Wavelet Transform

Wavelet Transform

The wavelet transform corresponds to the decomposition of a quadratic integrable function s(x) ε $L^2(R)$ in a family of scaled and translated functions $\Psi_{k,l}(t)$,

$$\psi_{k,l}(t) = k^{-1/2} \psi\left(\frac{t-l}{k}\right)$$

The function Ψ(x) is called wavelet function and shows band-pass behavior. The wavelet coefficients $d_{a,b}$ are derived as follows:

$$d_{k,l} = \frac{1}{\sqrt{k}} \int_{-\infty}^{\infty} s(x) \Psi^{*}\left(\frac{x-l}{k}\right) dx$$

where k ε R^+, l ε R

and * denotes the complex conjugate function

The discrete wavelet transform (DWT) represents a 1-D signal s(t) in terms of shifted versions of a lowpass scaling function φ(t) and shifted and dilated versions of a prototype bandpass wavelet function ψ(t).

For special choices of $\phi(t)$ and $\psi(t)$, the functions:

$$\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k),$$

$$\phi_{j,k}(t) = 2^{-j}\phi(2^{-j}t - k)$$

for j and k ε Z, form an orthonormal basis, and we have the representation:

$$z(t) = \sum_{k} u_{j0,k}\phi_{j0,k}(t) + \sum_{j=j_0}^{\infty}\sum_{k}\omega_{j,k}\psi_{j,k}(t)$$

where,

$$u_{j,k} = \int s(t)\phi^{*}_{j,k}(t)\,dt \quad \textbf{and} \quad \omega_{j,k} = \int s(t)\psi^{*}_{j,k}(t)\,dt$$

Relook at F.T. expressions:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi xu} dx \qquad f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi xu} du$$

Thus f(x) is represented here as a linear combination of the basis functions: exp(jωx)

Wavelet transform on the other hand, represents f(x) (or f(t)) as a linear combination of:

$$\psi_{kl}(t) = 2^{-k/2} \psi(2^{-k} t - l)$$

where ψ(t) is called the mother wavelet.

Parameters k and l are integers – which generates the basis functions as the dilated and shifted variations of the mother wavelet.

The parameter k plays the role of frequency and l plays the role of time. Hence by varying k and l, we have different frequency and different time or space – hence the term ***multi-channel multi-resolution approach***.

Compare in discrete case:

The F.T.:

$$f(x) = \frac{1}{N}\sum_{u=0}^{N-1} F(u)e^{j2\pi ux/N};\ x = 0,1,....,(N-1)$$

The DWT:

$$f(t) = \sum_{k}\sum_{l} X_{DWT}(k,l)[2^{-k/2}\psi(2^{-k}t-l)]$$

where,

$$X_{CWT}(k,l) = \frac{1}{\sqrt{k}}\int_{-\infty}^{\infty} x(t)\psi(\frac{t-l}{k})dt$$

DWT – Discrete Wavelet Transform:

Forward:

$$X_{DWT}(k,l) = a^{-k/2} \int_{-\infty}^{\infty} x(t) h(a^{-k} t - lT) dt$$

and Inverse:

$$x(t) = \sum_{k} \sum_{l} X_{DWT}(k,l) [a^{-k/2} f(a^{-k} t - lT)]$$

Take, T = 1 and time is continuous.

Analysis filters:

$$h_k(t) = a^{-k/2} h(a^{-k} t)$$

Synthesis filters:

$$f_k(t) = a^{-k/2} f(a^{-k} t)$$

Functions h(t) and f(t) are derived by dilation of a single filter. Thus the basis functions are dilated ($t \rightarrow a^{-k}t$) and shifted ($t \rightarrow t - la^{-k}t$) versions of:

$$f(t) = \psi_{kl}(t) = a^{-k/2} \psi(a^{-k} t - lT)$$

Synthesis filters for perfect reconstruction: $f_k(t) = h_k^*(-t)$

Visualize pseudo-frequency corresponding to a scale. Assume a center frequency F_c of the wavelet and use the following relationship:

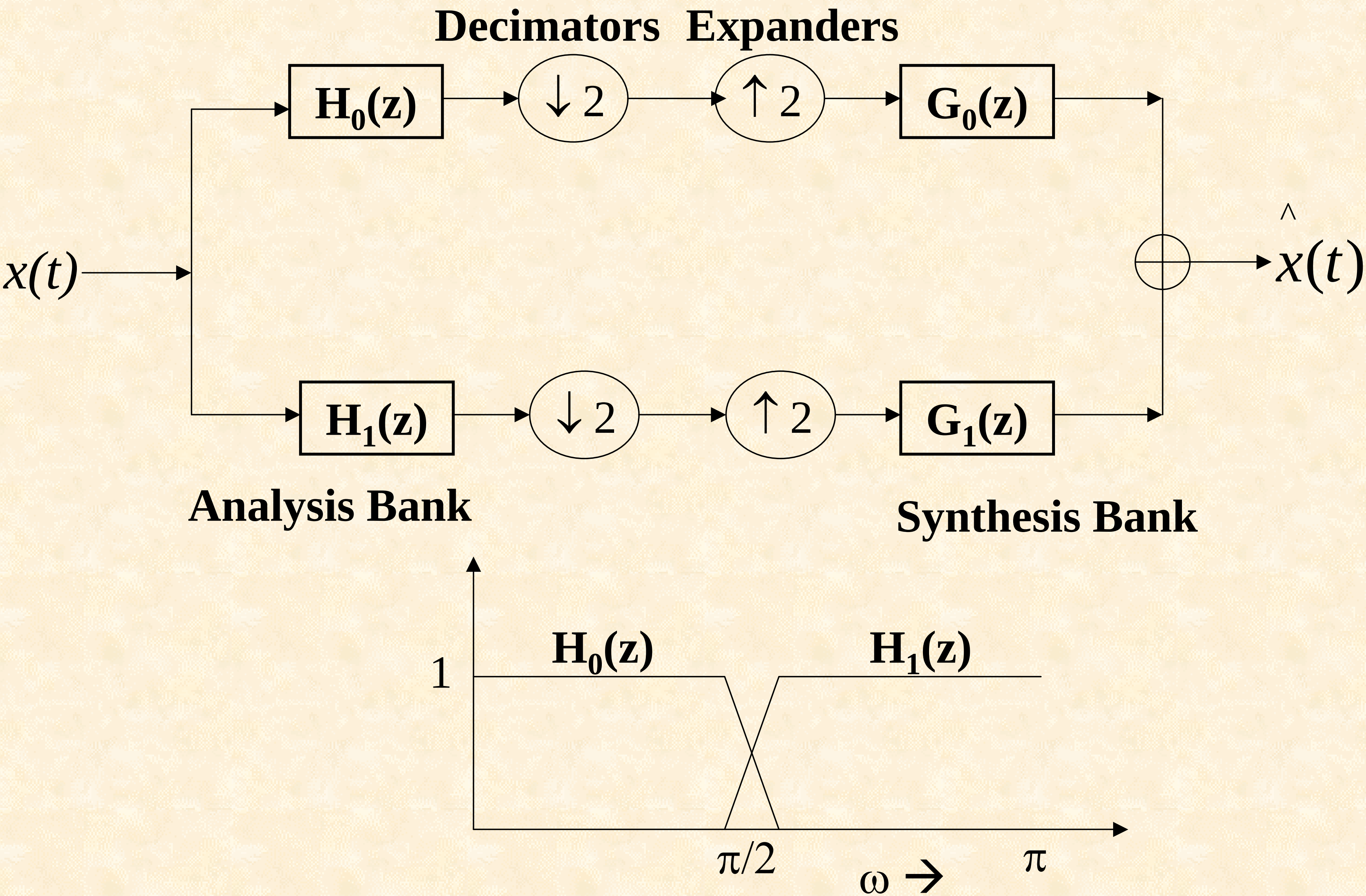
$$F_a = \frac{F_c}{a.\Delta}$$

where a is the scale. Δ is the sampling period and F_c is the center frequency of a wavelet in Hz. F_a is the pseudo-frequency corresponding to the scale a, in Hz.

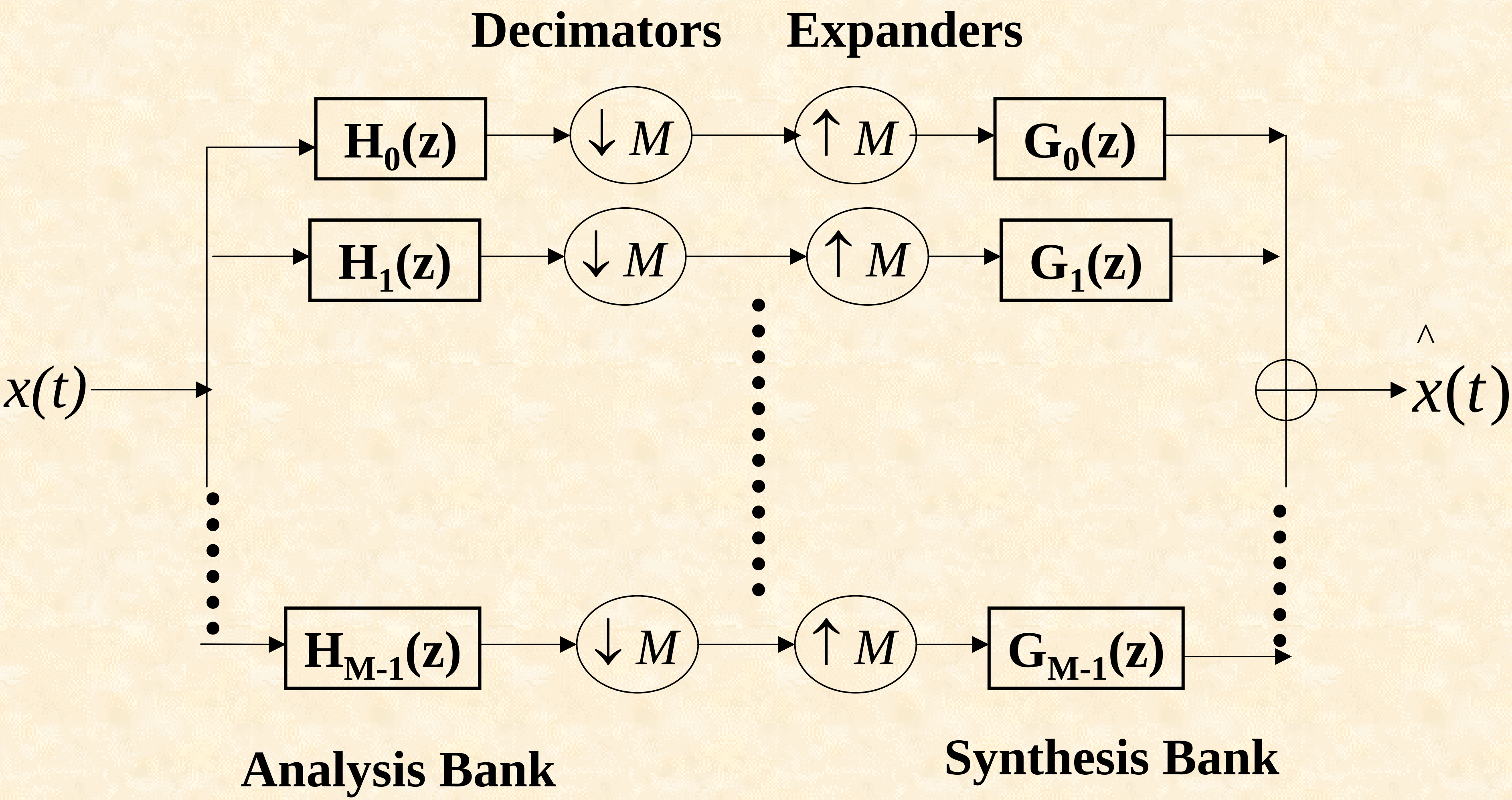
The highpass and lowpass filters are not independent of each other, and they are related by the following expression:

$$g[L-1-N] = (-1)^n . h(n)$$

QMF bank and typical magnitude responses



M-channel (M-band) QMF bank



•The DWT analyzes the signal at different frequency bands with different resolutions by decomposing the signal into ***coarse approximation and detail information***.

•DWT employs two sets of functions, called scaling functions and wavelet functions, which are associated with **low pass and highpass filters**, respectively.

•The decomposition of the signal into **different frequency bands** is simply obtained by **successive highpass and lowpass filtering** of the time domain signal.

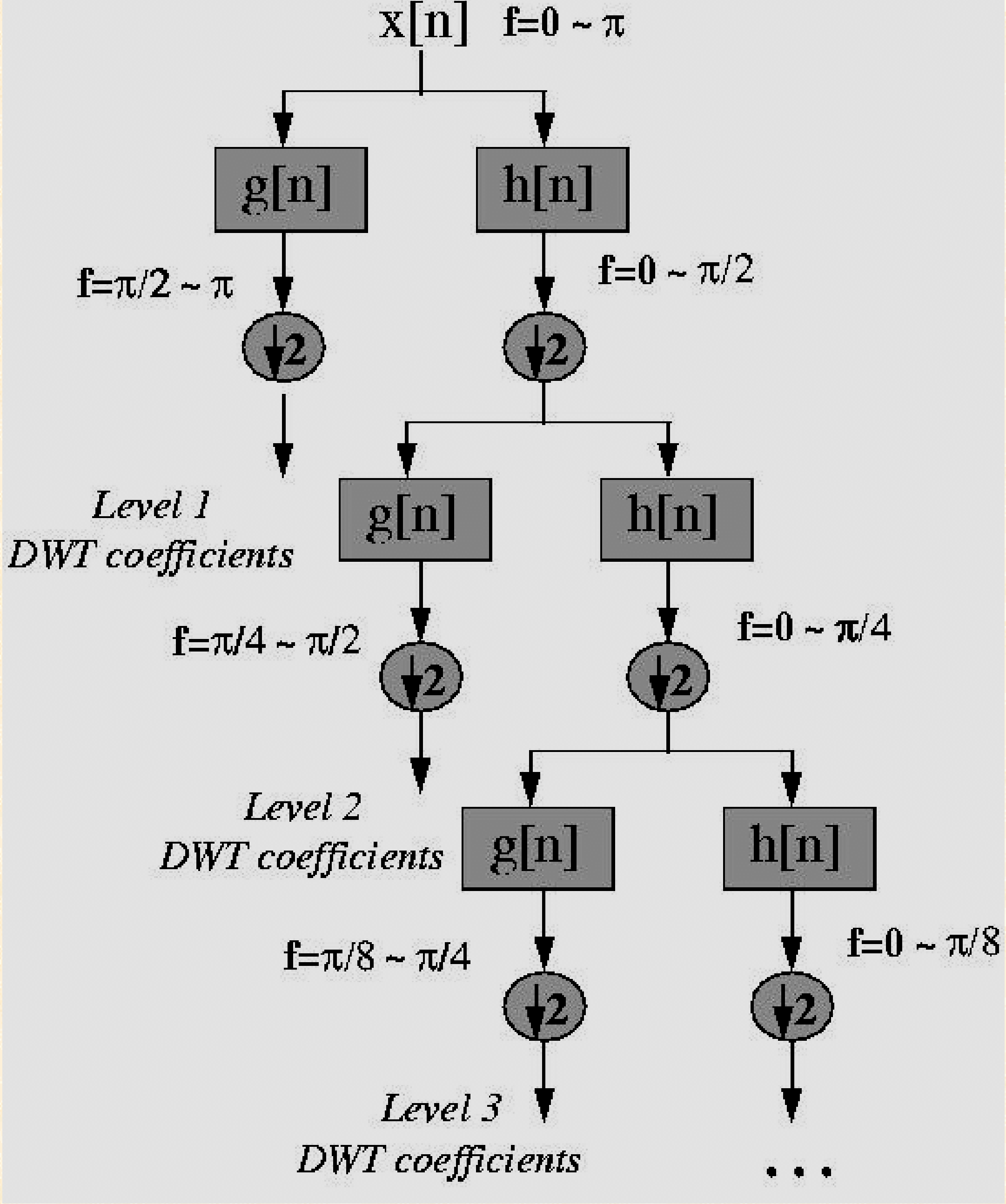
•The original signal x[n] is first passed through a **half-band highpass filter g[n] and a lowpass filter h[n]**.

•After the filtering, **half of the samples can be eliminated** (according to the Nyquist's rule) since the signal now has a highest frequency of $f_{max}/2$ radians instead of f_{max}.

•The **signal can therefore be sub-sampled by 2**, simply by discarding every other sample. This constitutes one level of decomposition and can mathematically be expressed as follows:

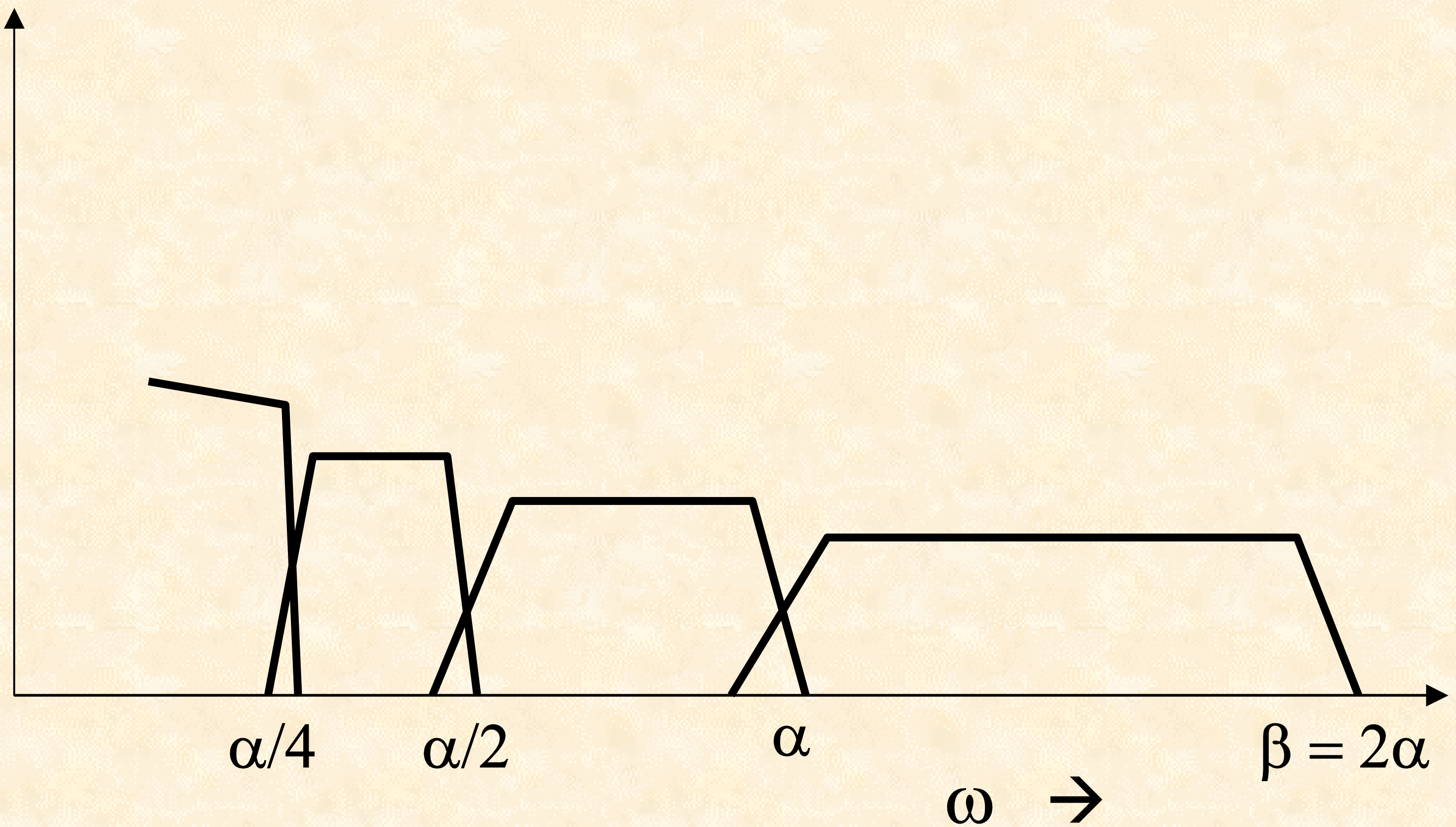
$$y_{hi}[k] = \sum x[n].g[2k-n]$$

$$y_{lo}[k] = \sum x[n].h[2k-n]$$



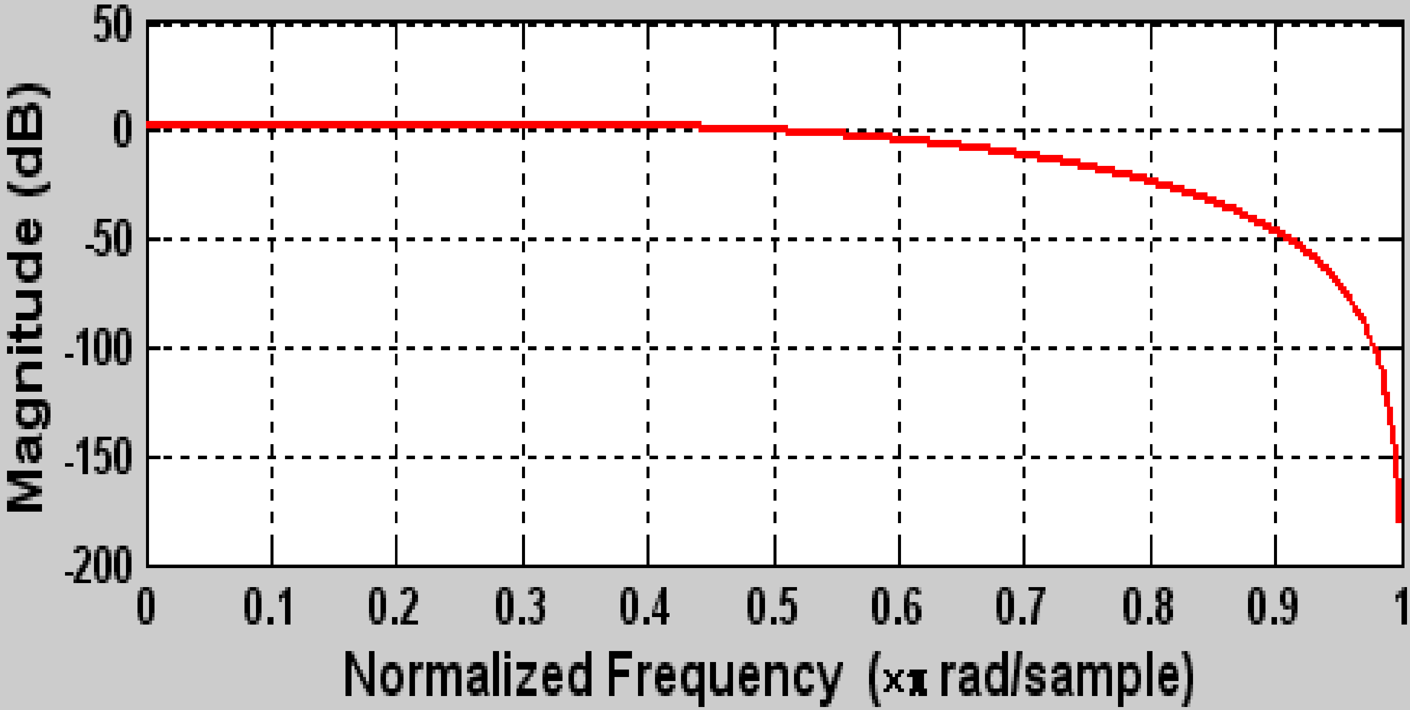
Block diagram of the methodology of 1-D DWT.

Frequency responses (bandwidths) of the different output channels of the wavelet filter bank, for a = 2 and three or more levels of decomposition

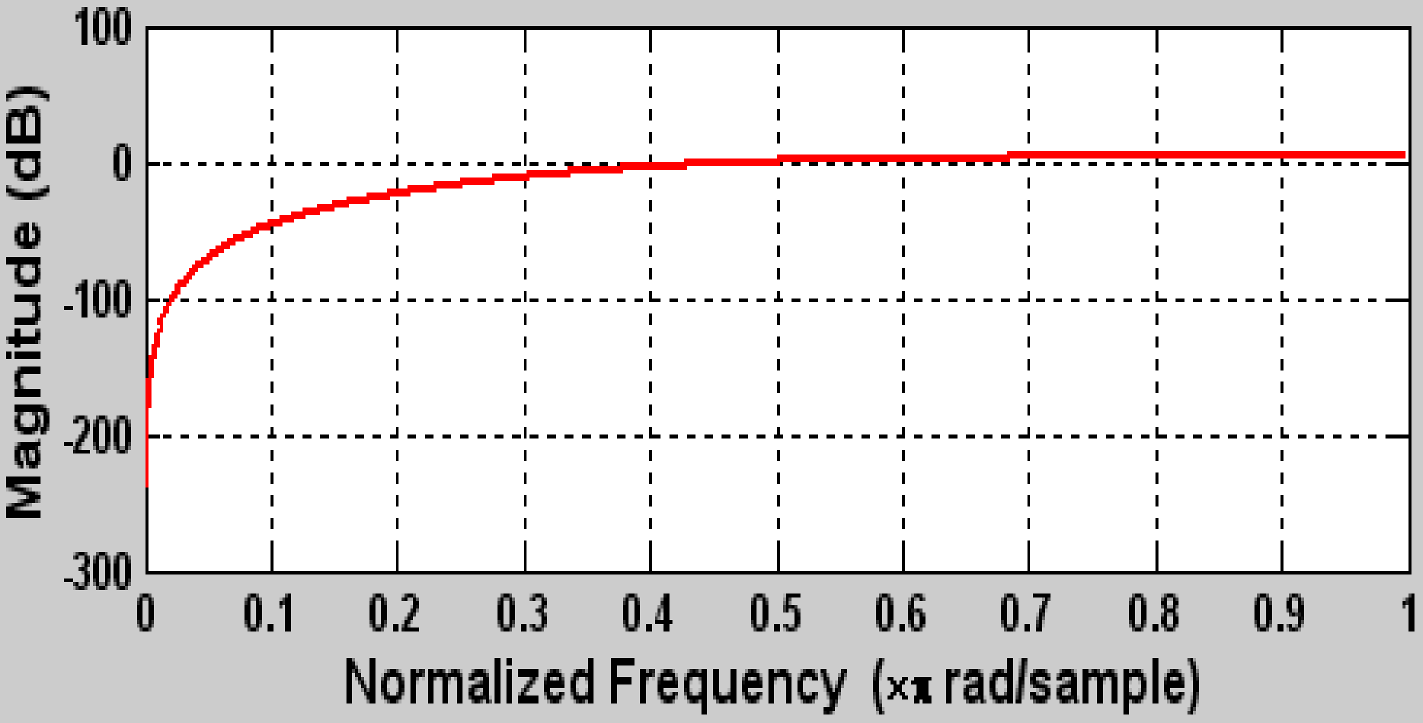


Frequency Response of 2-channel Daubeschies 8-tap orthogonal wavelet filters.

Low-Pass

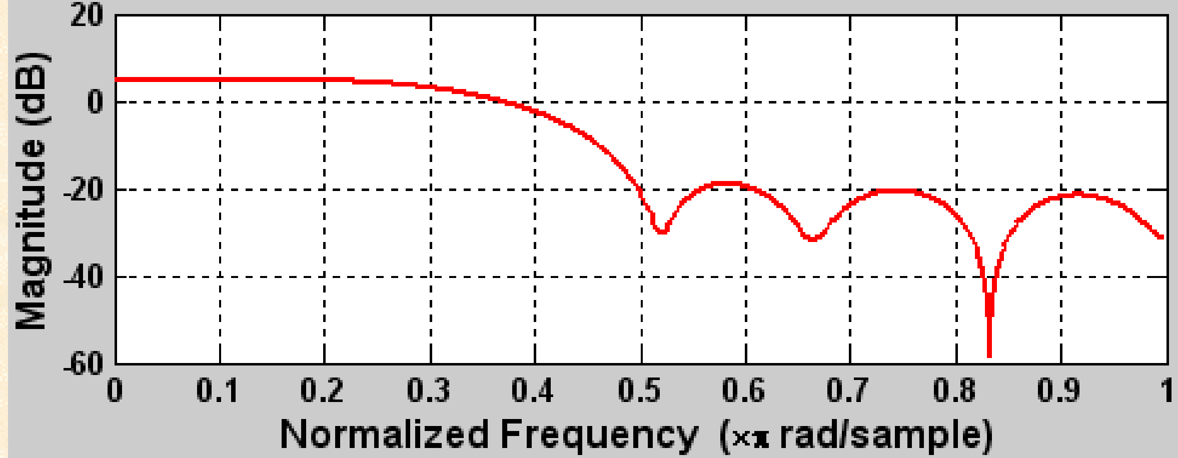


High-Pass

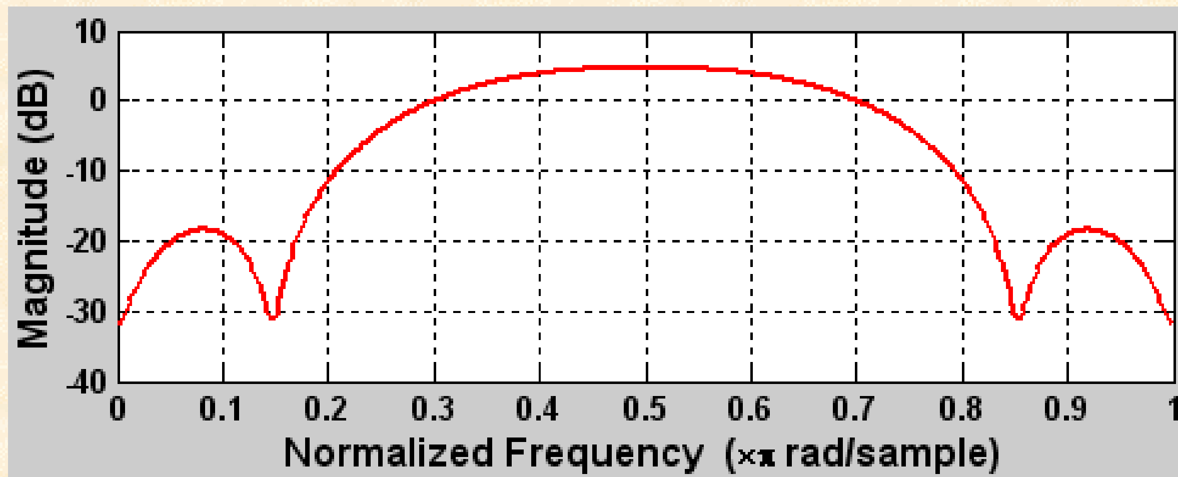


Frequency Response of a 3-channel orthogonal wavelet filters.

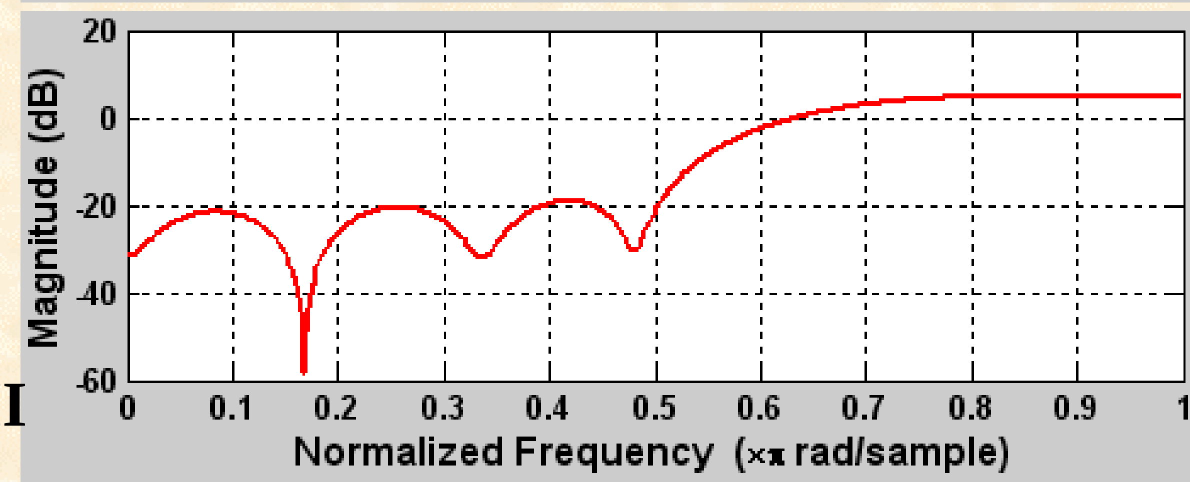
Channel - I



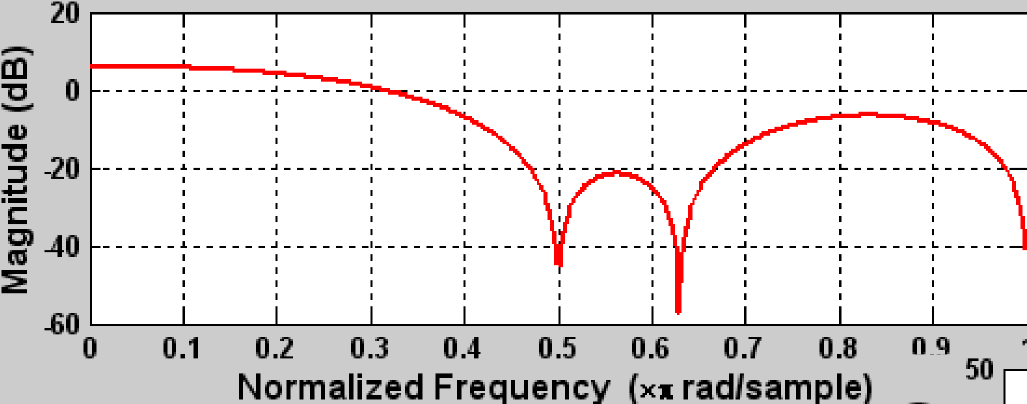
Channel - II



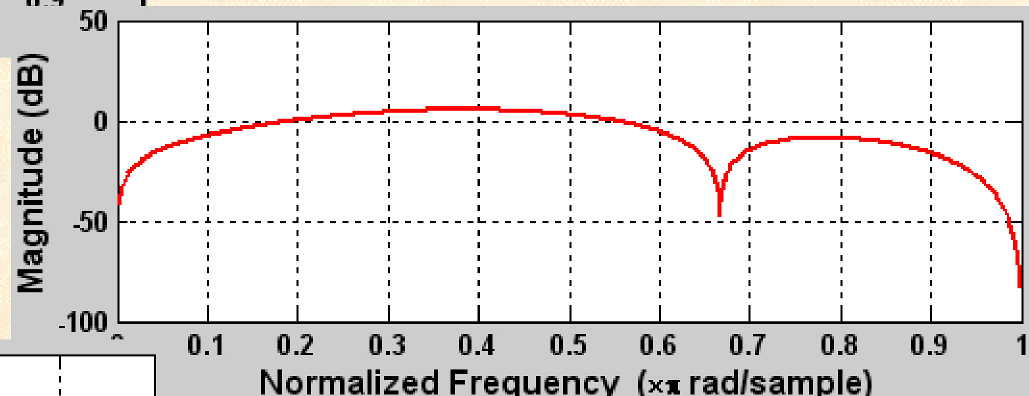
Channel - III



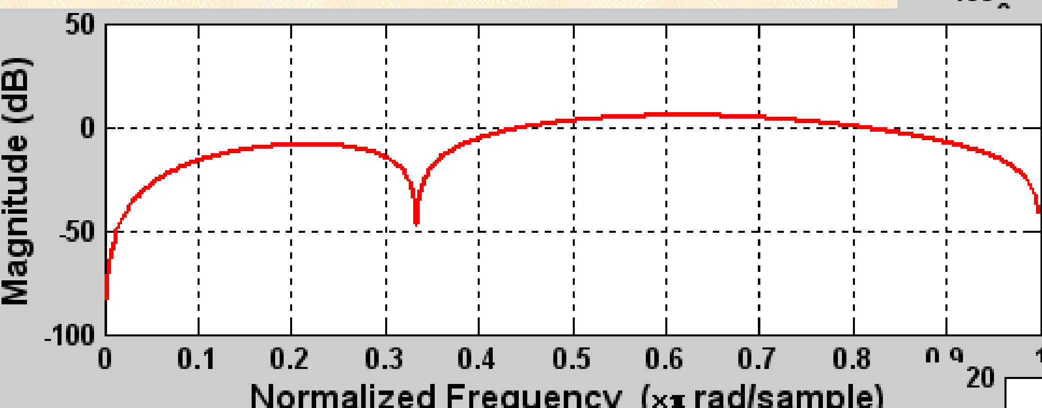
Frequency Response of a 4-channel orthogonal wavelet filters.



Channel - I

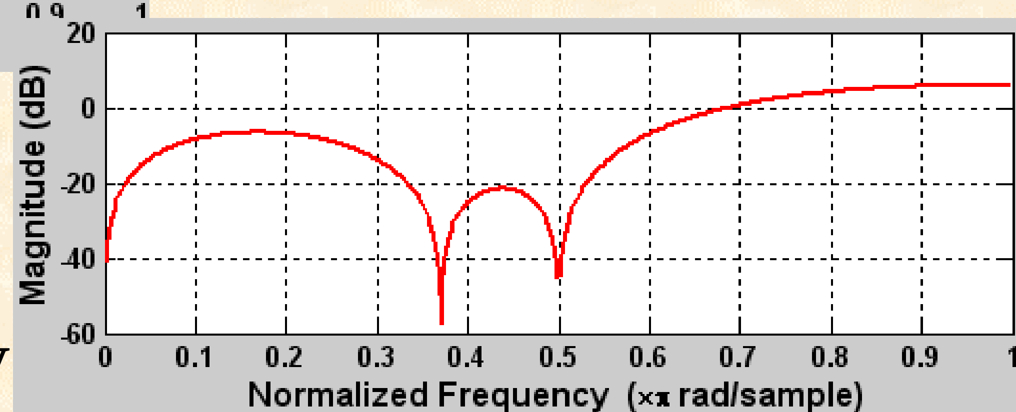


Channel - II

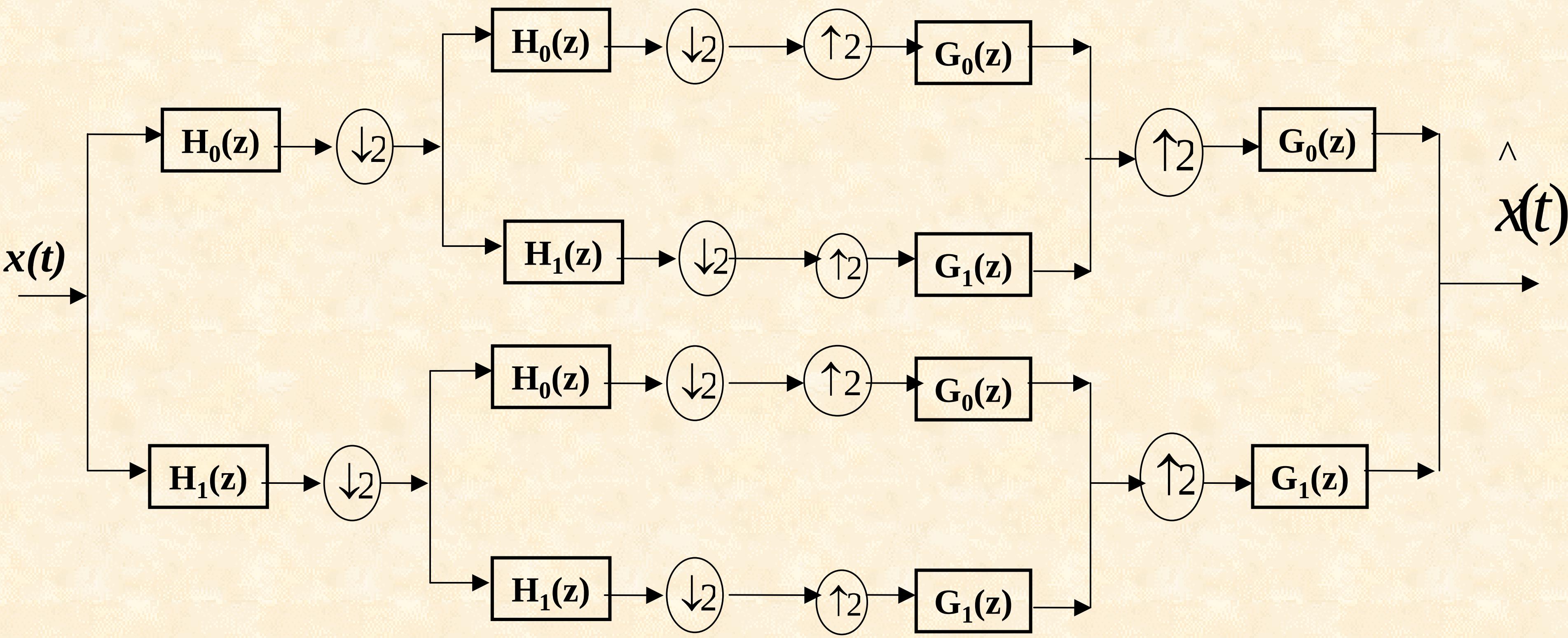


Channel - III

Channel - IV



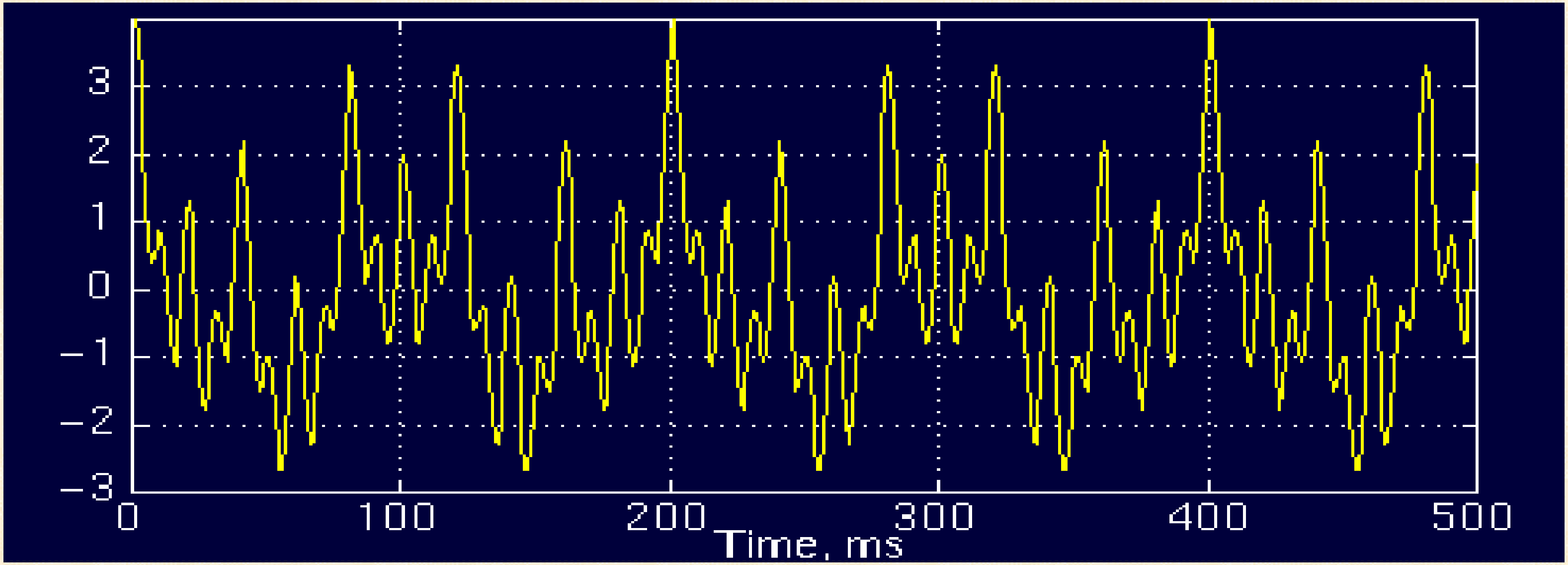
Two-level maximally decimated filter bank

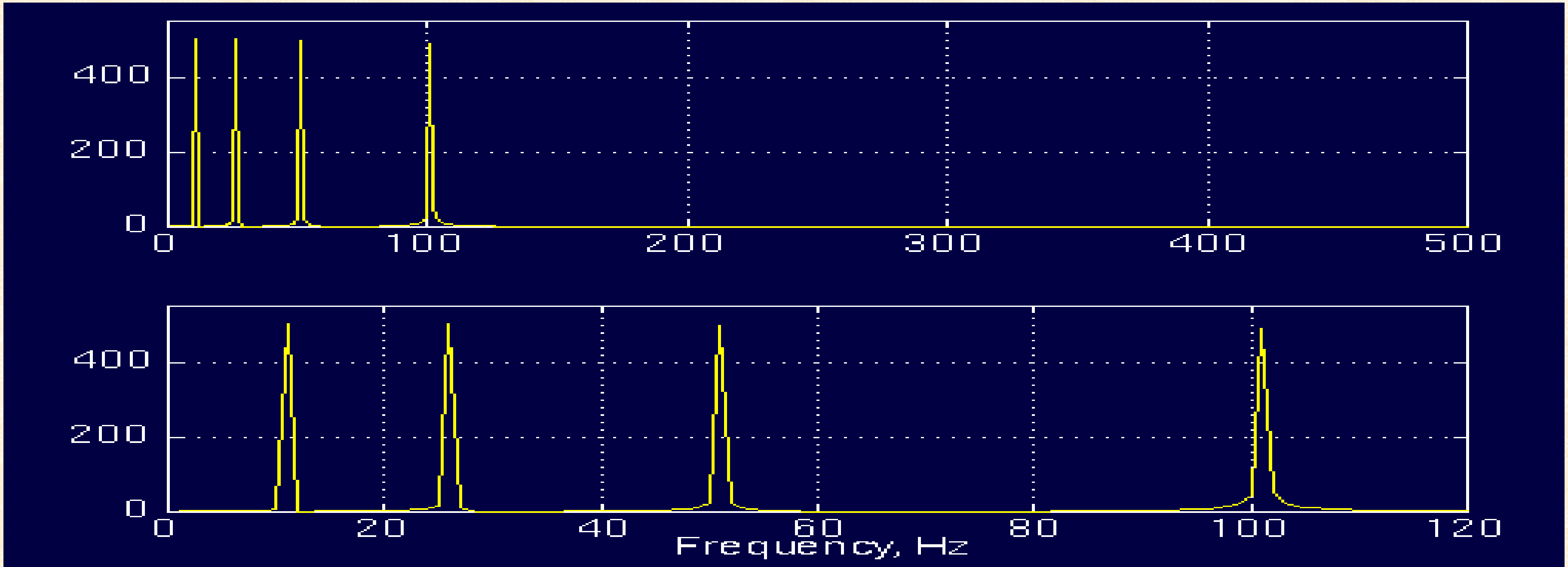


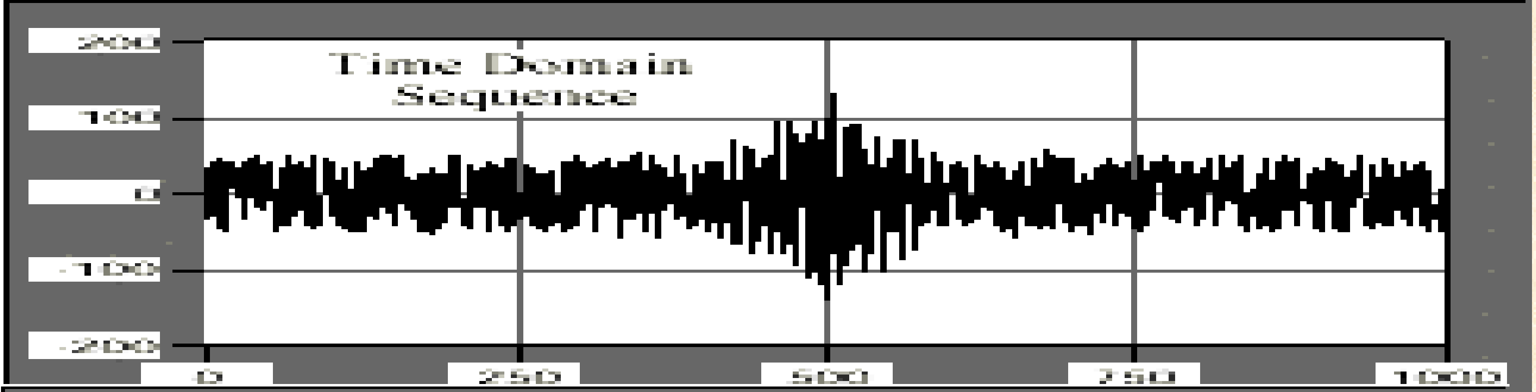
Illustrations to demonstrate the difference between:

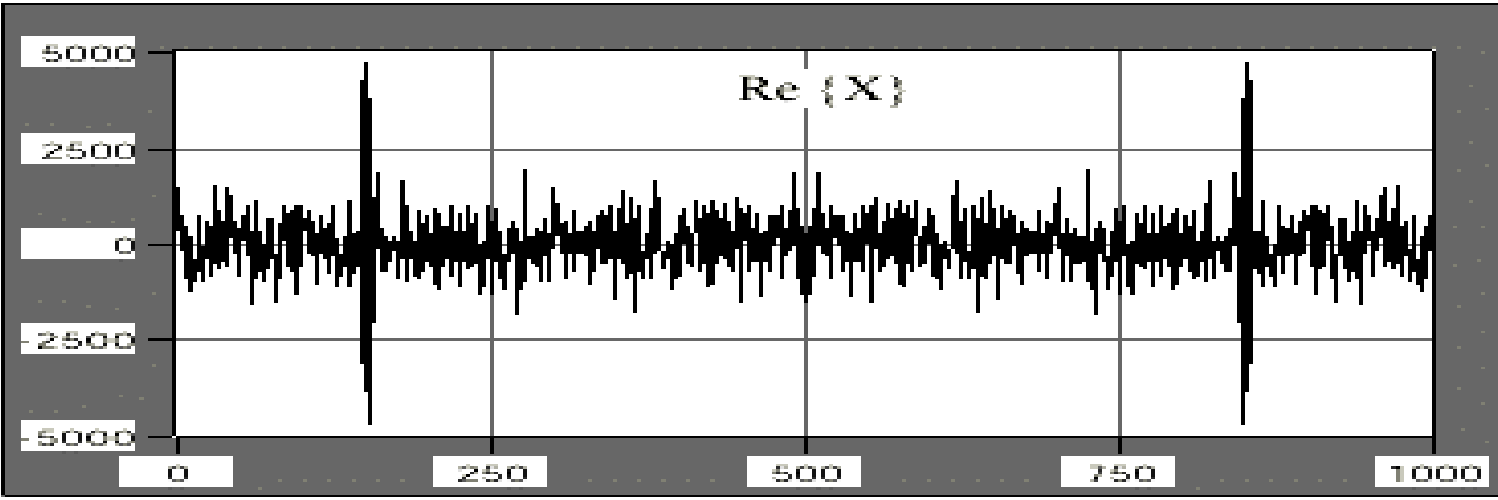
FT, STFT and WT

$$x(t) = \cos(2\pi 10t) + \cos(2\pi 25t) + \cos(2\pi 50t) + \cos(2\pi 100t)$$

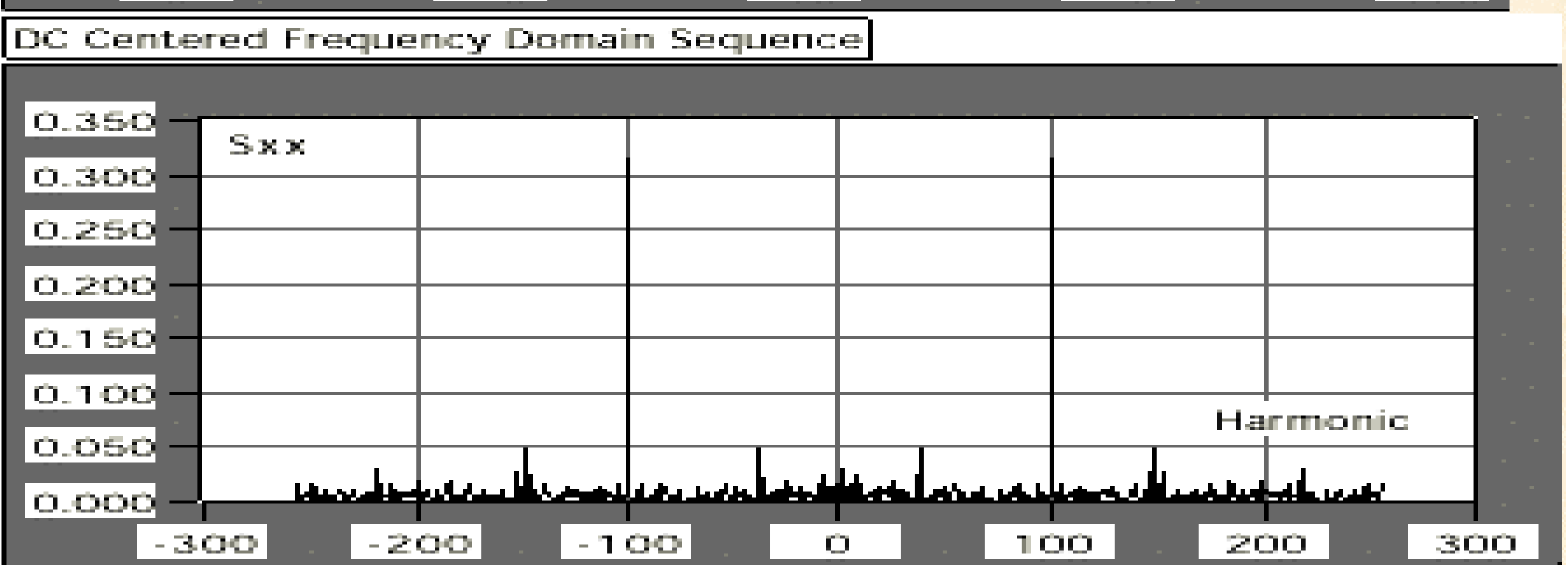


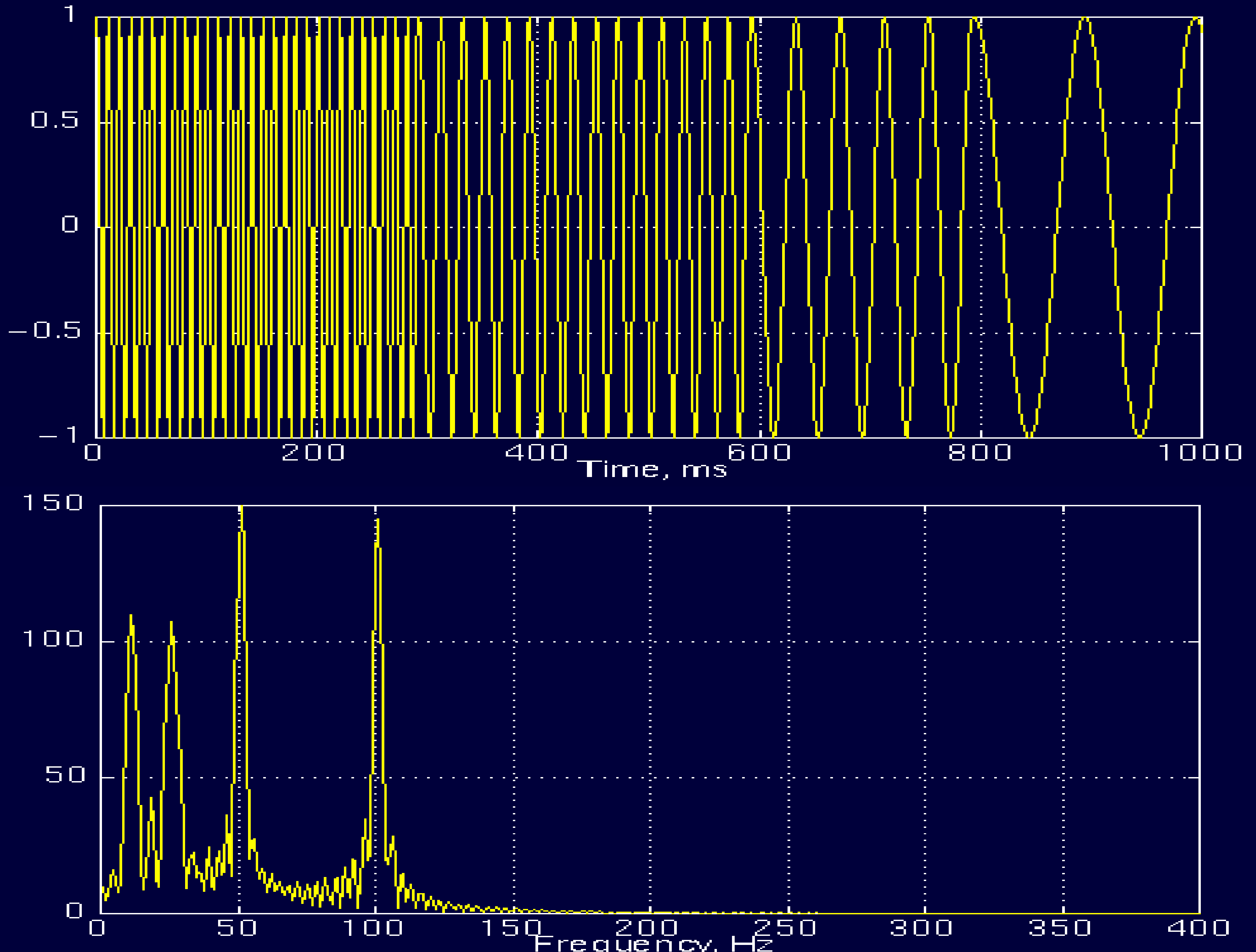



Time Domain Sequence
200
100
0
-100
-200
0
250
500
750
1000


Re{X}
5000
2500
0
-2500
-5000
0
250
500
750
1000

$$X_{n-i} = X_{-i} ,$$


DC Centered Frequency Domain Sequence
Sxx
Harmonic
0.350
0.300
0.250
0.200
0.150
0.100
0.050
0.000
-300
-200
-100
0
100
200
300



1
0.5
0
−0.5
−1
0
200
400
600
800
1000
Time, ms
150
100
50
0
0
50
100
150
200
250
300
350
400
Frequency, Hz

Note that the FT gives what frequency components (spectral components) exist in the signal. Nothing more, nothing less.

When the <u>time localization</u> of the spectral components are needed, a transform giving the <u>TIME-FREQUENCY REPRESENTATION</u> of the signal is needed.

What is Wavelet Transform and how does it solve the problem?

View WT as a plot on a 3-D graph, where time is one axis, frequency the second and amplitude is the third axis.

This will show us what frequencies, f, exist at which time, T.

There is an issue, called "*<u>uncertainty principle</u>*", which states that, we cannot exactly know what frequency exists at what time instance , but we can only know what frequency bands exist at what time intervals.

The uncertainty principle, originally found and formulated by Heisenberg, states that, the momentum and the position of a moving particle cannot be known simultaneously. This applies to our subject as follows:

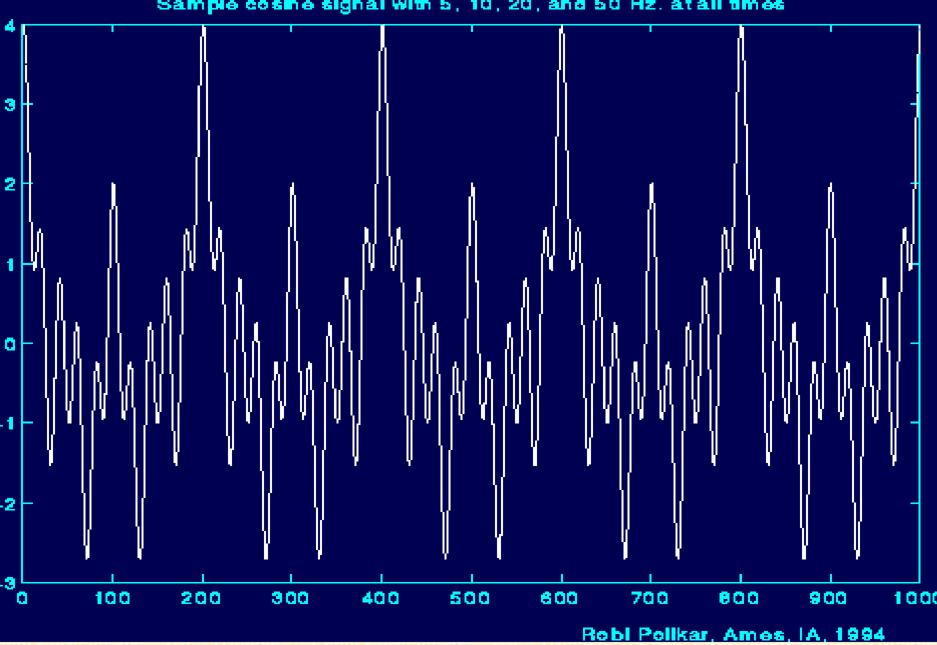
The frequency and time information of a signal at some certain point in the time-frequency plane cannot be known.

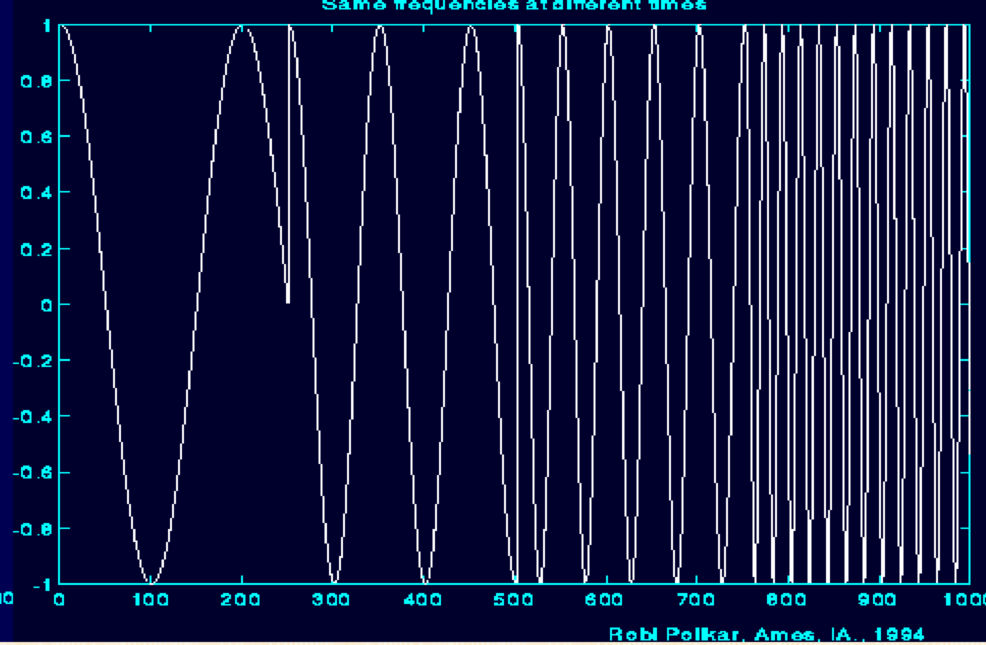
In other words: We cannot know what spectral component exists at any given time instant. The best we can do is to investigate what spectral components exist at any given interval of time.

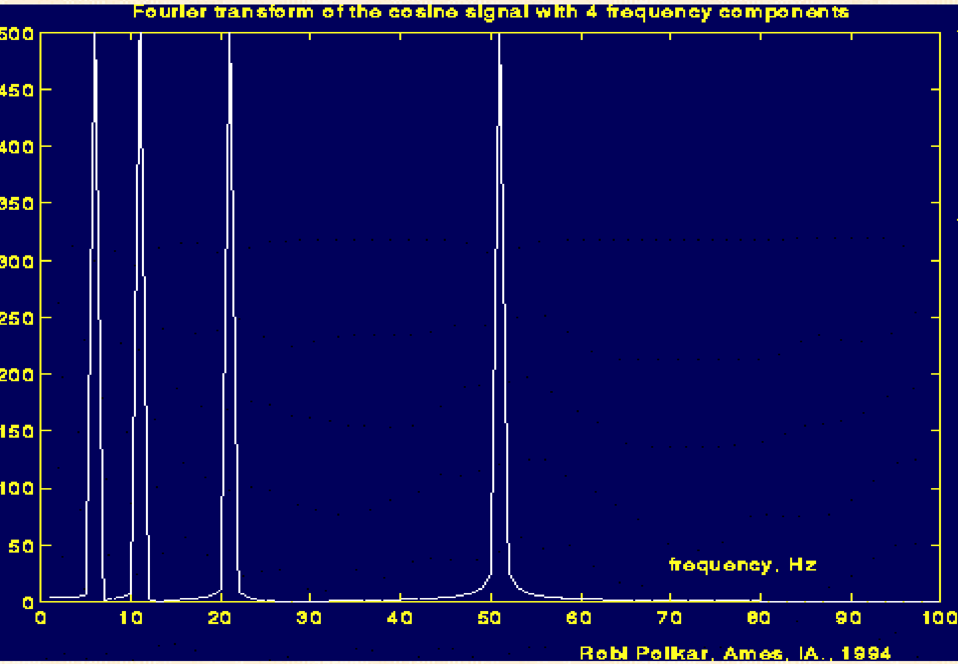
This is a problem of resolution, and it is the main reason why researchers have switched from STFT to WT.

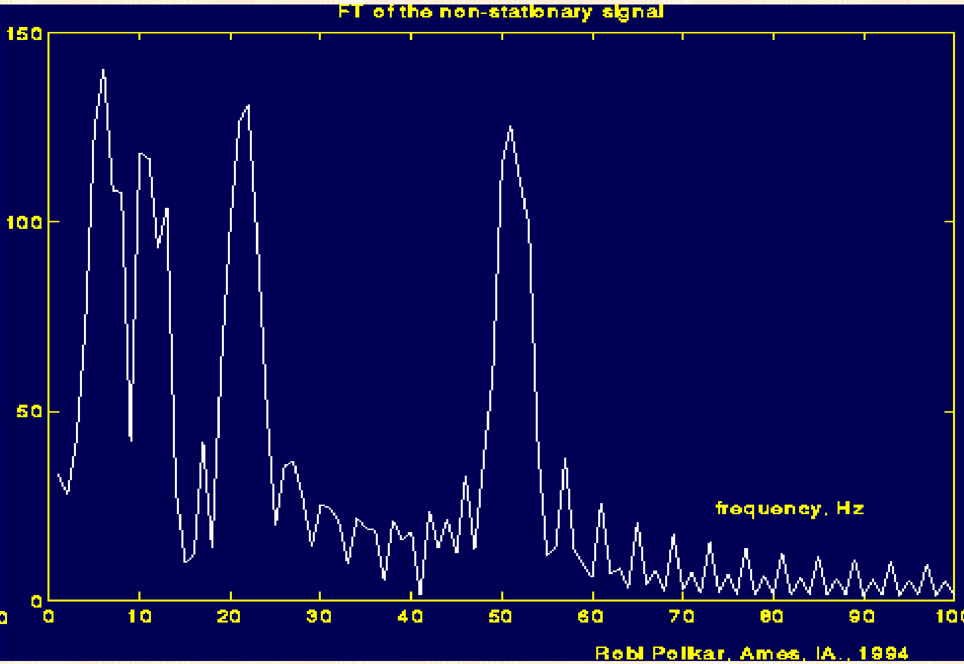
STFT gives a fixed resolution at all times, whereas WT gives a variable (or suitable) resolution as follows:
Higher frequencies are better resolved in time, and lower frequencies are better resolved in frequency.

This means that, a certain high frequency component can be located better in time (with less relative error) than a low frequency component. On the contrary, a low frequency component can be located better in frequency compared to high frequency component

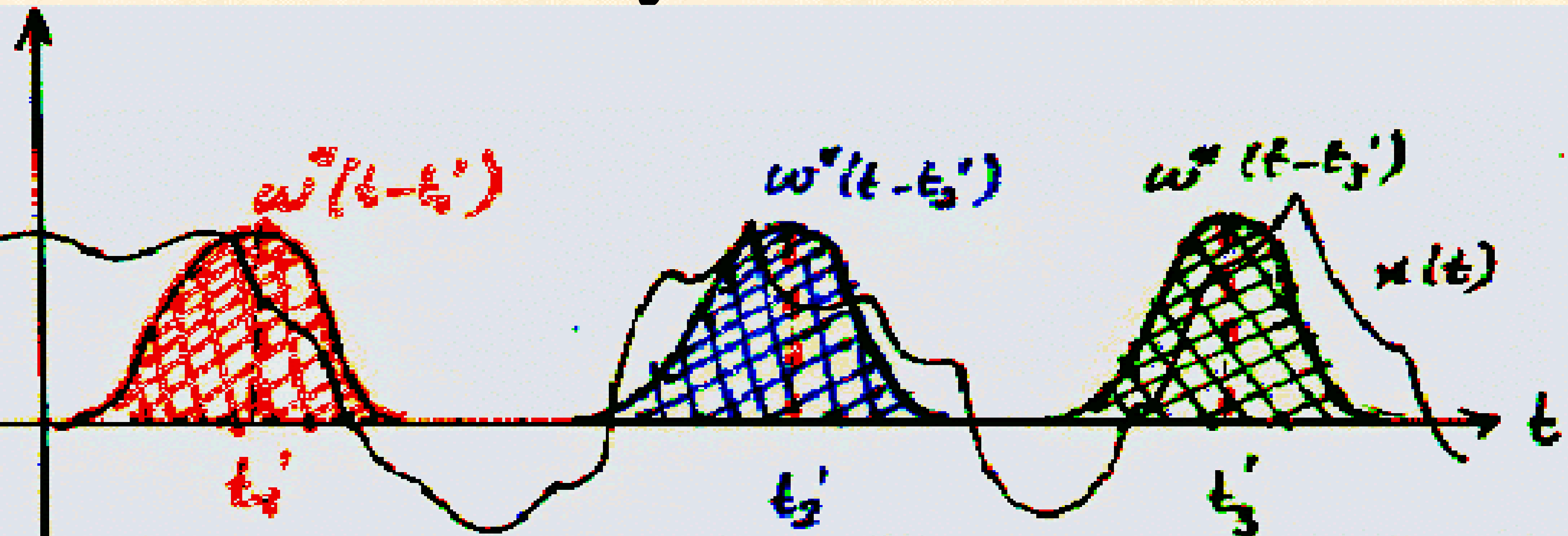

Sample cosine signal with 5, 10, 20, and 50 Hz. at all times
Robi Polikar, Ames, IA, 1994


Same frequencies at different times
Robi Polikar, Ames, IA., 1994

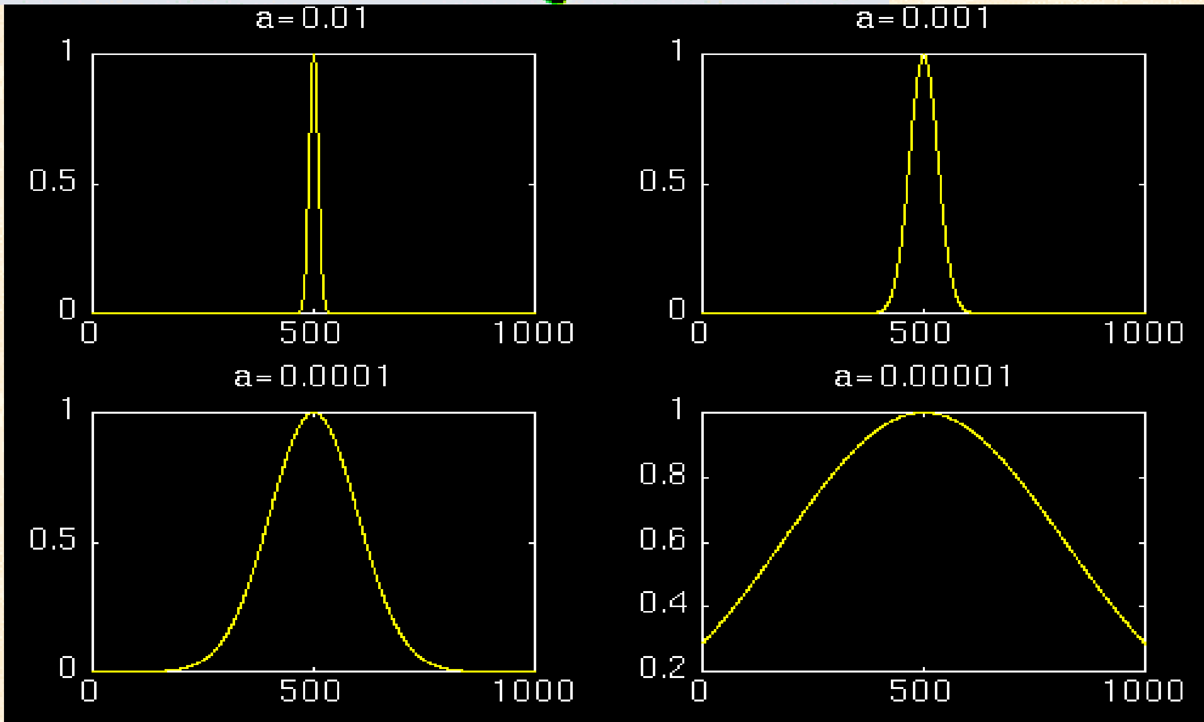

Fourier transform of the cosine signal with 4 frequency components
frequency, Hz
Robi Polikar, Ames, IA., 1994

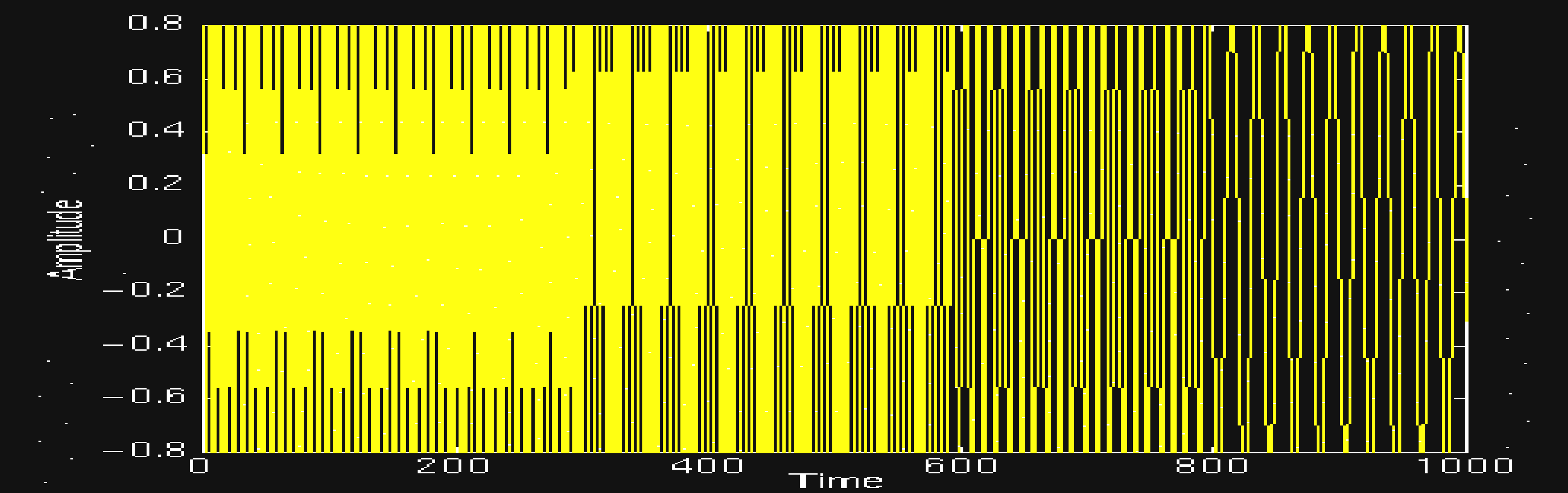

FT of the non-stationary signal
frequency, Hz
Robi Polikar, Ames, IA., 1994

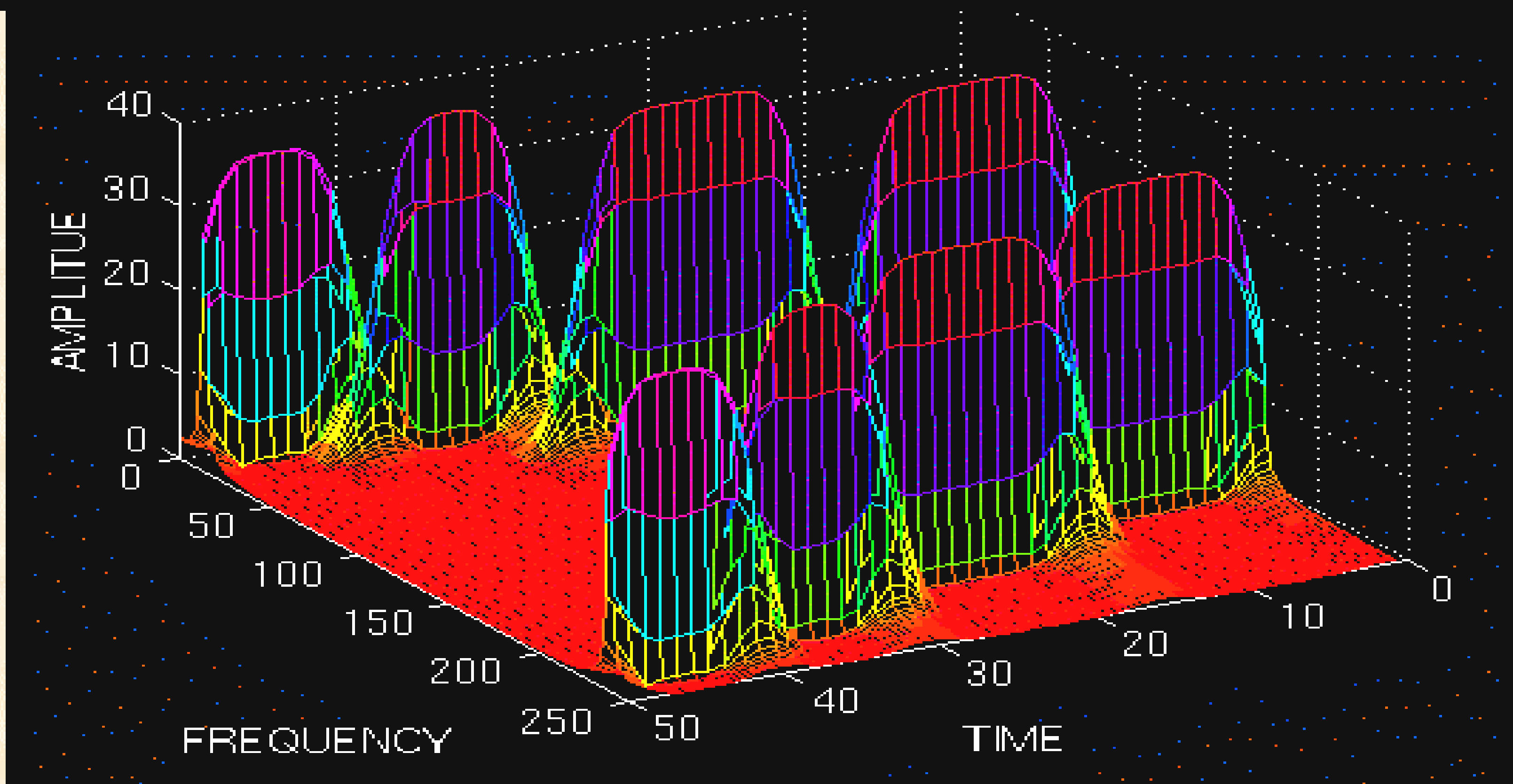
$$STFT_x^{\omega}(t, f) = \int [x(t).\omega(t - t')] \exp(-j2\pi ft) dt$$

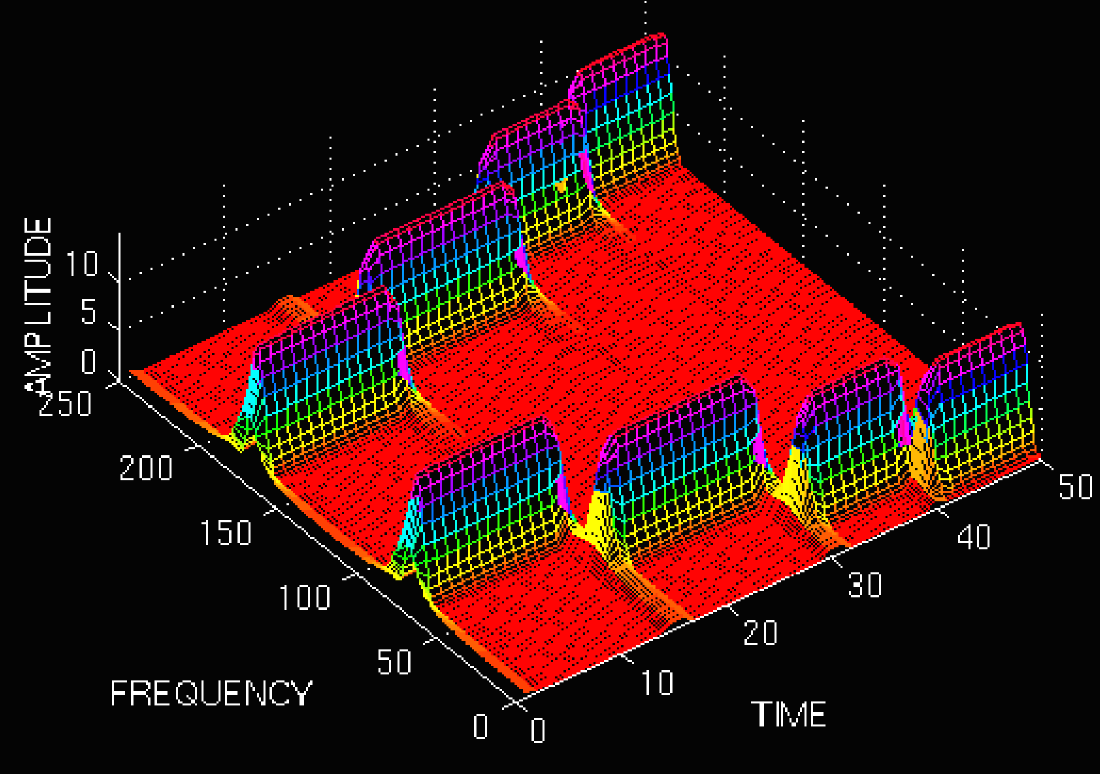


$$w(t) = \exp(\frac{-a * t^2}{2})$$



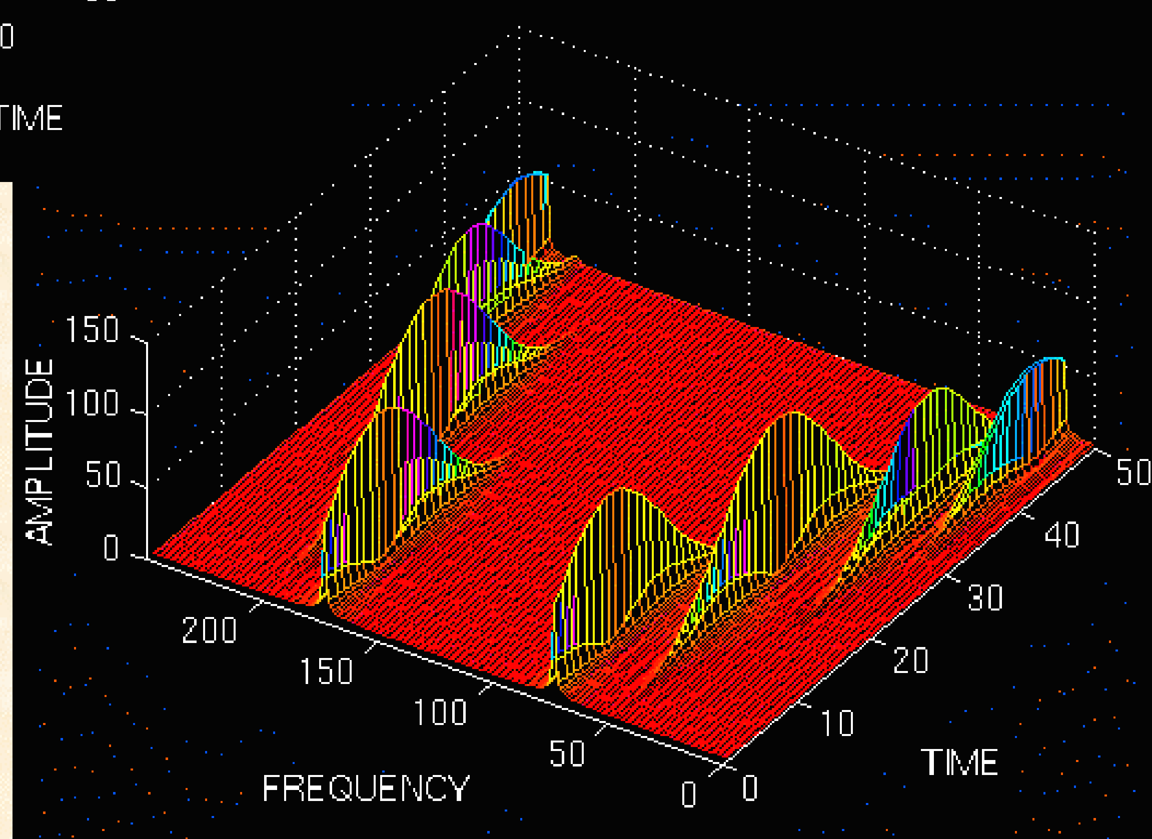

Amplitude
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
200
400
600
800
1000
Time

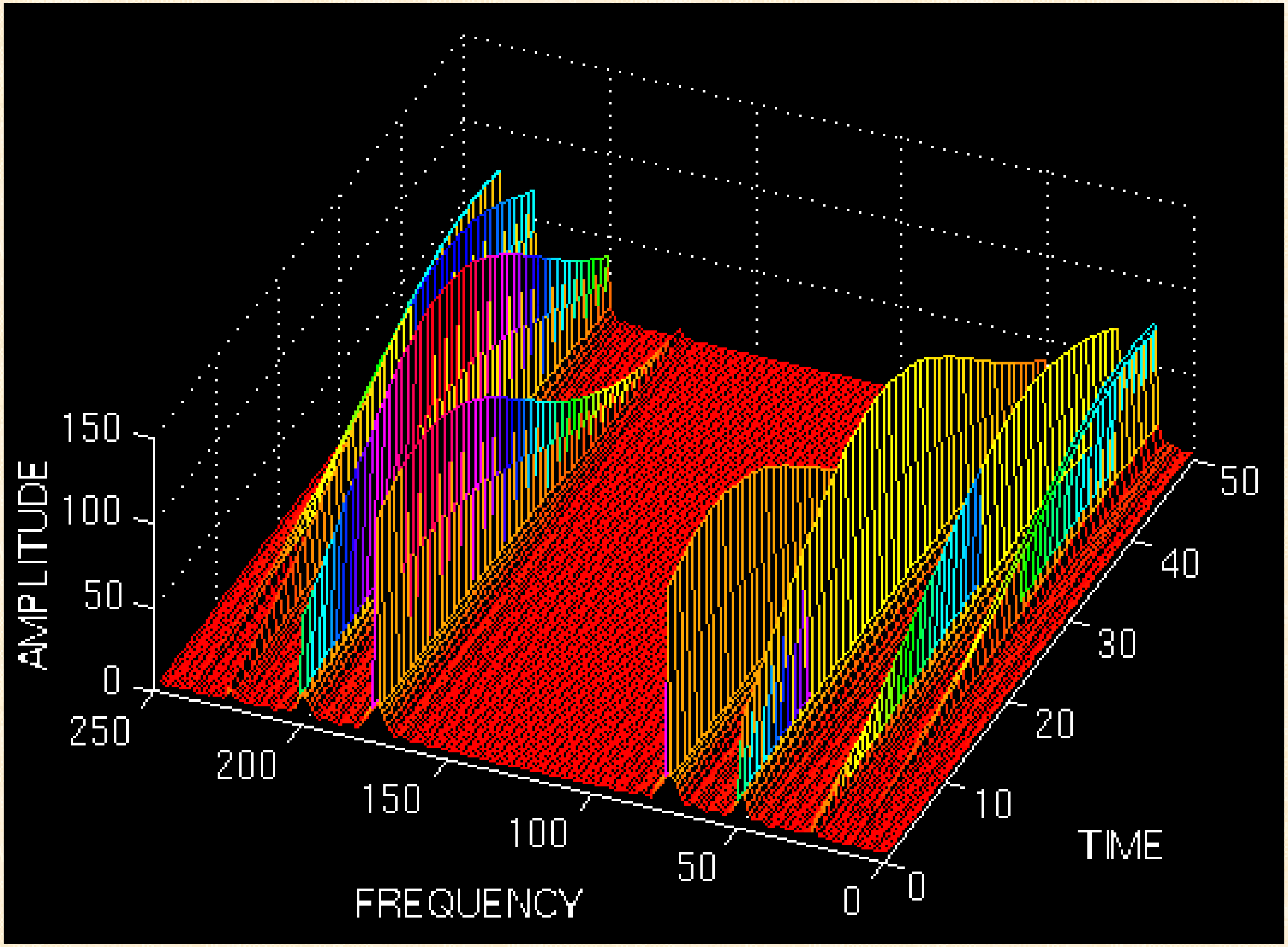

AMPLITUE
40
30
20
10
0
FREQUENCY
0
50
100
150
200
250
TIME
0
10
20
30
40
50



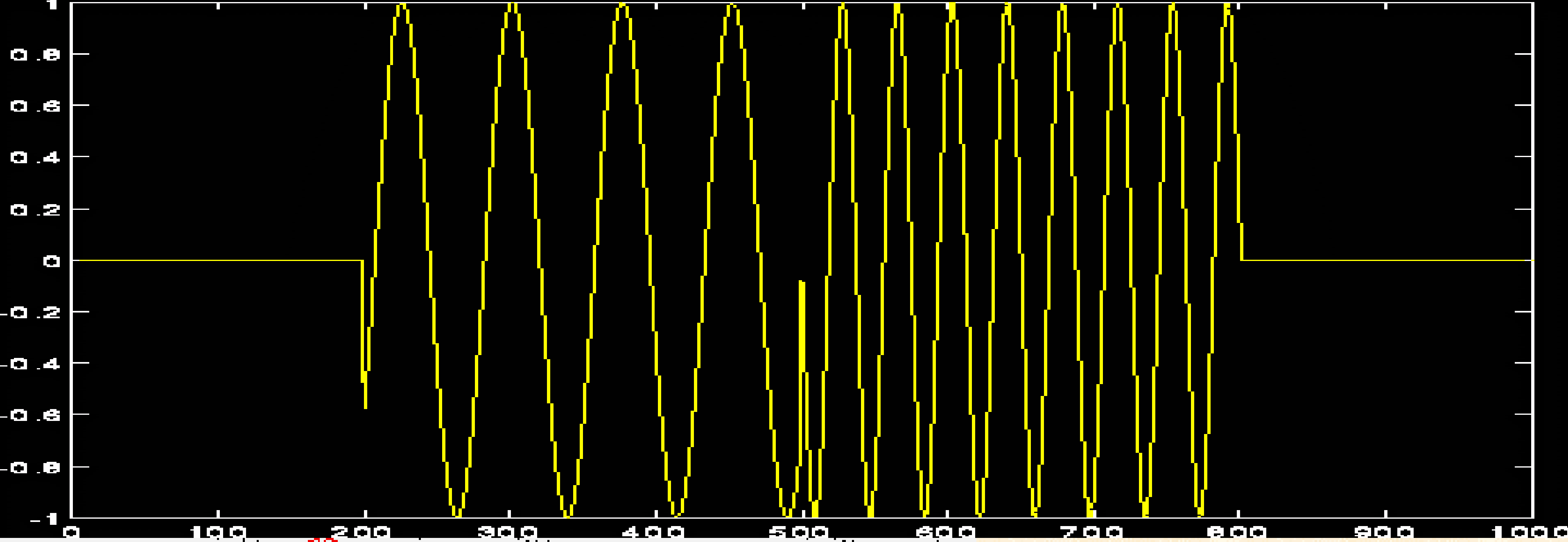
Broader Window, *w*

Narrow Window, *w*

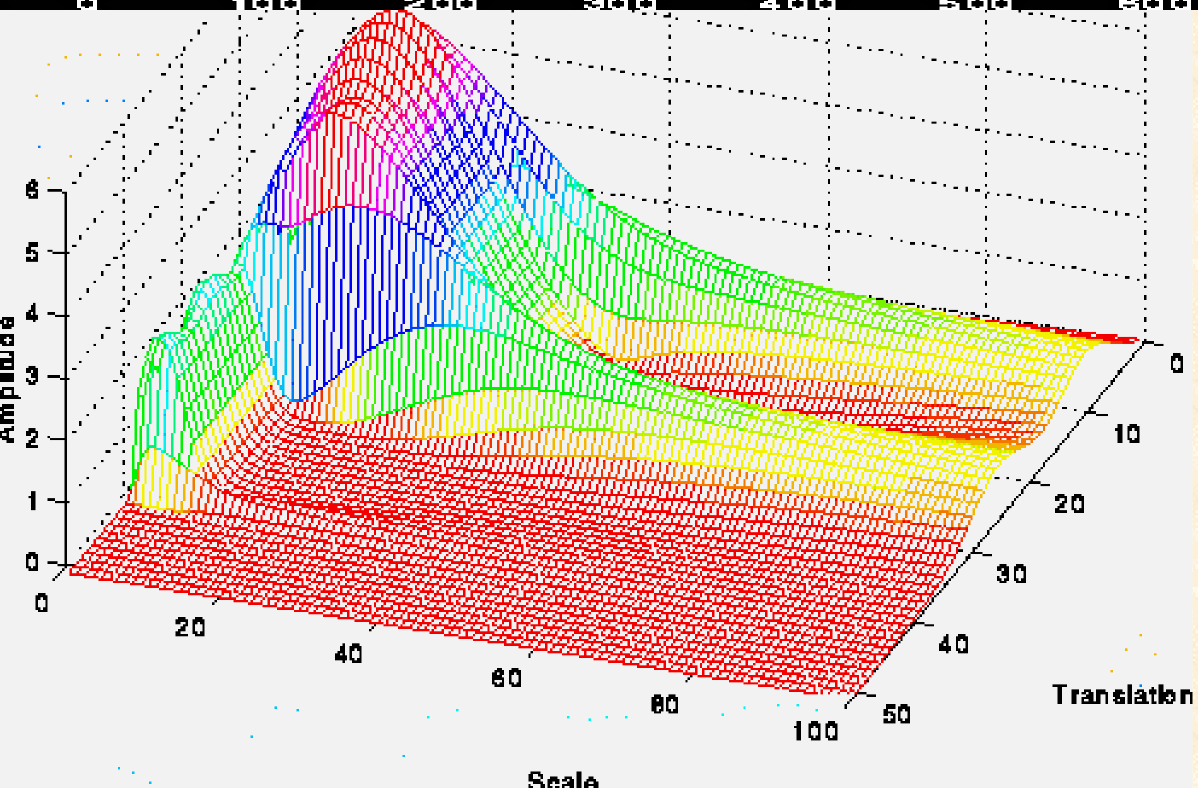




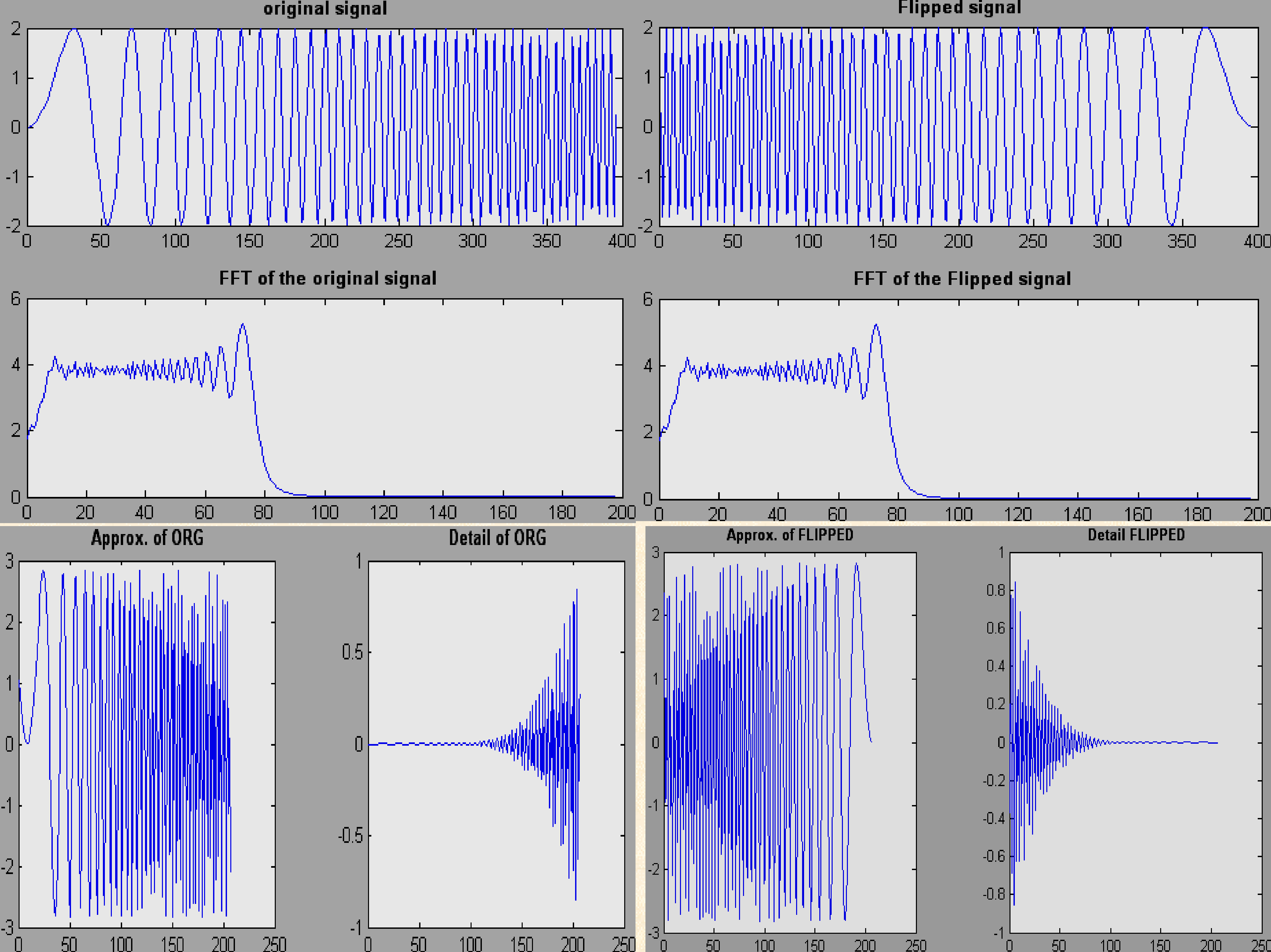
Still larger window, *w*

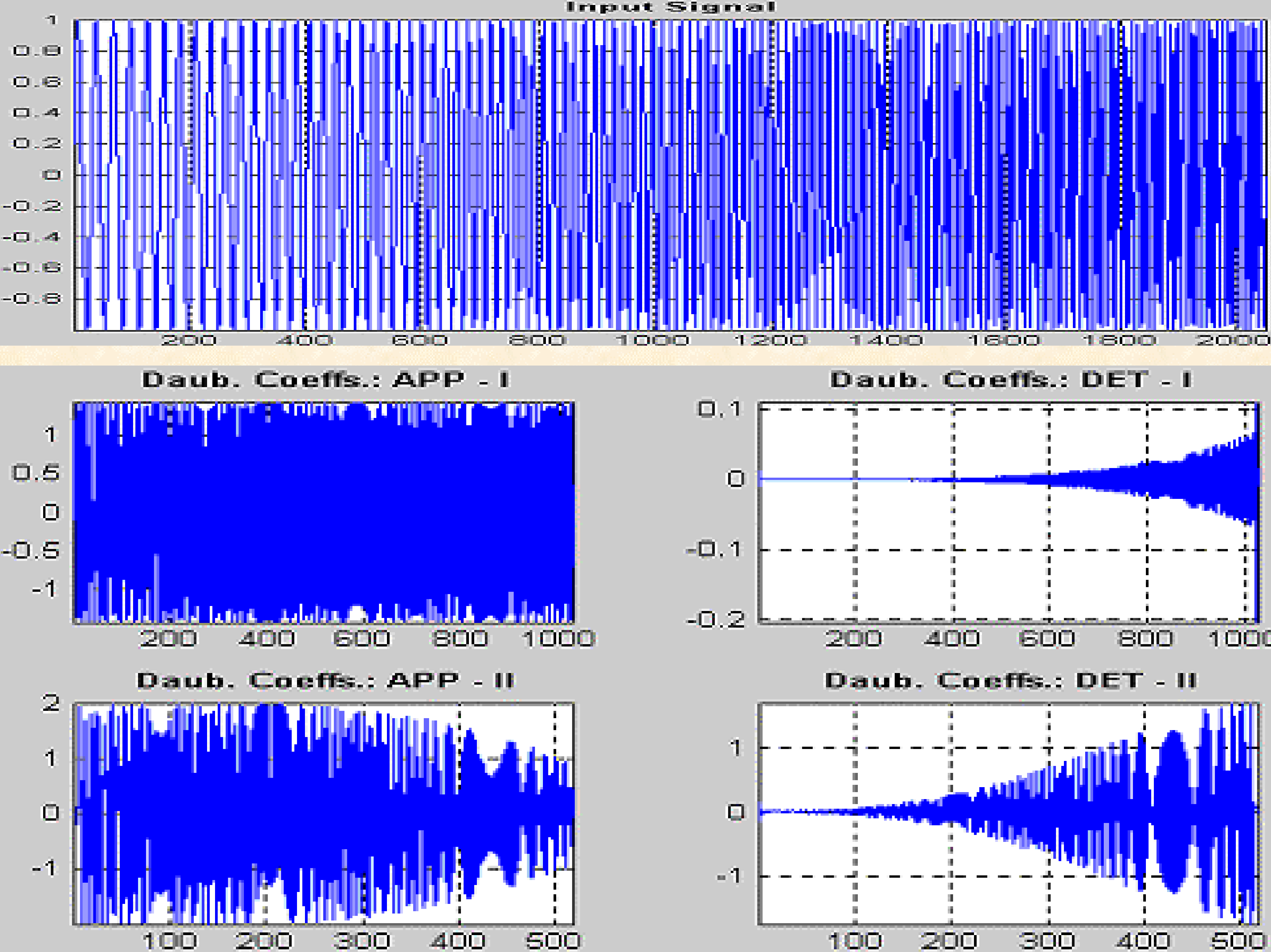


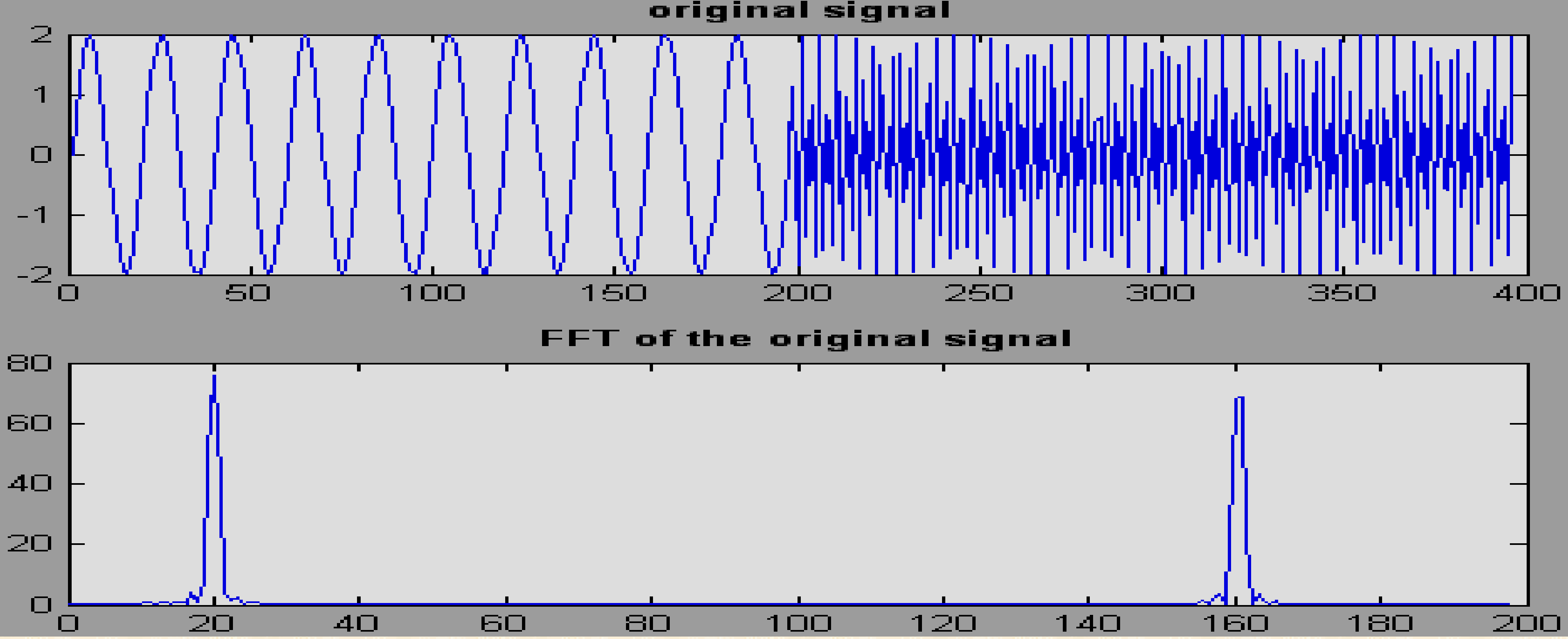
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
100
200
300
400
500
600
700
800
900
1000

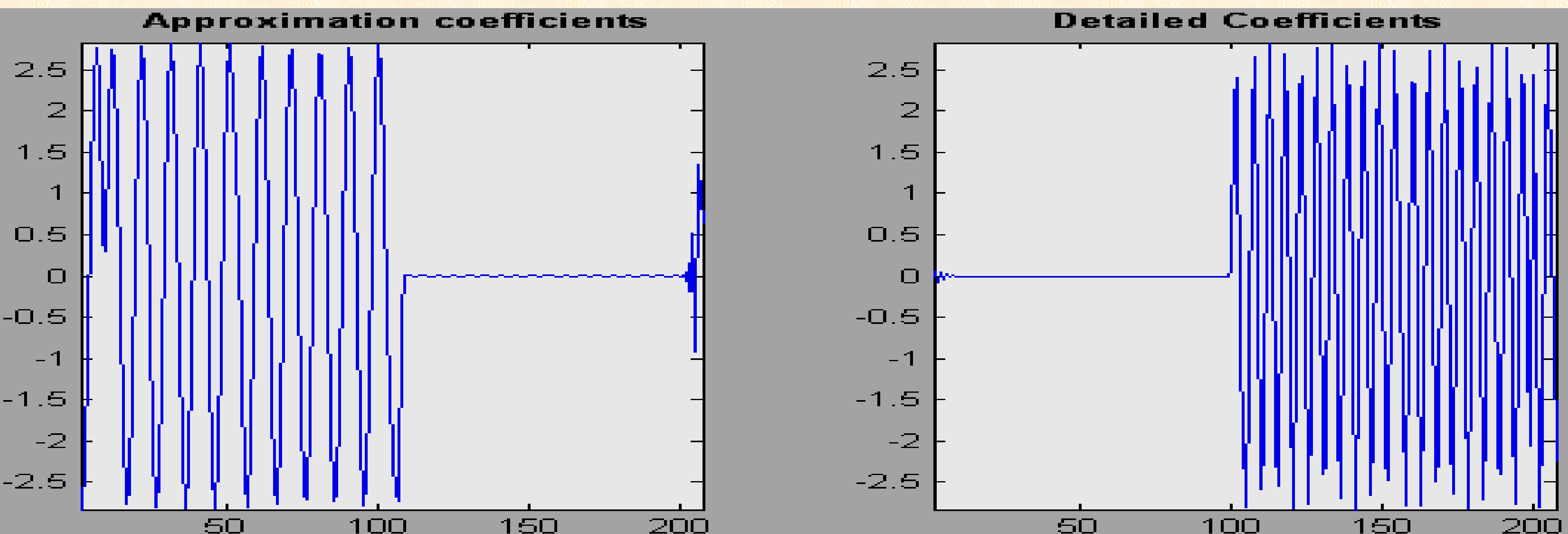


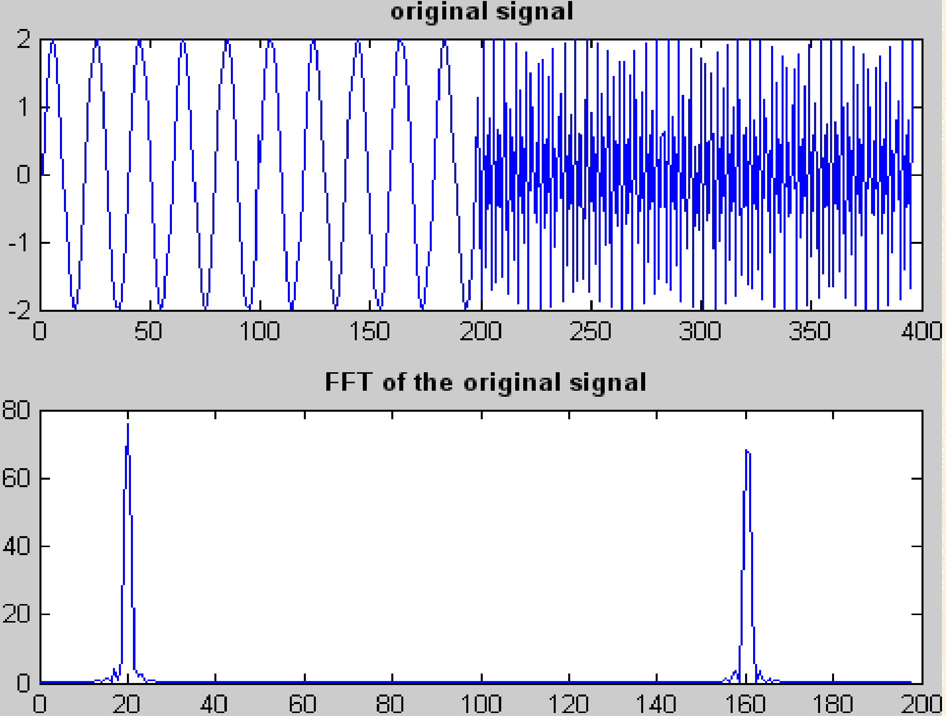
Amplitude
Scale
Translation

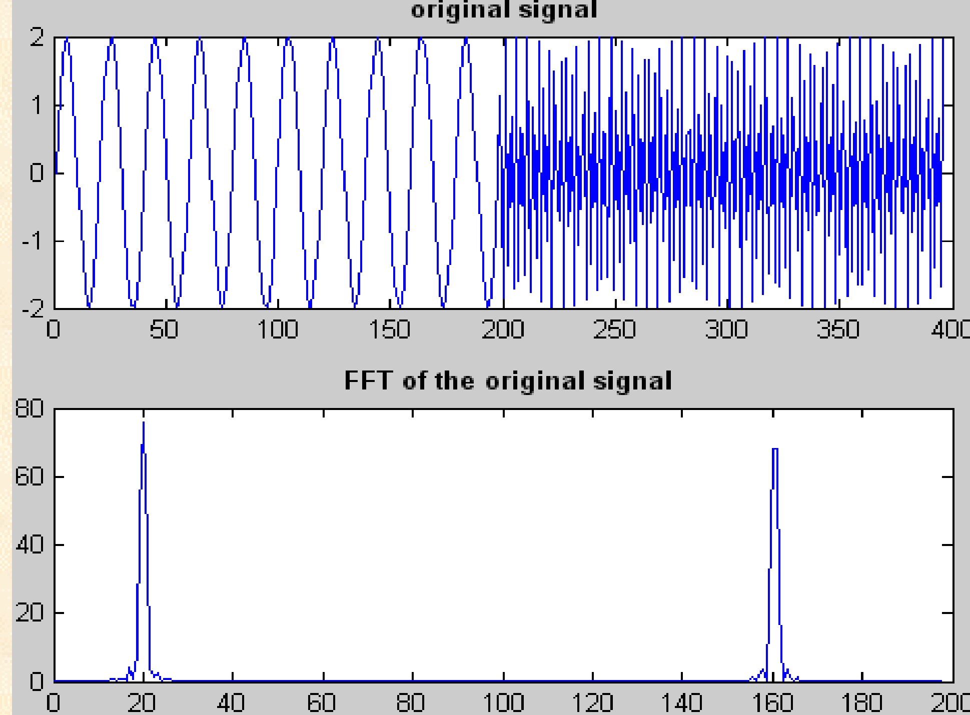

original signal
Flipped signal
FFT of the original signal
FFT of the Flipped signal
Approx. of ORG
Detail of ORG
Approx. of FLIPPED
Detail FLIPPED

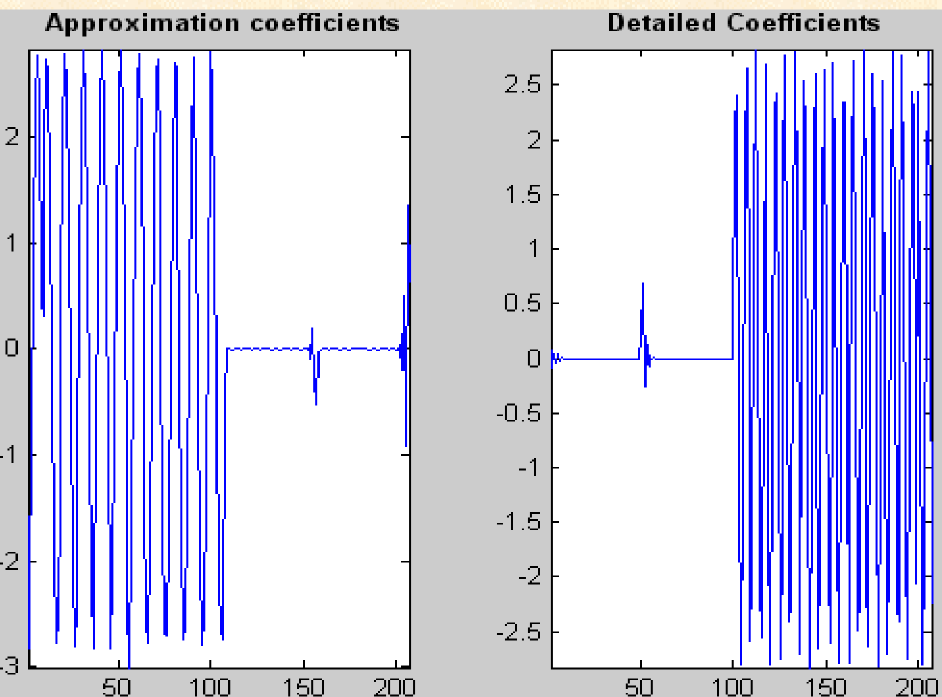

Input Signal
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
200
400
600
800
1000
1200
1400
1600
1800
2000
Daub. Coeffs.: APP - I
1
0.5
0
-0.5
-1
200
400
600
800
1000
Daub. Coeffs.: DET - I
0.1
0
-0.1
-0.2
200
400
600
800
1000
Daub. Coeffs.: APP - II
2
1
0
-1
100
200
300
400
500
Daub. Coeffs.: DET - II
1
0
-1
100
200
300
400
500

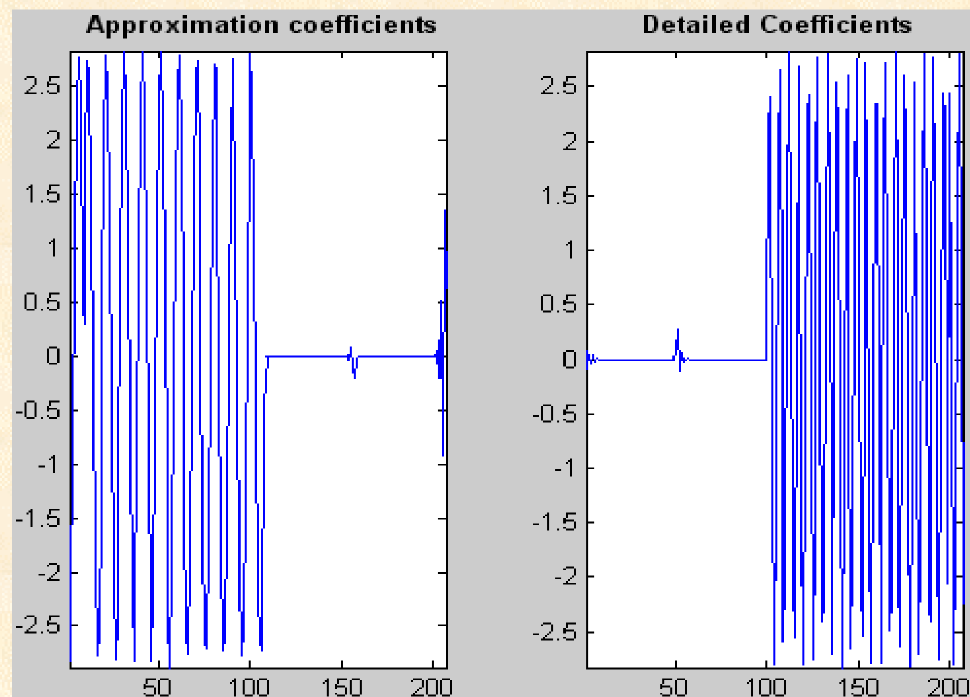

original signal
2
1
0
-1
-2
0
50
100
150
200
250
300
350
400
FFT of the original signal
80
60
40
20
0
0
20
40
60
80
100
120
140
160
180
200


Approximation coefficients
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
50
100
150
200
Detailed Coefficients
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
50
100
150
200

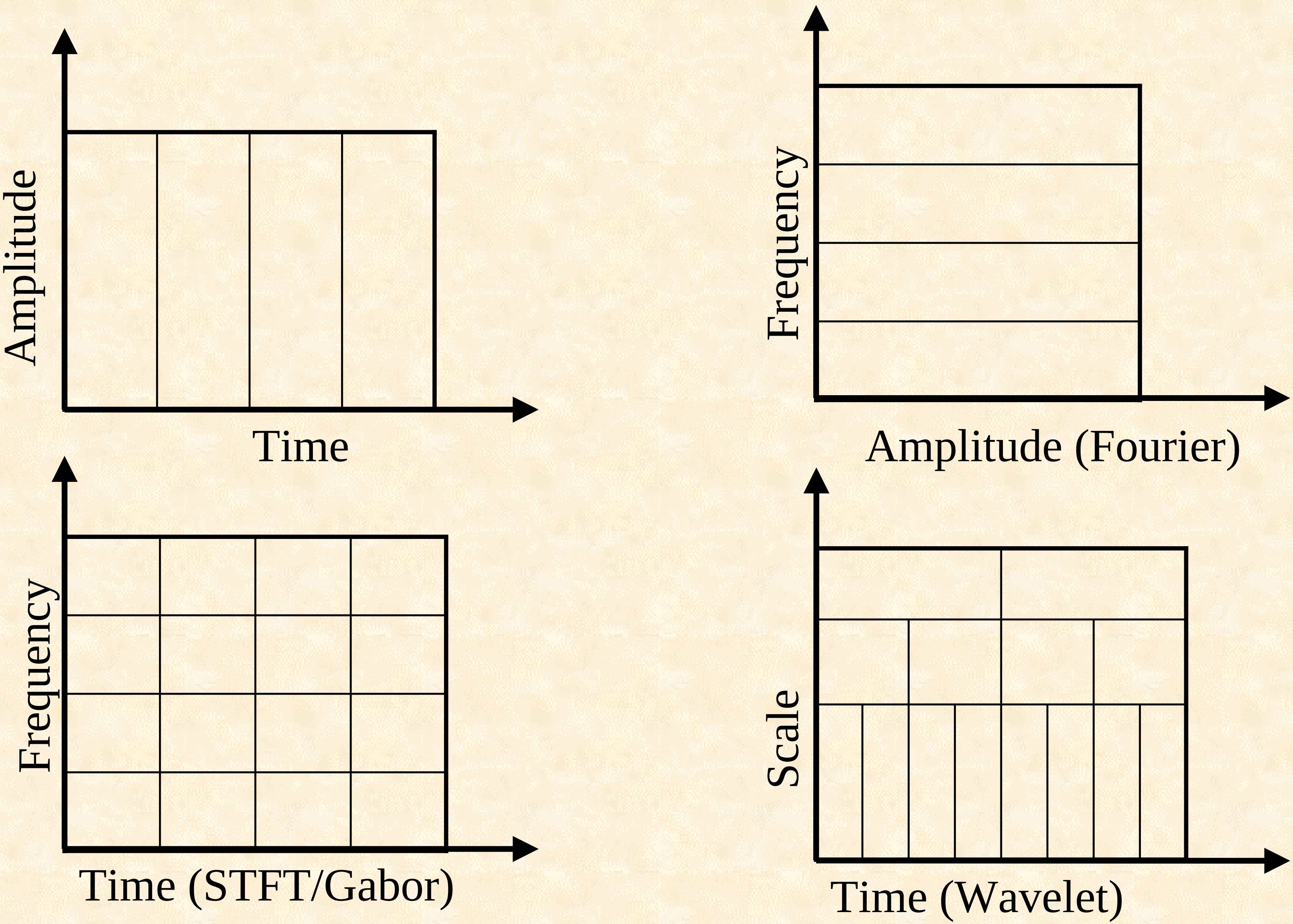

original signal
FFT of the original signal

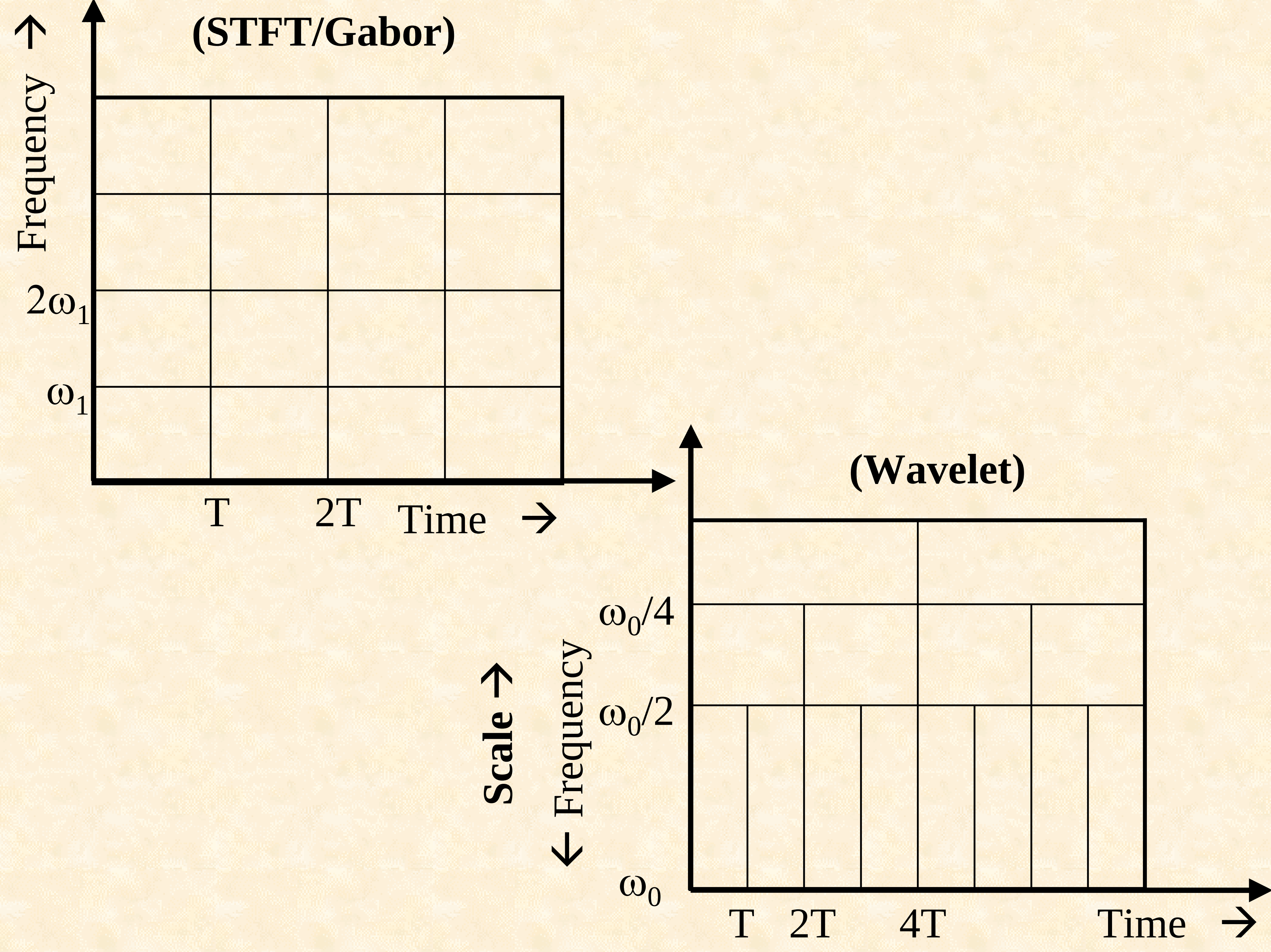

original signal
FFT of the original signal


Approximation coefficients
Detailed Coefficients

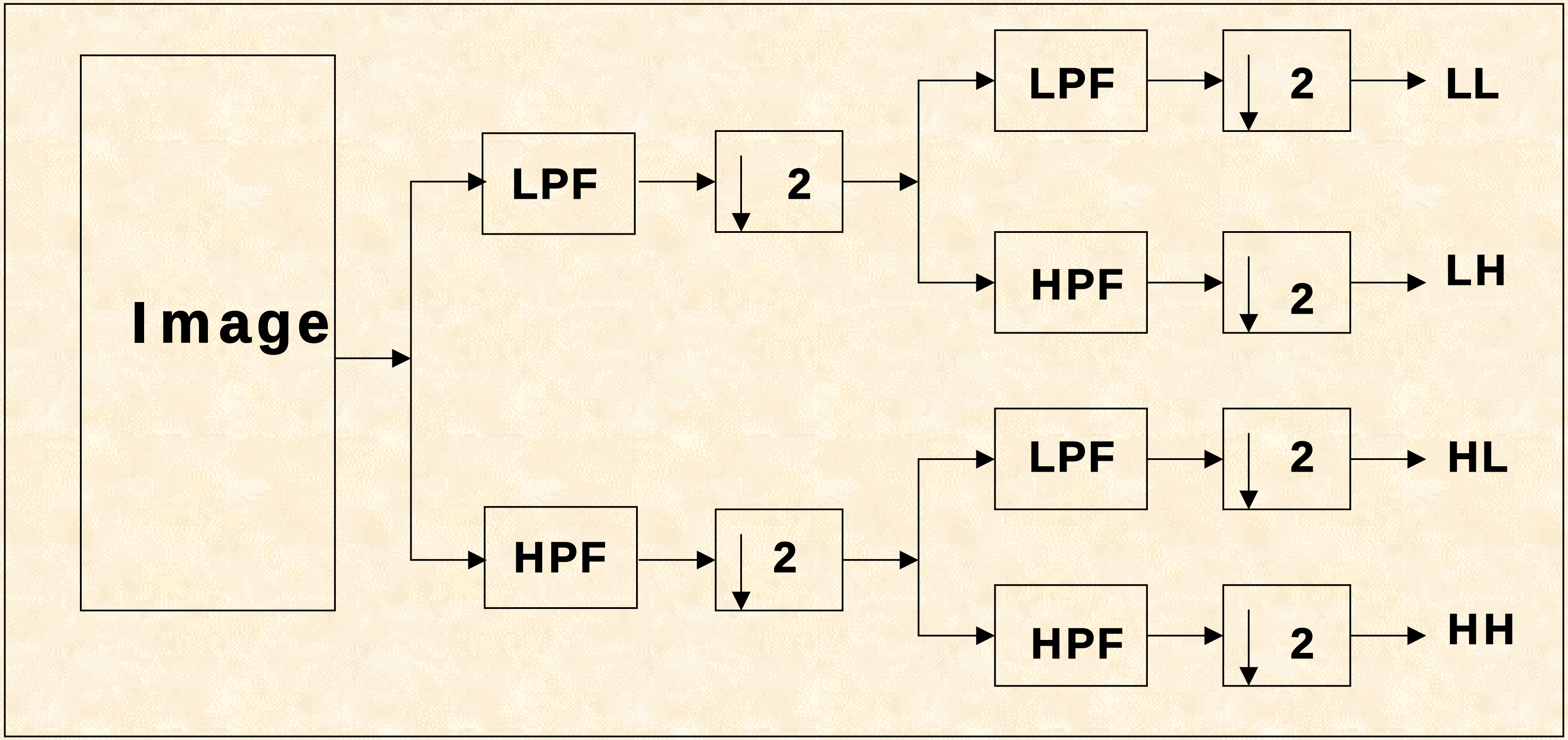

Approximation coefficients
Detailed Coefficients

Time and Frequency Resolutions




(STFT/Gabor)
Frequency →
2ω1
ω1
T
2T
Time →
(Wavelet)
ω0/4
ω0/2
ω0
Scale →
← Frequency
T
2T
4T
Time →

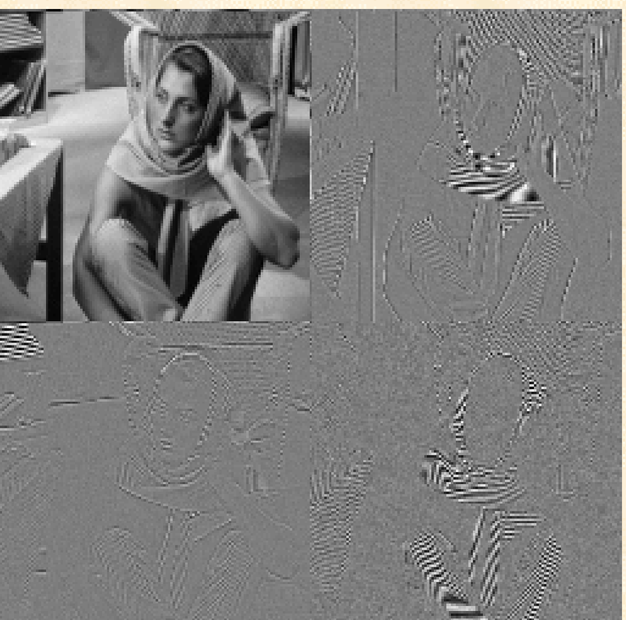
Two-dimensional Wavelet Transform



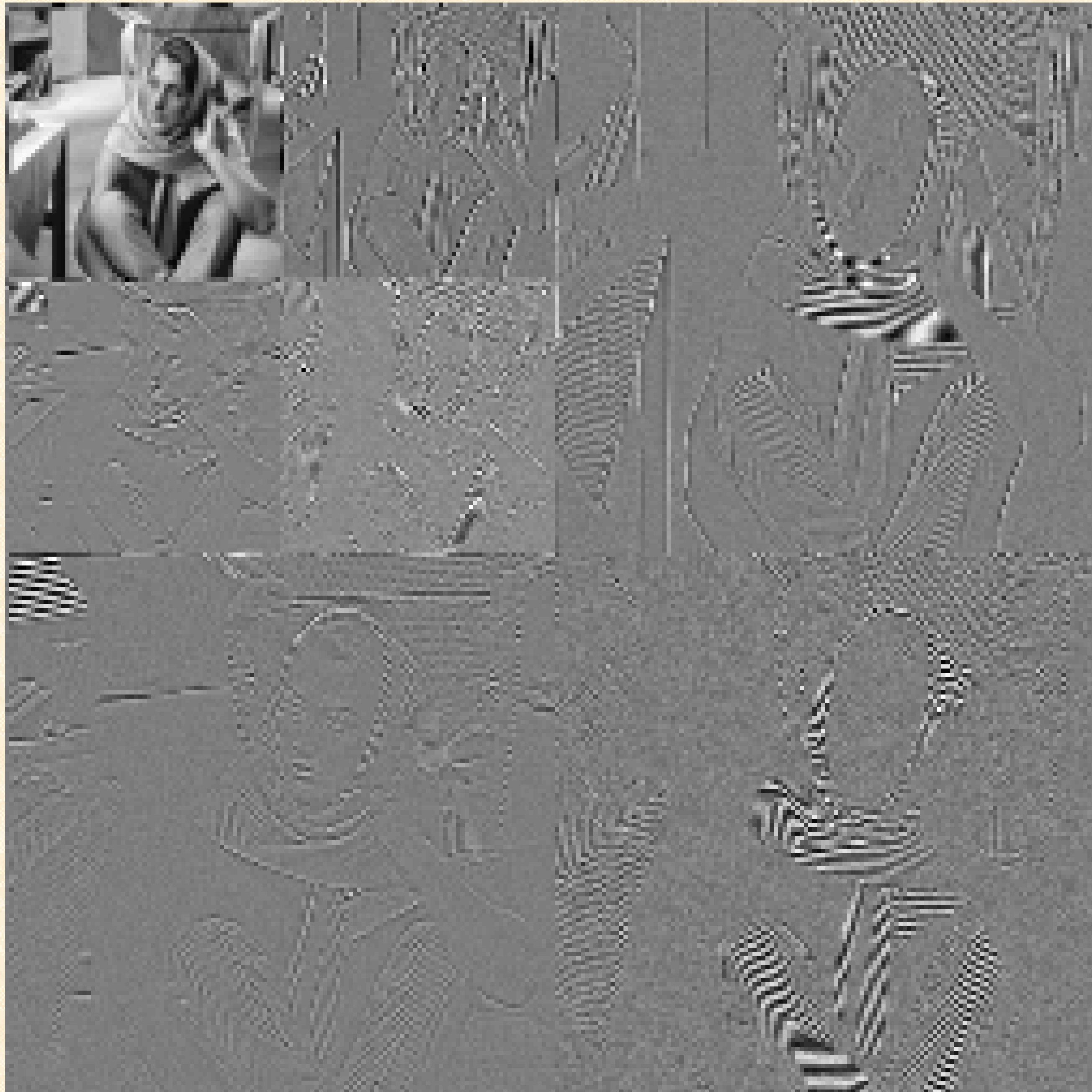
LL	HL
LH	HH

approximation	vertical details
horizontal details	diagonal details

Level I wavelet decomposition of an image



Level II wavelet decomposition of an image



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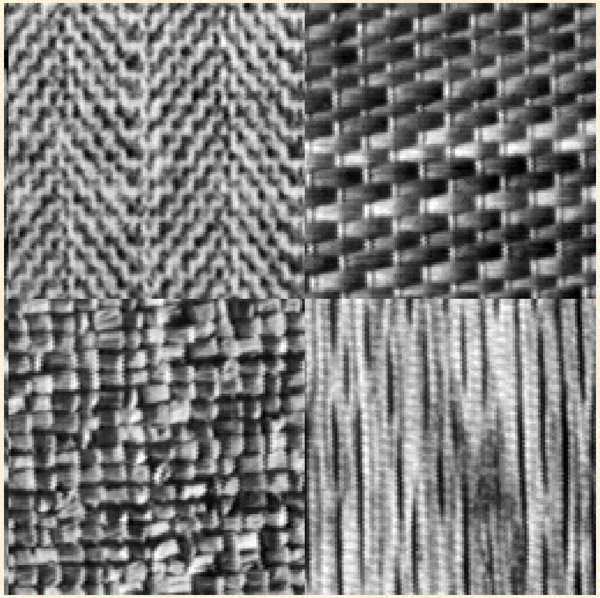
Wavelet based analysis

of texture Images

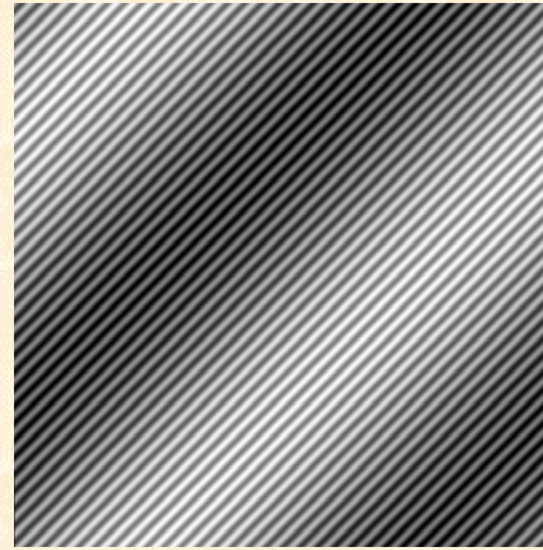
Problem of

Shape from

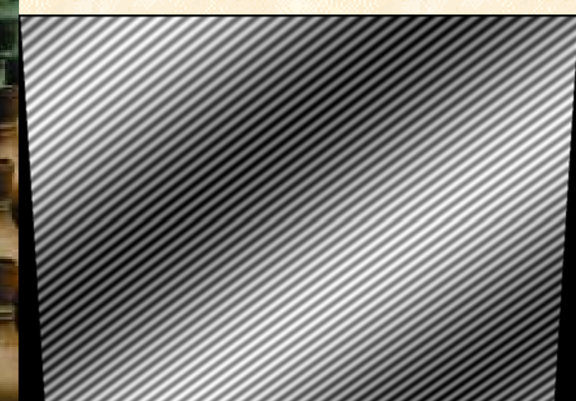
3-D Textures

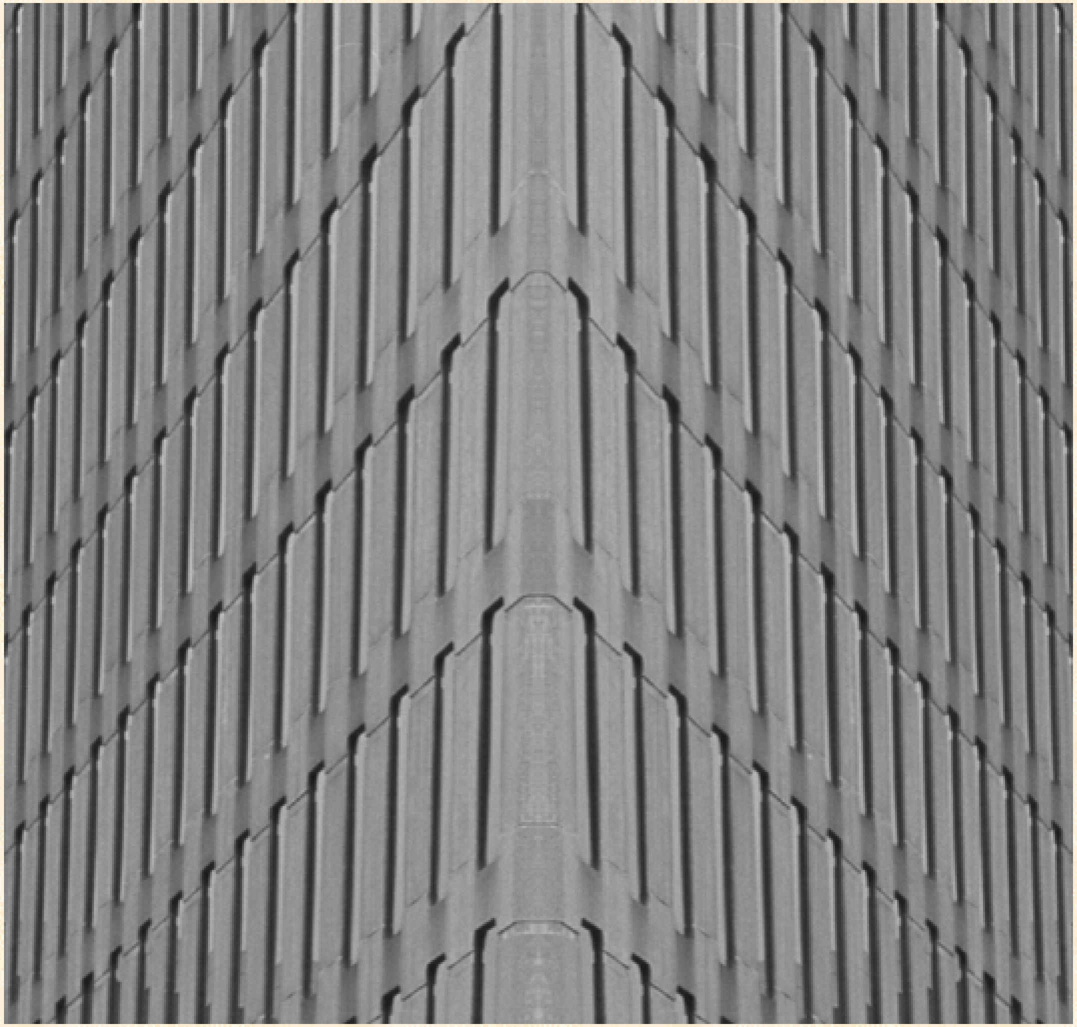


2-D Textures



3-D Textures





Real world 3-D Texture image

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