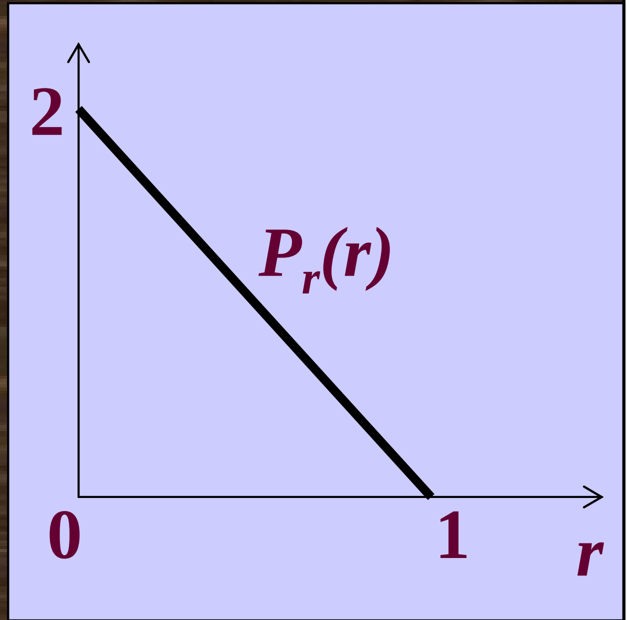
ALGORITHM

INPUT: ***Input image***
OUTPUT: ***Output image after equalization***

1. **Compute histogram** $h(x_i)$
2. **Calculate normalized sum of histogram (CDF)**
3. **Transform input image to output image**

Here follows an example on how to perform histogram equalization on an image. (Next page)

Example on histogram equalization (Continuous Case)



PDF of input image:-

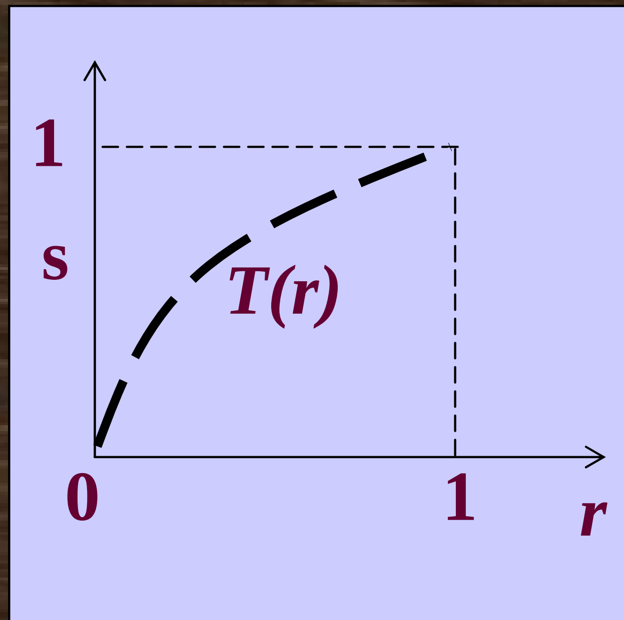
$$p_r(r) = -2r + 2 \;,\; 0 \le r \le 1$$

CDF is calculated as :-

$$s = T(r) = \int_0^r (-2w + 2)\,dw$$

$$= -r^2 + 2r$$

$$gives \quad r^2 - 2r + s = 0$$



Example on histogram equalization (Continuous Case)

$$r = \frac{2 \pm \sqrt{4 - 4s}}{2} = 1 \pm \sqrt{1 - s}$$

$$r = 1 - \sqrt{1 - s}\,, \quad \frac{dr}{ds} = \frac{1}{2\sqrt{1 - s}}$$

(Taking derivative with respect to *s*)

$$p_r(r) = -2(1 - \sqrt{1 - s}) + 2, \quad p_r(r) = 2\sqrt{1 - s}$$

$$p_s(s) = p_r(r)\frac{dr}{ds}, \quad p_s(s) = 1$$ **(Uniform pdf)**

Example on histogram equalization (Discrete case)

(a) r_k	(b) n_k	(c) $p_r(r_k)$
0	790	0.19
1/7	1023	0.25
2/7	850	0.21
3/7	656	0.16
4/7	329	0.08
5/7	245	0.06
6/7	122	0.03
1	81	0.02
total	4096	1.00

(d) Cdf = s_k	(e) Quant. Values
0.19	1/7
0.44	3/7
0.65	5/7
0.81	6/7
0.89	6/7
0.95	1
0.98	1
1.00	1

(a) Quantized Gray levels; (b) a sample histogram; (c) its pdf; (d) Computed CDF and (e) approximated to the nearest gray level.

Example on histogram equalization (Discrete case)(Contd..)

(a) r_k	(b) n_k	(c) $p_r(r_k)$
0	790	0.19
1/7	1023	0.25
2/7	850	0.21
3/7	656	0.16
4/7	329	0.08
5/7	245	0.06
6/7	122	0.03
1	81	0.02
total	4096	1.00

Original Histogram

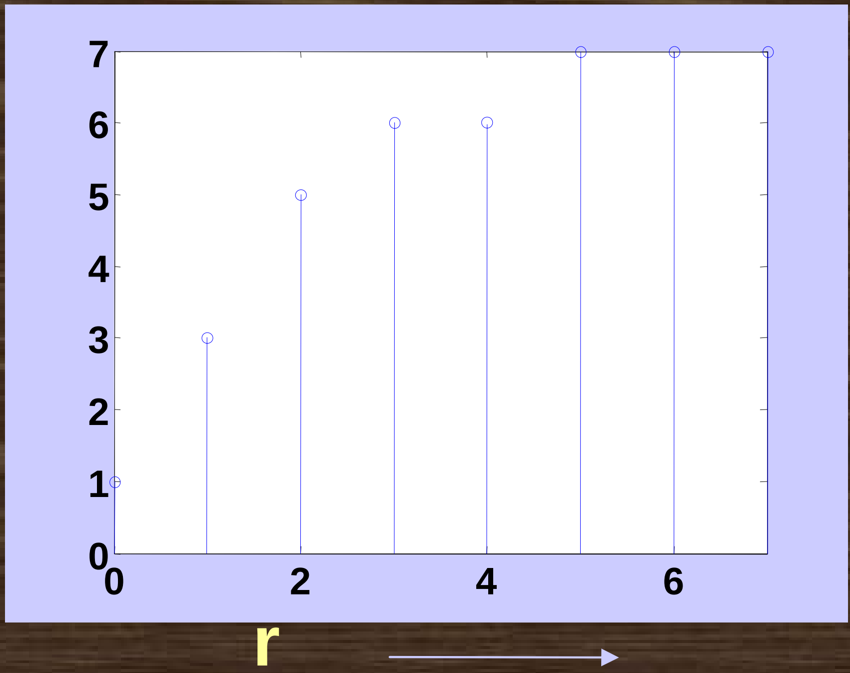
r_k	$s_k = T(r)$
0	1/7
1/7	3/7
2/7	5/7
3/7	6/7
4/7	6/7
5/7	7/7
6/7	7/7
1	7/7

Transformation function

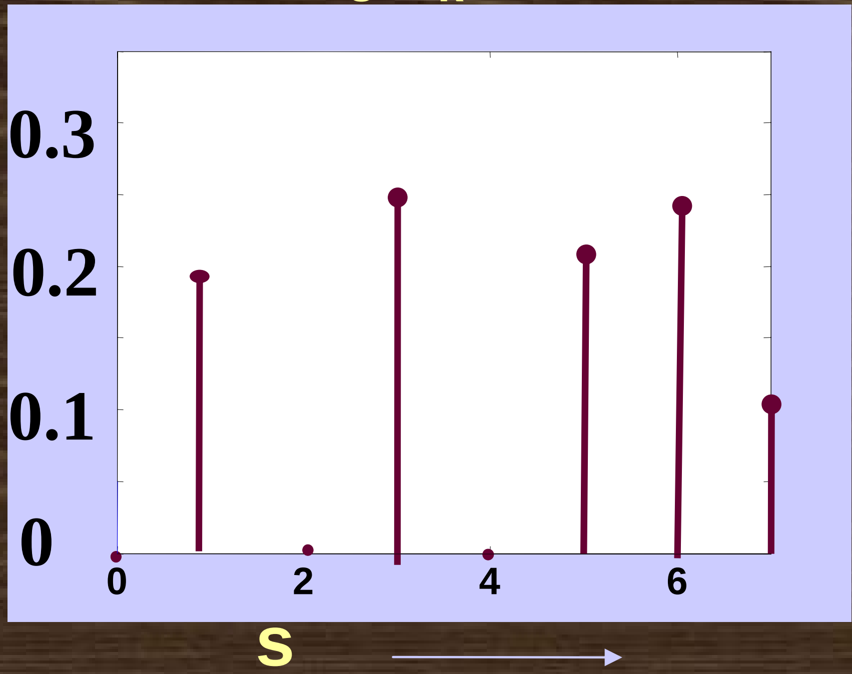
s_k	n_k	$p_s(s_k)$
0	0	0
1/7	790	0.19
2/7	0	0
3/7	1023	0.25
4/7	0	0
5/7	850	0.21
6/7	985	0.24
7/7=1	448	0.11

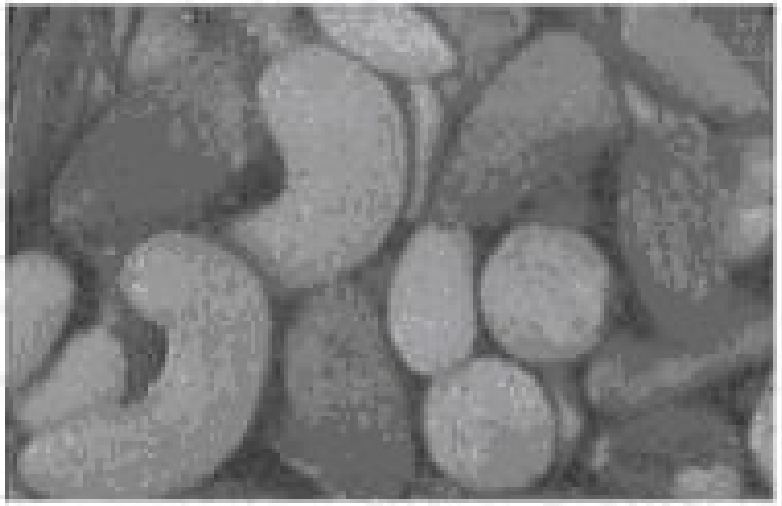
The new histogram

T(r)

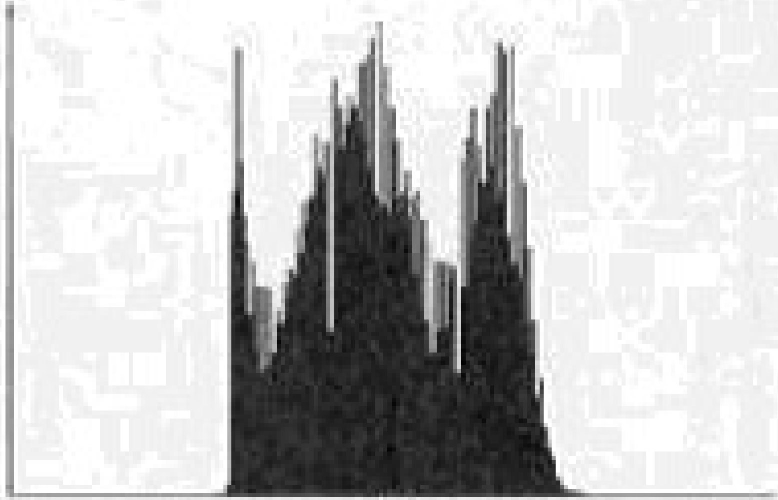


$p_s(s_k)$



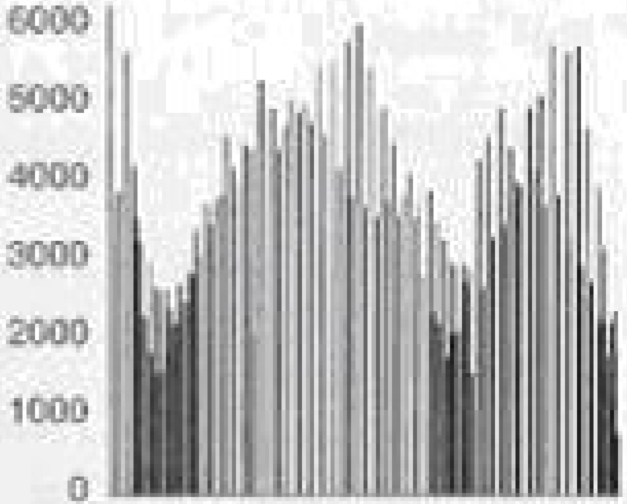


(a)

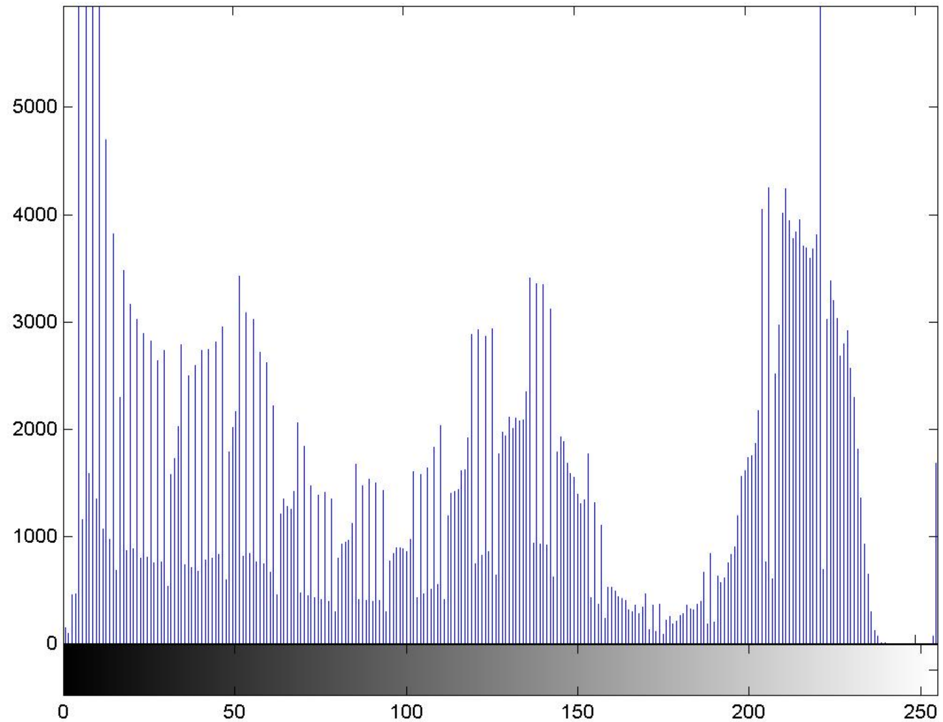


(b)

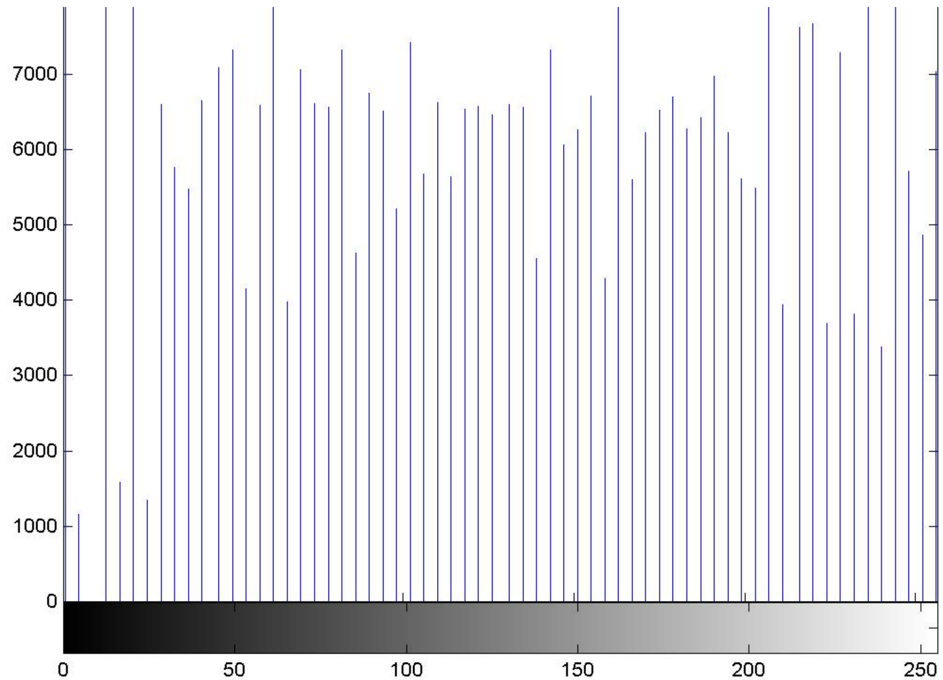
(c)



(d)



5000
4000
3000
2000
1000
0
0
50
100
150
200
250



7000
6000
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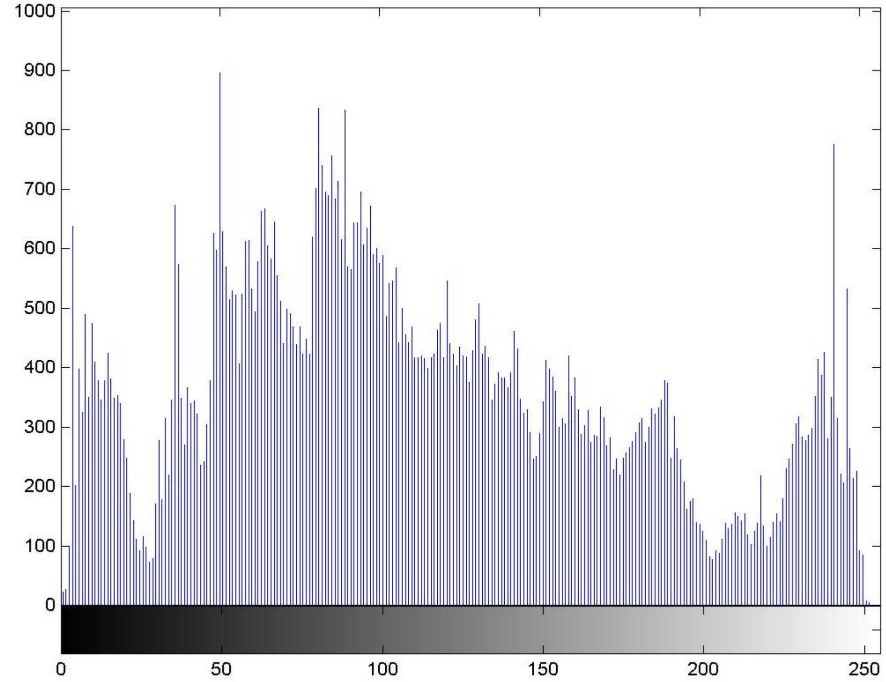
250


1000
900
800
700
600
500
400
300
200
100
0
0
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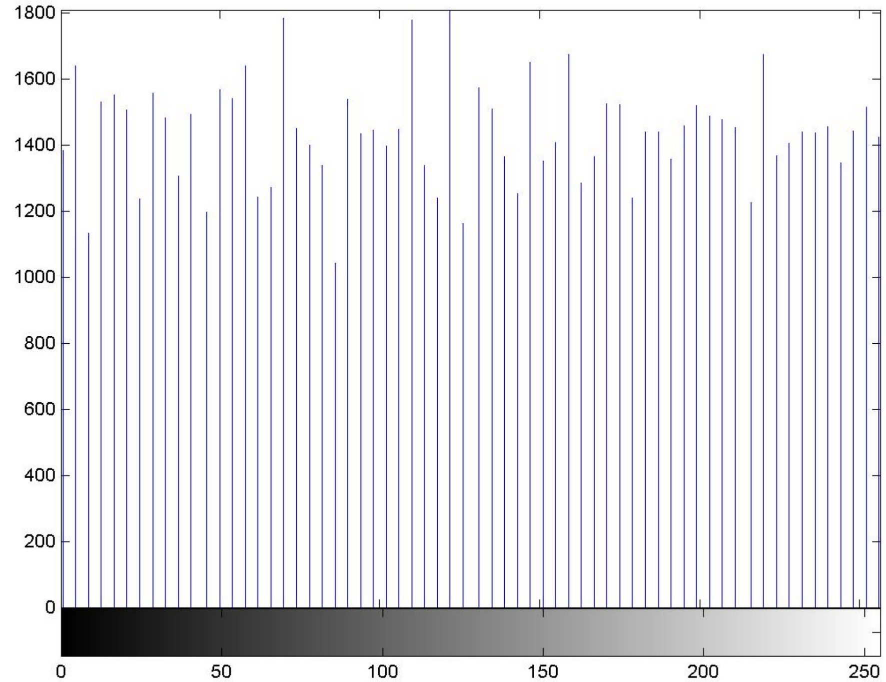
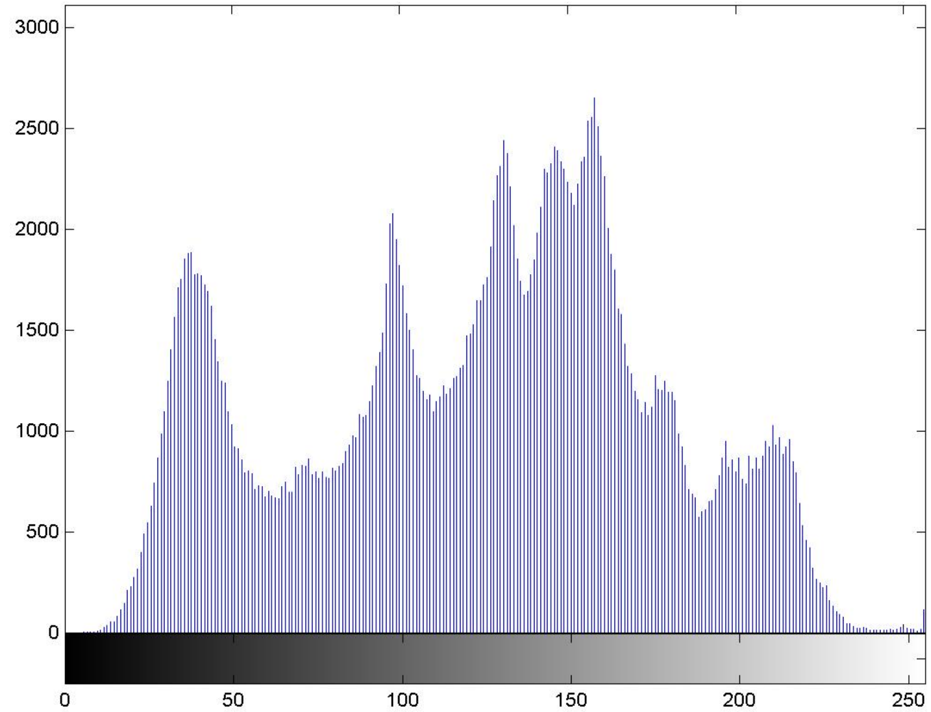
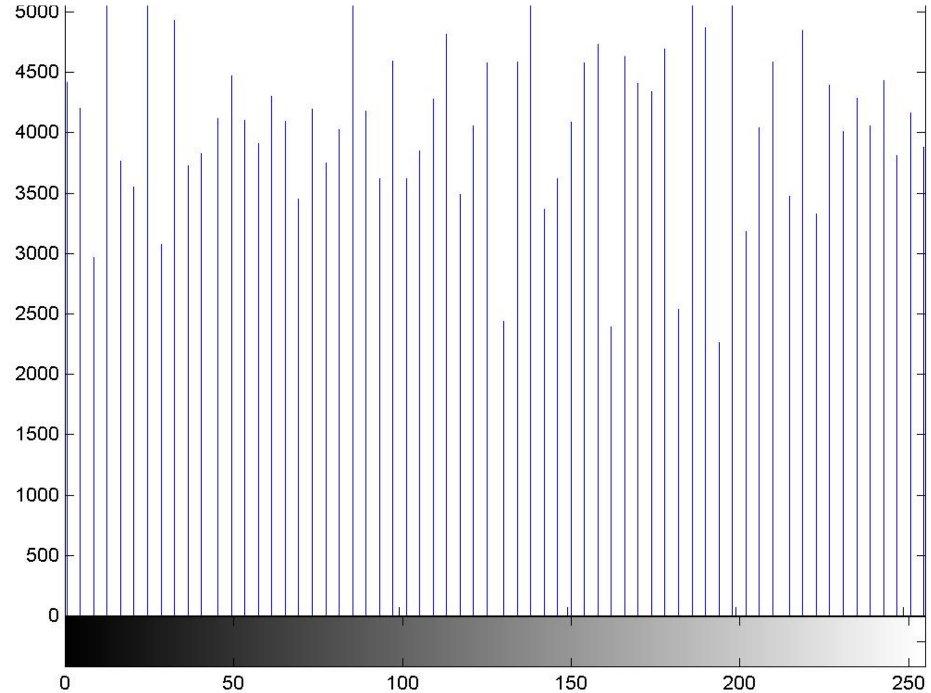
100
150
200
250


1800
1600
1400
1200
1000
800
600
400
200
0
0
50
100
150
200
250

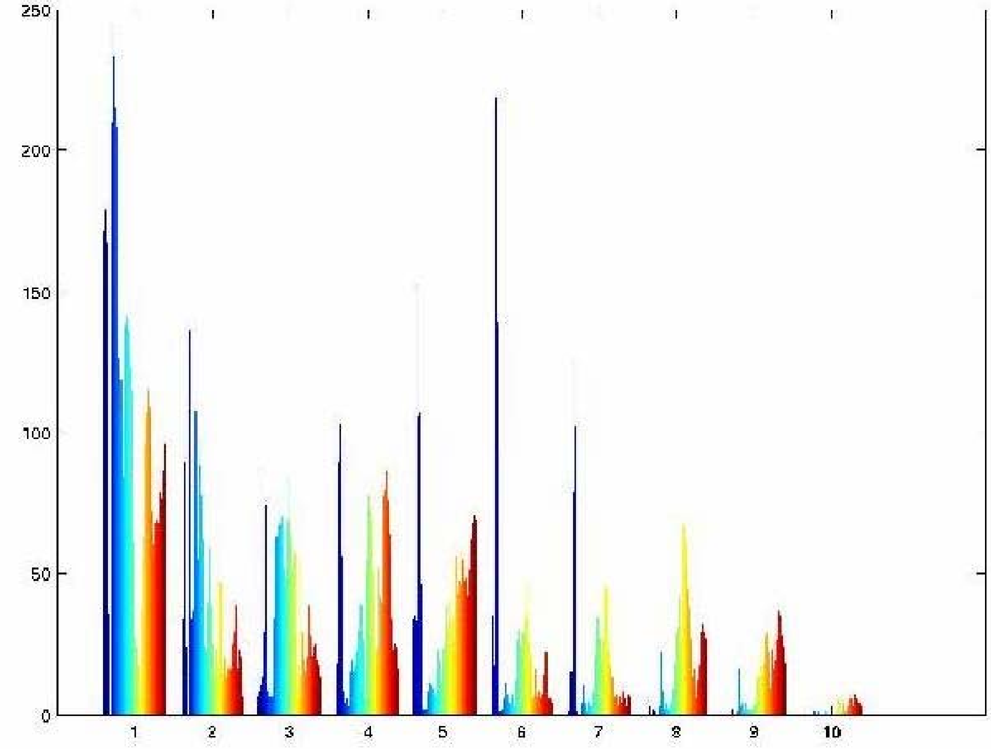

original image


3000
2500
2000
1500
1000
500
0
0
50
100
150
200
250

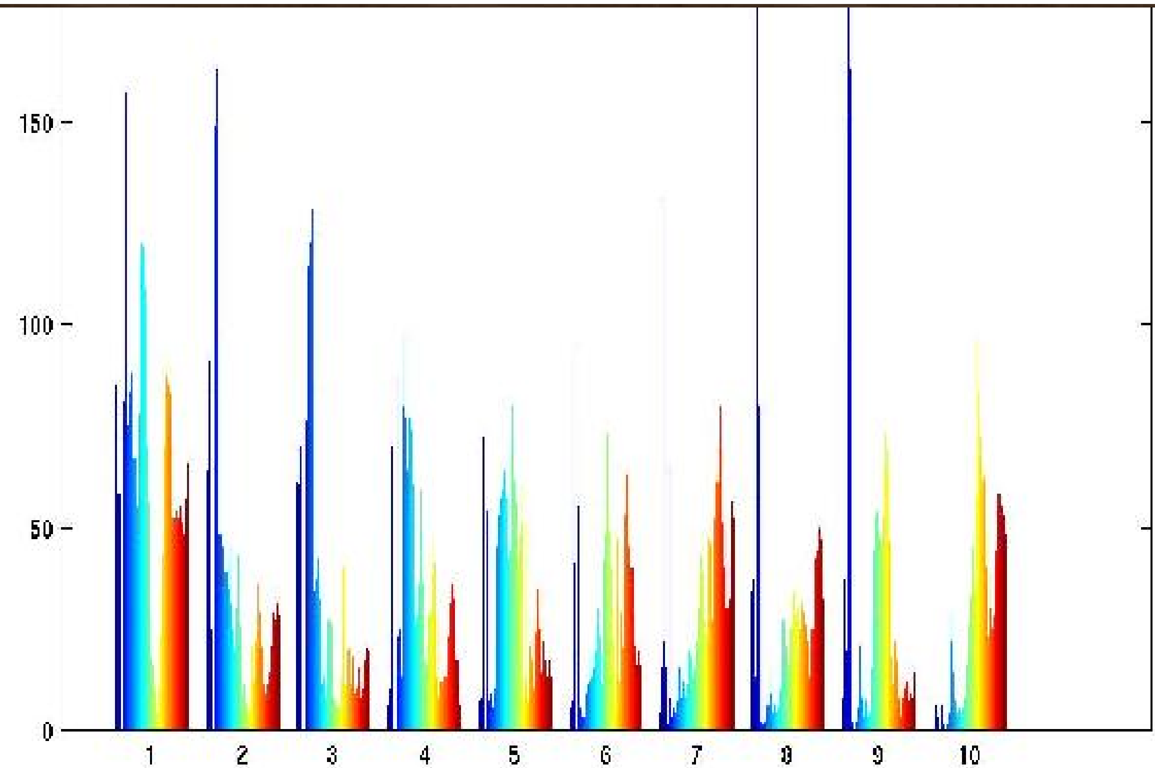

original image


5000
4500
4000
3500
3000
2500
2000
1500
1000
500
0
0
50
100
150
200
250


original image


250
200
150
100
50
0
1
2
3
4
5
6
7
8
9
10


original image


150
100
50
0
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6
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10

HISTOGRAM EQUALISATION - revisited

Here, the goal is to obtain a uniform histogram for the output image.

Some features of Histogram equalization are as follows:

- **Histogram equalization is a point process**

- **Histogram equalization causes a histogram with closely grouped values to spread out into a flat or equalized histogram.**

- **Spreading or flattening the histogram makes the dark pixels appear darker and the light pixels appear lighter.**

- **Histogram equalization does not operate on the histogram itself but uses the results of one histogram to transform the original image into an image that will have equalized histogram.**

- **Histogram equalization do not introduce new intensities in the image. Existing values will be mapped to new values keeping actual number of intensities in the resulting image equal or less than the original number of intensities.**

HISTOGRAM MODIFICATION

$$s = T(r); \qquad r = T^{-1}(s);$$

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| = \frac{p_r(r)}{ds/dr} = \frac{p_r(r)}{d(T(r))/dr}$$

$$p_s(s) = \frac{p_r(T^{-1}(s))}{T'(r)} = \frac{p_r(T^{-1}(s))}{T'(T^{-1}(s))}$$

Example:

Let, s = T(r) = ar + b; /* Linear case */

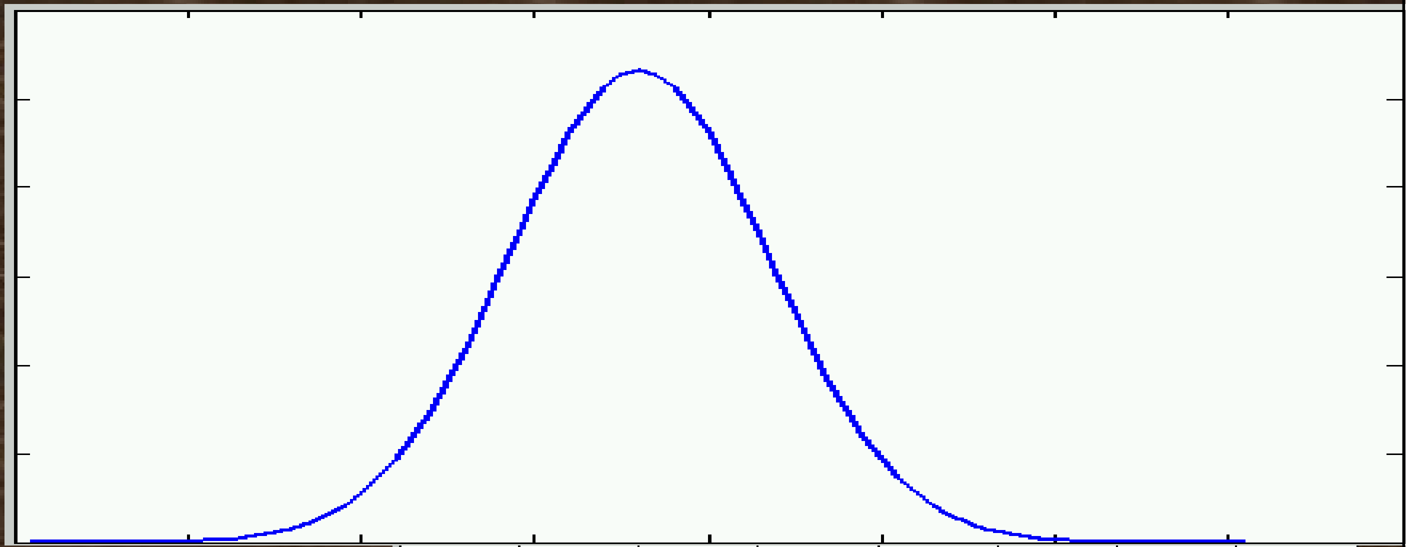
$$p_s(s) = a.p_r({s-b}/{a})$$

$$r = \frac{s-b}{a}\{= T^{-1}(s)\}$$

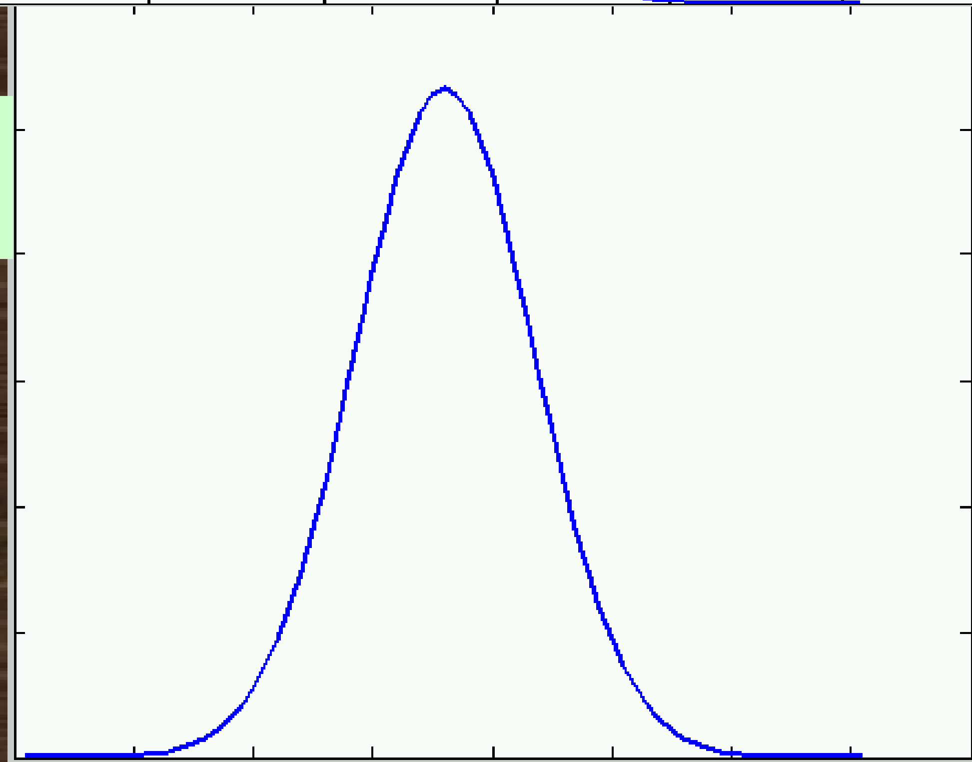
$$If, \quad p_r(r) = e^{-(r-c)^2} \quad /* Gaussian */$$

$$Then, \quad p_s(s) = ae^{-[s/a-(c+b/a)]^2} \quad /* Also\ a\ Gaussian */$$

$$If, \quad p_r(r) = e^{-(r-c)^2} \quad /* Gaussian */$$

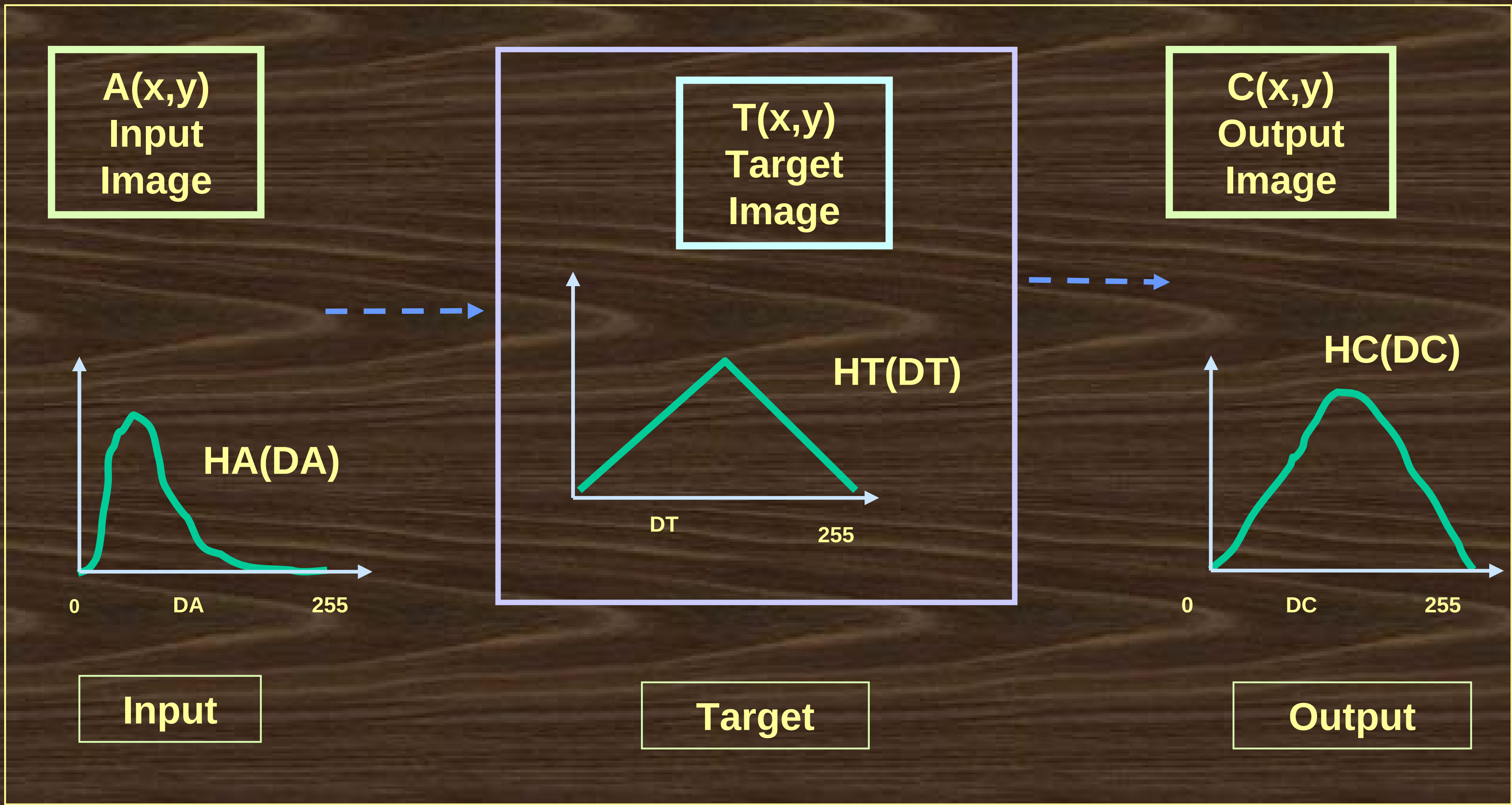


$$Then, \quad p_s(s) = a e^{-[s/a-(c+b/a)]^2}$$



Histogram specification

Histogram specification method develops a gray level transformation such that the histogram of the output image matches that of the pre-specified histogram of a target image. Figure below shows the flow diagram of histogram specification method



Development of the method (Continuous case)

- Continuous gray levels *r* and *z* of the input and the target image.
- Their corresponding pdfs are $p_r(r)$ and $p_z(z)$
- Let *s* be the random variable with the property

$$s = T(r) = \int_0^r p_r(w)\,dw$$

- Consider this

$$v = G(z) = \int_0^z p_z(t)\,dt = s$$

- Ideally we would want $G(z) = T(r)$
- So *z* must satisfy the condition

$$z = G^{-1}(s) = G^{-1}[T(r)]$$

Development of the method (Discrete case) (1)

- The discrete formulation of the method, histogram is first obtained for both the input and target image. The histogram is then equalized using the formula

$$s_k = T(r_k) = \sum_{j=0}^{k} n_j / n = \sum_{j=0}^{k} P_r(r_j)$$

- Where $0 \le r_k \le 1, \;\; k = 0,1,2,....,L-1$
  - –**n is the total number of pixels in the image**
  - –**n_j is the number of pixels with gray level r_j**
  - –**L is the number of gray levels**
- From the given $p_z(z_i)$ we can obtain

$$v_k = G(z_k) = \sum_{i=0}^{k} p_z(z_i) = s_k, \qquad k = 0,1....L-1$$

Development of the method (Discrete case) (2)

Hence z_k must satisfy the condition

$$z_k = G^{-1}(s_k) = G^{-1}\left[T(r_k)\right], \qquad k = 0,1...L-1$$

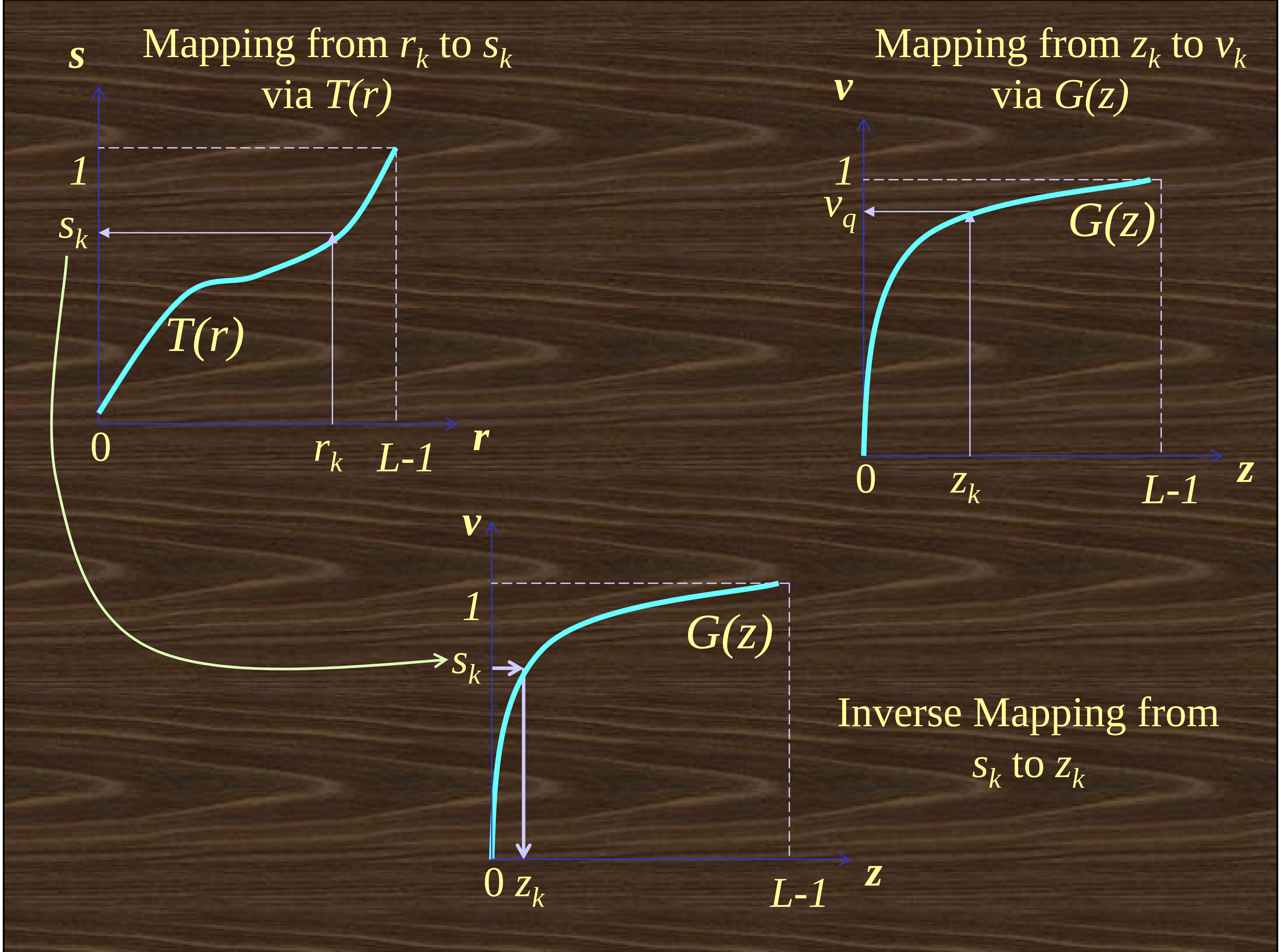
ALGORITHM

INPUT: Input image, Target image

OUTPUT: Output image that has the same characteristic as the target image

Steps:

- Read the Input image and the Target image.
- Obtain the histogram of the input image and the target image.
- Equalize the input and the target images using the equation (1).
- Calculate the transformation function G of the target Image.
- Map the original image gray level r_k to the final gray level z_k


Mapping from r_k to s_k via $T(r)$
s
1
s_k
$T(r)$
0
r_k
$L-1$
r
Mapping from z_k to v_k via $G(z)$
v
1
v_q
$G(z)$
0
z_k
$L-1$
z
v
1
$G(z)$
s_k
0
z_k
$L-1$
z
Inverse Mapping from s_k to z_k

A hand worked example (1)

- Two histograms are given to us

r_k	n_k	p_k
0/7	790	0.19
1/7	1023	0.25
2/7	850	0.21
3/7	656	0.16
4/7	329	0.08
5/7	245	0.06
6/7	122	0.03
7/7	81	0.02

Input histogram

z_k	$p(z_k)$
0/7	0
1/7	0
2/7	0
3/7	0.15
4/7	0.2
5/7	0.3
6/7	0.2
7/7	0.15

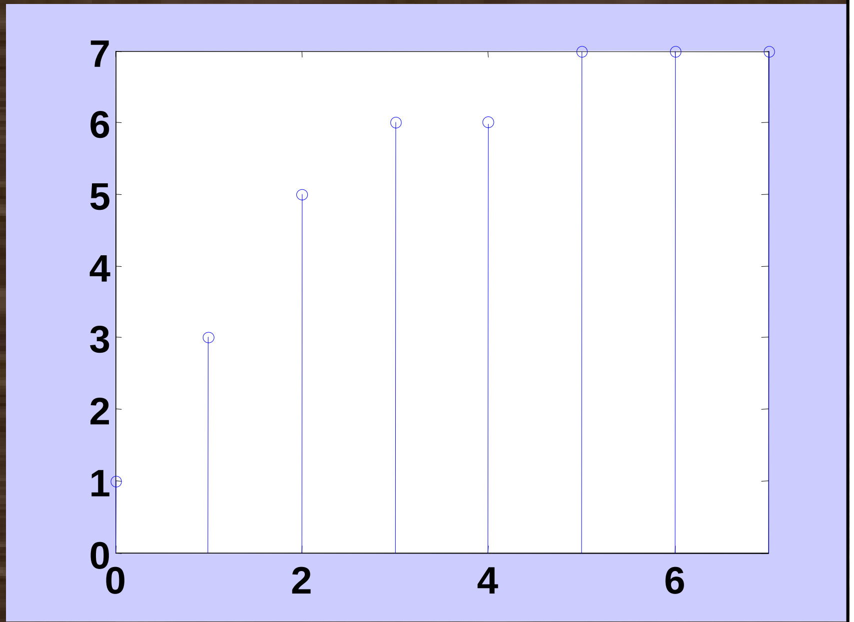
Target histogram

A hand worked example (2)

- **Equalizing both the histograms**
  - **The first histogram**

r_k	p_k	$cdf(p_k)$	Gray levels (s_k)
0/7	0.19	0.19	1/7
1/7	0.25	0.44	3/7
2/7	0.21	0.65	5/7
3/7	0.16	0.81	6/7
4/7	0.08	0.89	6/7
5/7	0.06	0.95	1
6/7	0.03	0.98	1
7/7	0.02	1	1

T(r)

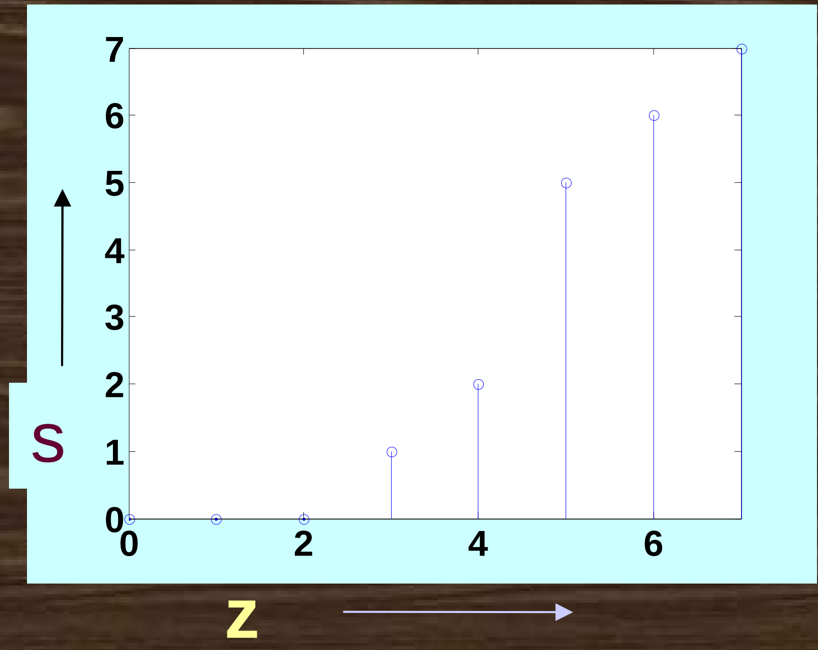


r

A hand-worked example (3)

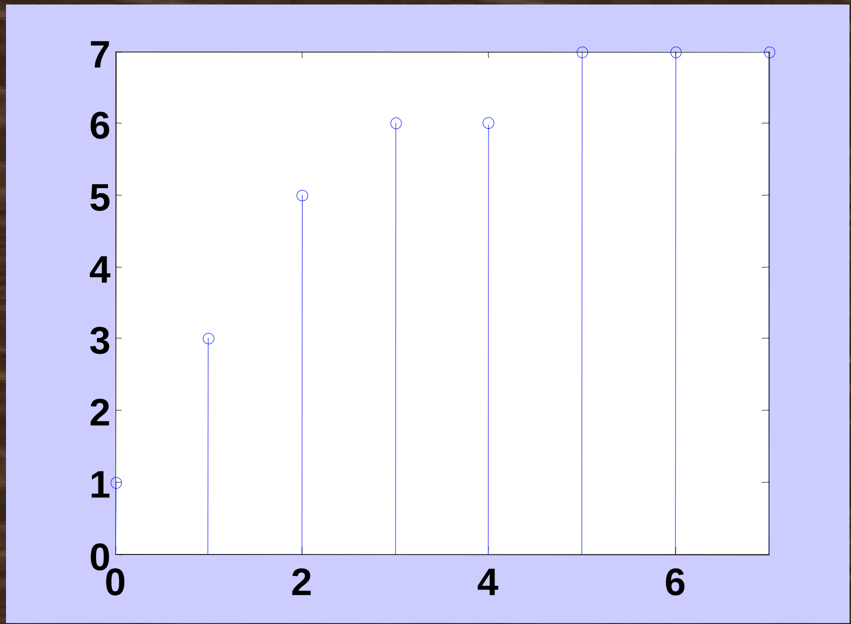
- **The second histogram**

z_k	$p(z_k)$	$cdf(z_K)$	Gray levels (v_k)
0/7	0	0	0/7
1/7	0	0	0/7
2/7	0	0	0/7
3/7	0.15	0.15	1/7
4/7	0.2	0.35	2/7
5/7	0.3	0.65	5/7
6/7	0.2	0.85	6/7
7/7	0.15	1	1



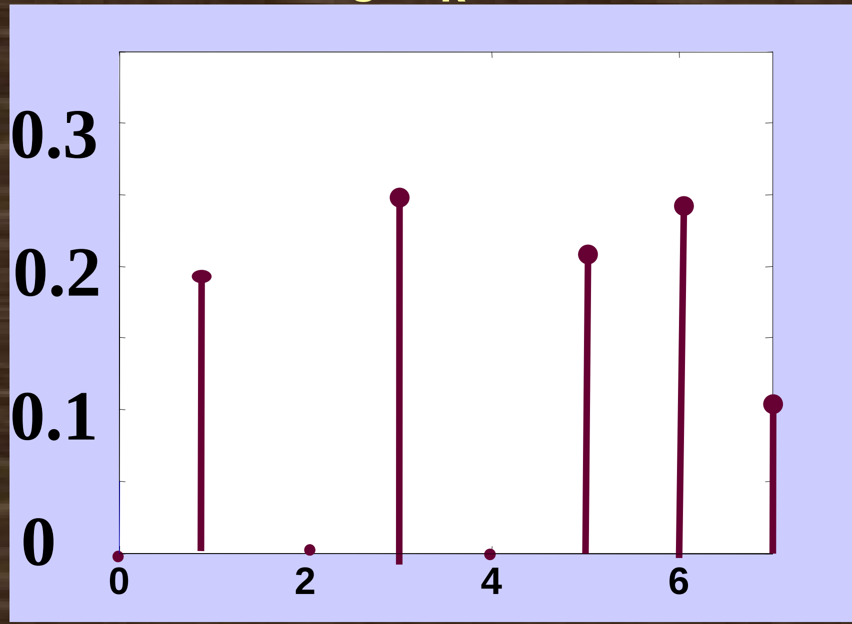
G(z)

T(r)



r

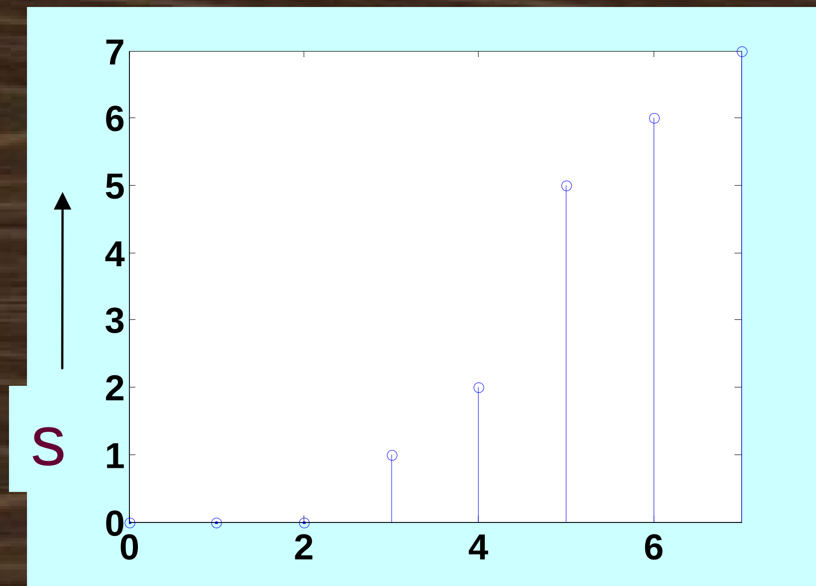
$p_s(s_k)$



s

p(z)

z



z

G(z)

A hand-worked example (4)

Mapping stage

r_k	s_k = cdf(p_k)
0/7	0.19
1/7	0.44
2/7	0.65
3/7	0.81
4/7	0.89
5/7	0.95
6/7	0.98
7/7	1

Equalized input histogram

$G(z_K)$	z_k
0	0/7
0	1/7
0	2/7
0.15	3/7
0.35	4/7
0.65	5/7
0.85	6/7
1	7/7

Equalized target histogram

Mapping function

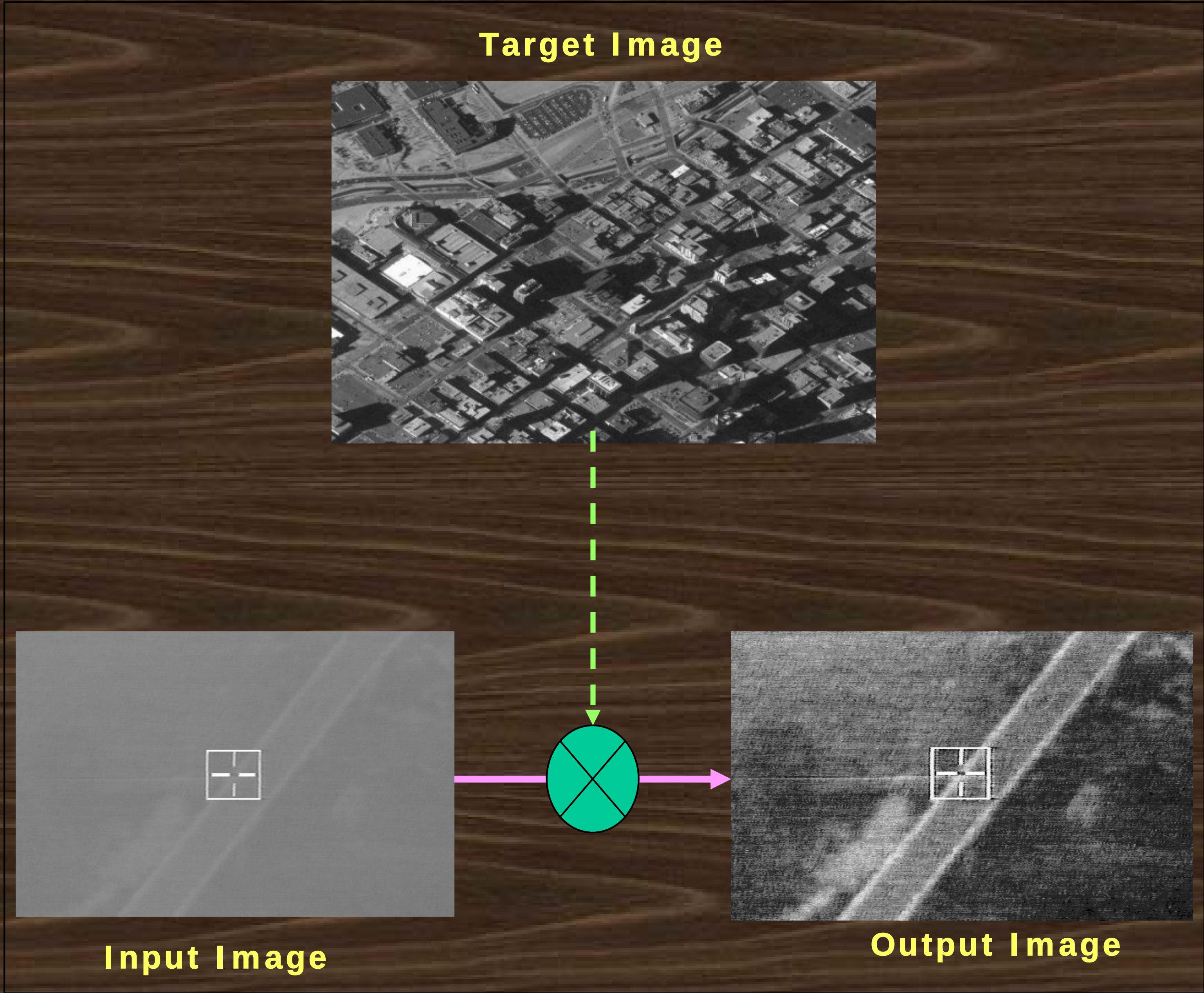
0 → 3

1 → 4

2 → 5

3,4 → 6

5,6,7 → 7


Target Image
Input Image
Output Image

Source Image

Direct Histogram Specification

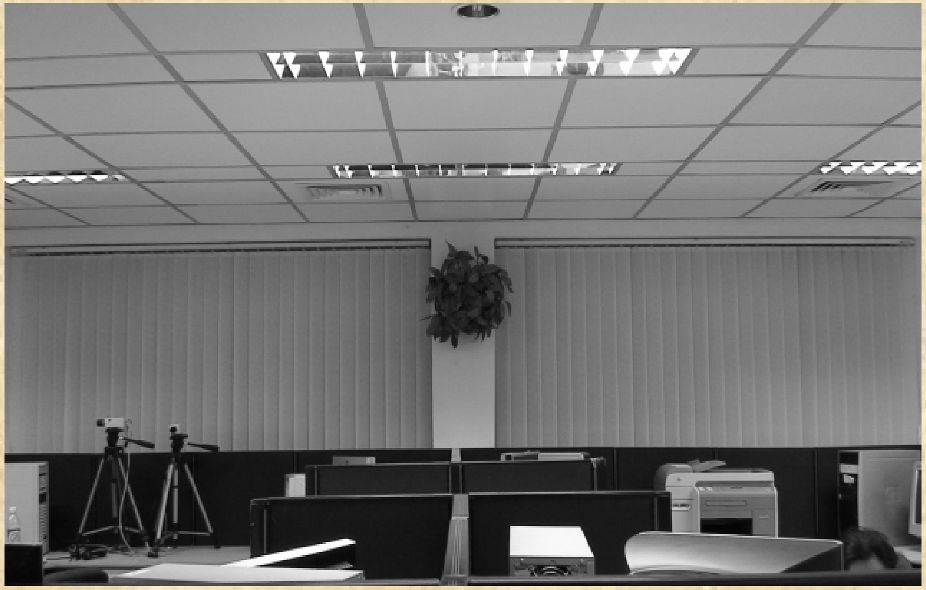
Target Image

DHS Output Image

Source Image

Direct Histogram Specification

Target Image

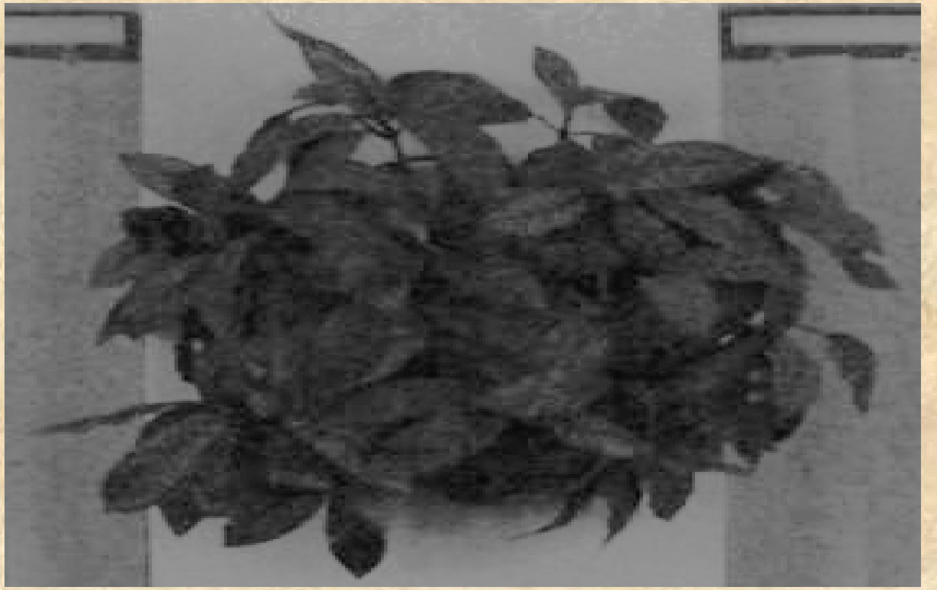


DHS Output Image

Source Image

Direct Histogram Specification

DHS Output Image



Target Image

End of Lectures - Histogram Processing