BASICS OF PROBABILITY

CHAPTER-1

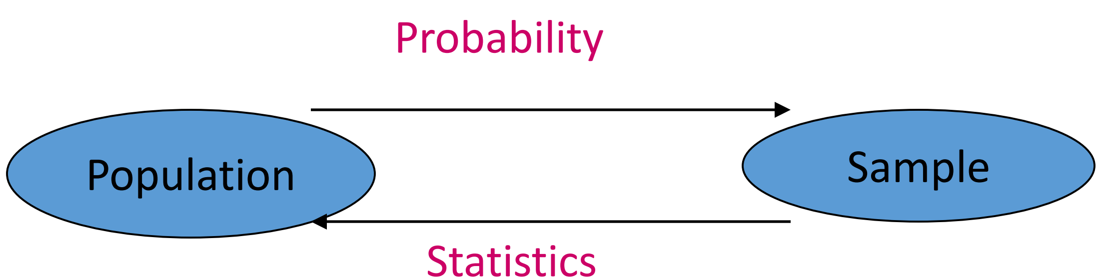
CS6015-LINEAR ALGEBRA AND RANDOM PROCESSES

Why Learn Probability?

- Nothing in life is certain. In everything we do, we gauge the chances of successful outcomes, from business to medicine to the weather

- Probability provides a quantitative description of the chances or likelihoods associated with various outcomes

- It provides a bridge between descriptive and inferential statistics



COMMON TERMS RELATED TO PROBABILITY

- **Probability** is the measure of the likelihood that an event will occur
- Probability values are between $\mathbf{0}$ (the event never occurs) and $\mathbf{1}$ (the event always occurs)
- **Random experiment**: It is a process whose outcome is uncertain
- **Outcome**: A possible result of a random experiment. These individual outcomes are also called as **simple events**.

- **Sample space**: The set of all possible outcomes of an experiment is called the **sample space** and is denoted by Ω.

- **Example:** Toss a coin.

  $\Omega = \{H, T\}$

  Some of the possible occurrences of events:

  (a) the outcome is a head.

  (b) the outcome is not a head.

  (c) the outcome is either a head or a tail.

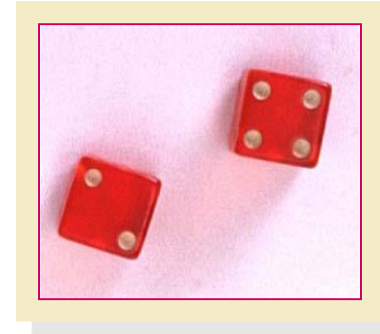
- They can be rewritten as:

  (a) $A = \{H\}$

  (b) $A = \{H\}^c$

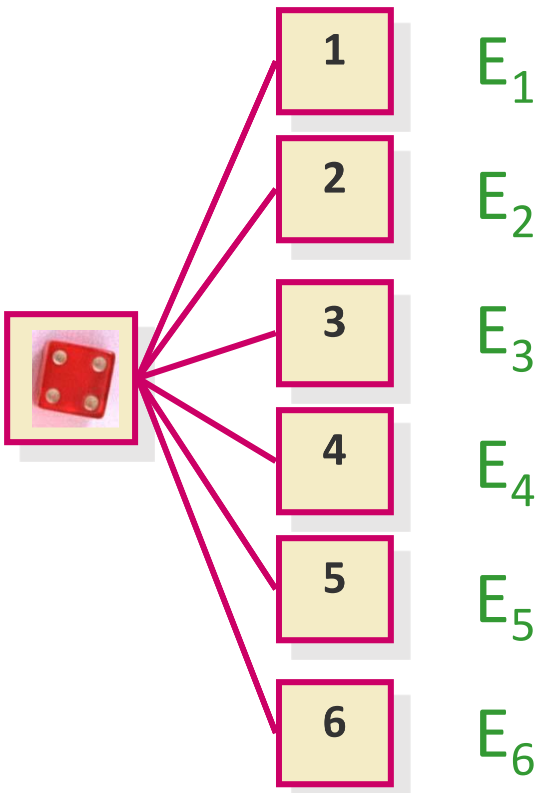
  (c) $A = \{H\} \cup \{T\}$

Example



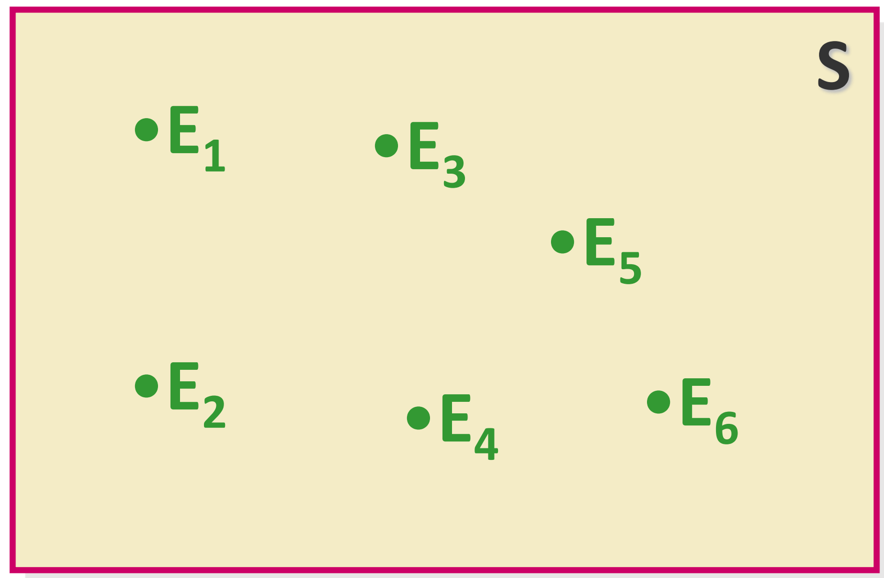
- **The die toss:**

Simple events:



Sample space:

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



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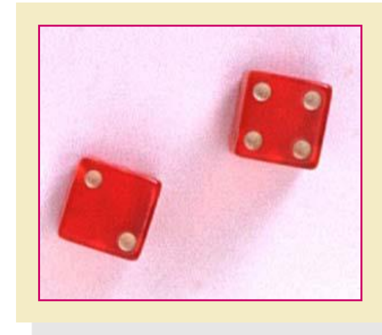
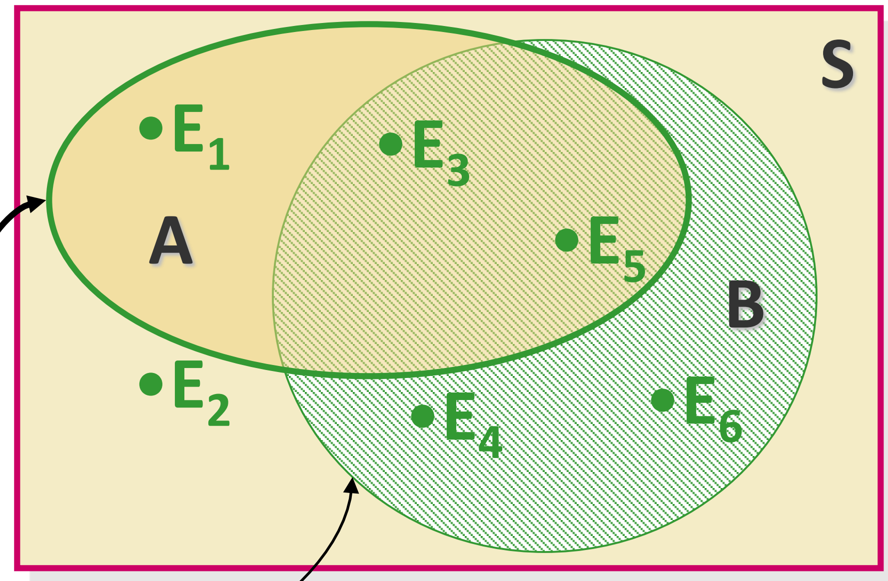
EVENTS

- Events are subsets of sample space.
- In other words, an event is a collection of one or more simple events.
- If A and B are two events, then $A \cup B$, $A \cap B, A^c$ are also events.
- Events A and B are called **disjoint** if $A \cap B = \emptyset$ (where $\emptyset$ is called the impossible event).
- The set Ω is called a **certain** event.

EVENTS



- **The die toss:**
  - –A: an odd number
  - –B: a number > 2



$A = \{E_1, E_3, E_5\}$

$B = \{E_3, E_4, E_5, E_6\}$

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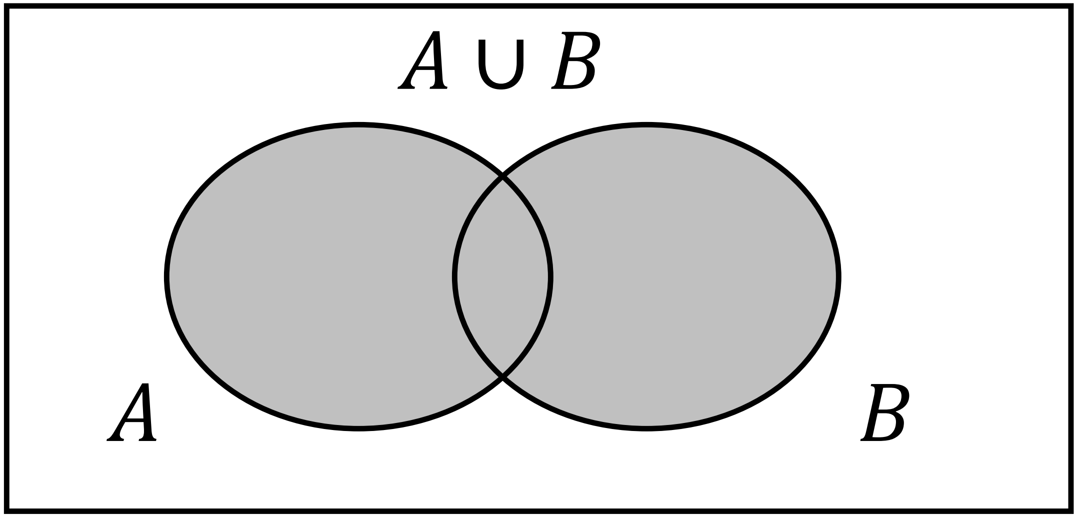
Set operations on Events

Union

Let A and B be two events, then the **union** of A and B is the event (denoted by $A \cup B$) defined as:

- $A \cup B$ = {e| e belongs to A **or** e belongs to B}
- The event $A \cup B$ **occurs** if the event A **occurs or** the event B **occurs** .



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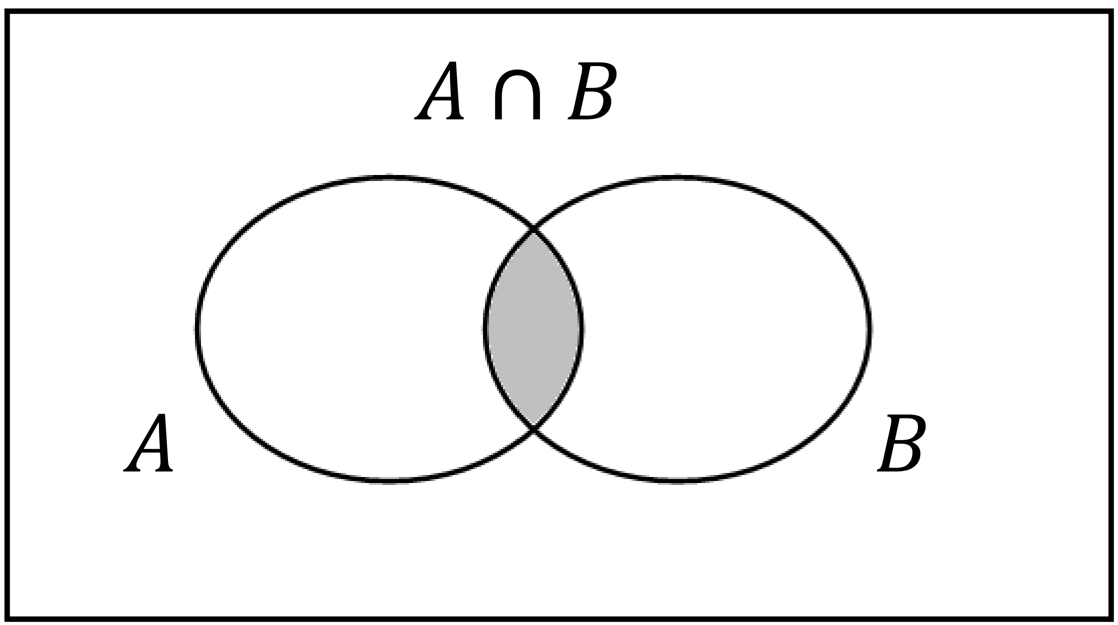
Intersection

Let A and B be two events, then the **intersection** of A and B is the event (denoted by $A \cap B$) defined as:

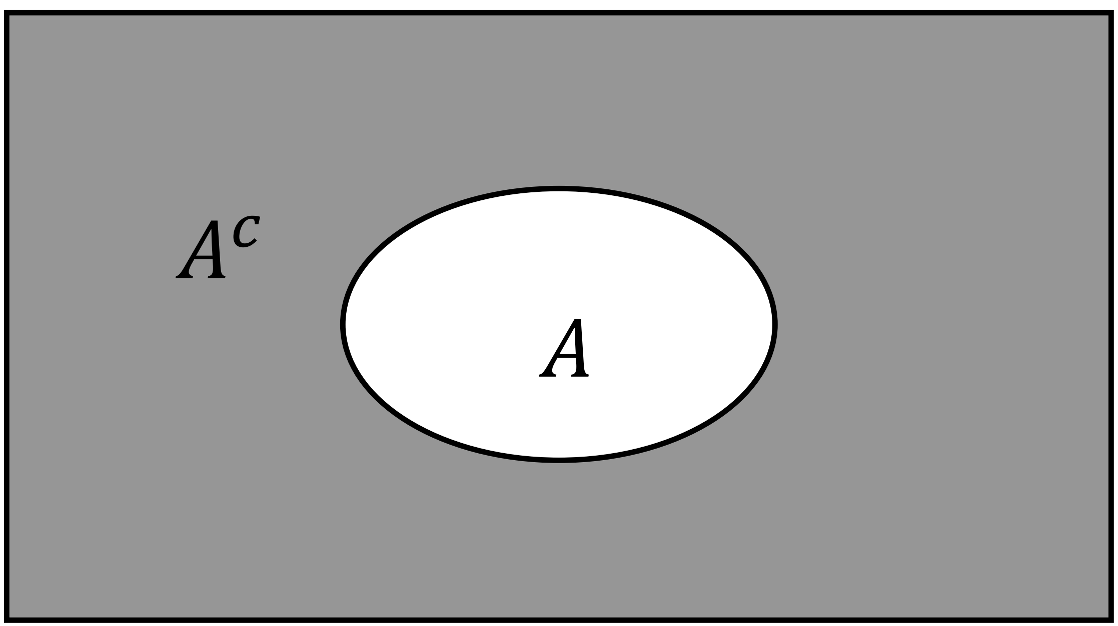
- $A \cap B = \{e \mid e$ belongs to A **and** e belongs to $B\}$
- The event $A \cap B$ **occurs** if the event A **occurs and** the event B **occurs** .



Complement

Let A be any event, then the **complement** of A (denoted by A^c) is defined as:

- $A^c = \{e |\ e$ **does not** belong to $A\}$
- The event A^c **occurs** if the event A **does not occur**

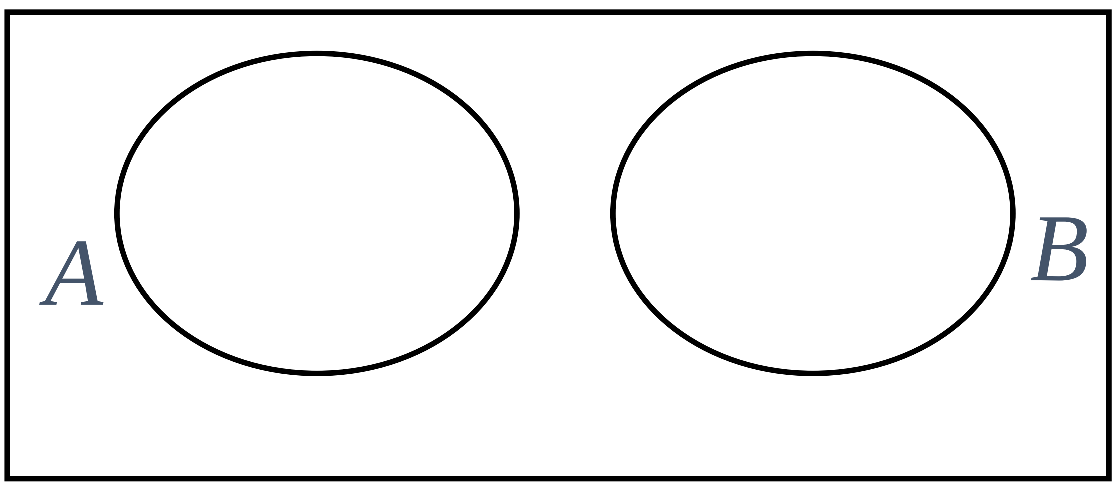


Mutually Exclusive Events

Two events A and B are called **mutually exclusive** if:

$$\boldsymbol{A \cap B = \phi}$$

In such a case, $P(A \cup B) = P(A) + P(B)$



Example: When a coin is tossed, occurrence of head and tail is mutually exclusive. These two events cannot happen at the same time.

FIELD and $\sigma-$field

- Let F be a collection of events that satisfies :

  (a) $A, B \in F \Rightarrow A \cup B, A \cap B \in F$

  (b) $A \in F \Rightarrow A^c \in F$

  (c) $\emptyset \in F$

Such an F is called a **field**.

- A collection F of subsets of Ω is called a ***σ*-field** if it satisfies the following:

  (a) $\emptyset \in F$

  (b) $A_1, A_2, \ldots \in F \Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$

  (c) $A \in F \Rightarrow A^c \in F$

Examples

- The smallest σ-field associated with Ω is the collection $F = \{\emptyset, \Omega\}$.

- $\{\emptyset, A, A^c, \Omega\}$ is a σ-field.

- The power set of Ω (in this case it is a finite set) which is written as $\{0,1\}^{\Omega}$ and contains all subsets of Ω, is a σ-field.

- Suppose we repeat an experiment N number of times and suppose A is some event which may or may not occur on each repetition, then

  probability $\boldsymbol{P(A) = N(A)/N}$

- If $A = \phi$, then $N(\phi) = 0$ and hence $P(\phi) = 0$
- If $A = \Omega$, then $N(\Omega) = N$ and hence $P(\Omega) = 1$
- If A and B are two disjoint events then,

$$N(A \cup B) = N(A) + N(B)$$

  Dividing by N on both sides

$$P(A \cup B) = P(A) + P(B)$$

- If $A_1, \dots, A_n$ are disjoint events, then

$$P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_n)$$

- A **probability measure** P on $\{\Omega, F\}$ is a function

  P: $F \rightarrow [0,1]$ satisfying:

  (a) $P(\phi) = 0$

  (b) $P(\Omega) = 1$

  (c) if $A_1, A_2, \ldots$ is a collection of disjoint members of F, in that $A_i \cap A_j = \phi$ for all pairs i, j satisfying $i \neq j$, then

$$\boldsymbol{P}(\cup_{i=1}^{\infty} \boldsymbol{A}_i) = \sum_{i=1}^{\infty} \boldsymbol{P}(\boldsymbol{A}_i)$$

- The triple (Ω, F, P) comprising a set Ω, a σ*-field* F of subsets of Ω and a probability measure P on (Ω, F), is called a **probability space**.

Example

- A coin, possible biased, is tossed once. We take $\Omega = \{H, T\}$ and $F = \{\phi, H, T, \Omega\}$ and $P\colon\ F \to\ [0,1]$ is given by
  $P(\phi) =\ 0$
  $P(H) =\ p$
  $P(T) =\ 1 - p$
  $P(\Omega)\ =\ 1$

  Where p is a fixed real number in the interval $[0,1]$. If $p = 1/2$, then the coin is *fair*, or *unbiased*.

Example problems

Toss a fair coin twice. What is the probability of observing at least one head?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

P(at least 1 head)

$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

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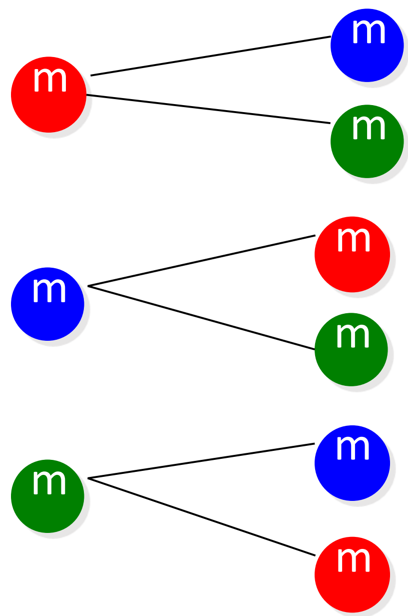
Example 2

A bowl contains three marbles, one red, one blue and one green. A child selects two marbles at random. What is the probability that at least one is red?

1st marble 2nd marble



E_i	$P(Ei)$
RB	1/6
RG	1/6
BR	1/6
BG	1/6
GB	1/6
GR	1/6

P(at least 1 red)

= $P(RB) + P(BR) + P(RG) + P(GR)$

= 4/6 = 2/3

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LEMMA

1. For any event A, $\boldsymbol{P(A^c) = 1 - P(A)}$

Proof:

$$A \cup A^c = \Omega$$

And $A \cap A^c = \phi$

So, $P(A \cup A^c) = P(A) + P(A^c) = 1$

2. For any event A, $\boldsymbol{P(A) \leq 1}$

Proof:

We know that $P(A) + P(A^c) = 1$

i.e., $P(A) = 1 - P(A^c)$

So, $P(A) \leq 1$

3. $\boldsymbol{P(\phi) = 0\ and\ P(\Omega) = 1}$

Proof:

$P(\phi) = 1 - P(\Omega) = 1 - 1 = 0$

Likewise, it can be shown that

$P(\Omega) = 1$

4. **If $\boldsymbol{A \subseteq B}$ then $\boldsymbol{P(A) \leq P(B)}$**

Proof:

$B = A \cup (B \cap A^c)$ This is the union of disjoint sets.

$P(B) = P(A) + P(B \cap A^c)$

$P(B \cap A^c) \geq 0$ (Since probability always lies from 0 to 1)

$So, P(B) = P(A) + P(B \cap A^c) \geq P(A)$

5. $\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

Proof:

$A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)$

Since they are disjoint sets,

$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$

Adding and subtracting $P(A \cap B)$ in the RHS,

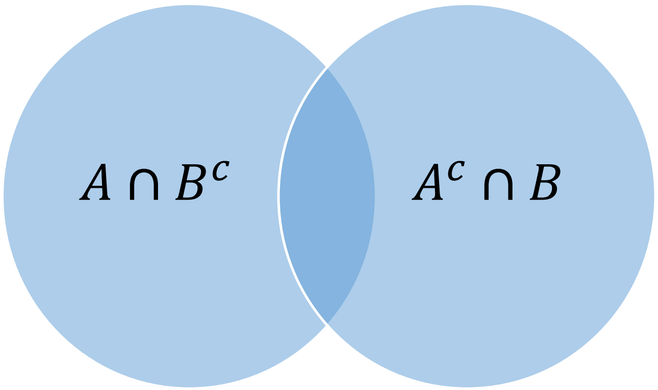
$$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B) + P(A \cap B) - P(A \cap B)$$

Rearranging the terms,

$$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B) + P(A \cap B) - P(A \cap B)$$

Hence,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



This is called the **additive rule**.

6. If $A_1, A_2, \dots, A_n$ are events,

$$\mathrm{P}\left(\bigcup_{i=1}^{n} \mathrm{A_i}\right) = \sum_{i<j} P\left(A_i \cap A_j\right) + \sum_{i<j<k} P\left(A_i \cap A_j \cap A_k\right) + \; \dots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

Proof: The proof is by induction similar to 5.

7. Let A_1, A_2 ...be an increasing sequence of events, so that $A_1 \subseteq A_2 \subseteq \cdots$and write A for their limit :

$$\mathrm{A} = \bigcup_{i=1}^{\infty} A_i = \lim_{i\to\infty} A_i$$

Then, $P(A) = \lim_{i\to\infty} P(A_i)$

Similarly B_1, B_2 ...be a decreasing sequence of events, so that $B_1 \supseteq B_2 \supseteq \cdots$and write B for their limit :

$$\mathrm{B} = \bigcap_{i=1}^{\infty} B_i = \lim_{i\to\infty} B_i$$

Then, $P(B) = \lim_{i\to\infty} P(B_i)$

Proof: $A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \cdots$ is the union of a disjoint family of events.

$$P(A) = P(A_1) + \sum_{i=1}^{\infty} P(A_{i+1} \setminus A_i)$$

$$= P(A_1) + \lim_{n\to\infty} \sum_{i=1}^{n-1} P(A_{i+1} \setminus A_i)$$

$$= P(A_1) + \lim_{n\to\infty} \sum_{i=1}^{n-1} [P(A_{i+1}) - P(A_i)]$$

Since $P(A \setminus B) = P(A) - P(B)$

$$= P(A_1) + \lim_{n\to\infty} P(A_n) - P(A_1) = \lim_{(n\to\infty)} \mathbf{P}(\mathbf{A_n})$$

To show the result for decreasing families of events, take complements and use the first part.

- If P(A)=0, then it is called a **null event**. Note that this is different from an **impossible event**.

Conditional Probability

- If $P(B) > 0$ then the conditional probability that A occurs given B occurs is

$$\boldsymbol{P(A|B) = \frac{P(A \cap B) \dashrightarrow Joint\ Probability}{P(B) \dashrightarrow Marginal\ Probability}}$$

- **<u>Example</u>**: A family has two children. What is the probability that both are boys, given that at least one is a boy?

Answer: The older and younger child may each be male or female, so there are four possible combinations, which we assume to be equally likely. Hence we can represent the sample space in the obvious way as:

$$\Omega = \{GG, GB, BG, BB\}$$

where $P(GG) = P(BB) = P(GB) = P(BG)$ =1/4. From the definition of conditional probability,

$$P(BB|one\ boy\ atleast) = P(BB|GB \cup BG \cup BB)$$

$$= \frac{P(BB \cap (GB \cup BG \cup BB))}{P(GB \cup BG \cup BB)} = \frac{\frac{1}{4}}{\frac{3}{4}} = 1/3.$$

- For a family with two children, what is the probability that both are boys given that the younger is a boy?

Answer: $P(BB|younger\ is\ a\ boy)$

$$= P(BB\ |GB \cup BB)$$

$$= \frac{P(BB \cap (GB \cup BB))}{P(GB \cup BB)}$$

$$= \frac{P(BB)}{P(GB \cup BB)} = \frac{\left(\frac{1}{4}\right)}{\frac{1}{2}} = \frac{1}{2}$$

- For any events A and B such that $0 < P(B) < 1$,

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Proof: $A = A \cap (B \cup B^c) = (\mathrm{A} \cap B) \cup (A \cap B^c)$

$$P(A) = P(A \cap B) + P(A \cap B^c)$$
$$= P(A|B)P(B) + P(A|B^c)P(B^c)$$

More generally, let $B_1, B_2, \ldots, B_n$ be a partition of Ω such that $P(B_i) > 0\ \forall i$. Then,

$$\boldsymbol{P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Example

- Only two factories manufacture goggles. 20 per cent of the goggles from factory I and 5 per cent from factory II are defective. Factory I produces twice as many goggles as factory II each week. What is the probability that a goggle, randomly chosen from a week's production, is satisfactory?

Answer:

Let A be the event that the chosen goggle is satisfactory, and let B be the event that it was made in factory I.

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

$$= \frac{4}{5} . \frac{2}{3} + \frac{19}{20} . \frac{1}{3} = \frac{51}{60}$$

- If the chosen goggle is defective, what is the probability that it came from factory I?

Answer:

In our notation this is just $P(B\,|A^c)$.

$$P(B\,|A^c) = \frac{P(B \cap A^c)}{P(A^c)} = \frac{P(A^c \cap B)P(B)}{P(A^c)} = \frac{\frac{1}{5}\cdot\frac{2}{3}}{1 - \frac{51}{60}} = \frac{8}{9}$$

Example

The academy awards is soon to be shown.

For a specific married couple the probability that the husband watches the show is 80%, the probability that his wife watches the show is 65%, while the probability that they both watch the show is 60%.If the husband is watching the show, what is the probability that his wife is also watching the show?

Solution: Let *B* = the event that the husband watches the show

$$P(B) = 0.80$$

Let *A* = the event that his wife watches the show

$$P(A) = 0.65 \quad P(A \cap B) = 0.60$$

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.60}{0.80} = 0.75}$$

Example

- Instructor has a list of 100 questions for quiz. Student has to answer 3 questions picked randomly from the list of 100 questions. If the student answers all 3 questions, he passes else he fails. What is the probability of the student passing given he knows answers to 90 questions?

Answer : Let A_1 be the probability that the student gets first answer right.

So, $P(A_1) = \frac{90}{100}$

Likewise, let A_2 be the probability that the student gets second answer right and so on.

$$P(A_2|A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{89}{99}$$

$$P(A_3|A_1, A_2) = \frac{88}{98}$$

So, $P(A) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$

$$= \frac{90}{100} \times \frac{89}{99} \times \frac{88}{98}$$

Independence

- $P(A \mid B) = P(A)$, then we call A and B **'independent'**.
- This is well defined only if $P(B) > 0$.
- <u>**Definition**</u> : Events A and B are called **independent** if

$$P(A \cap B) = P(A)P(B)$$

- More generally, a family $\{A_i : i \in I\}$ is called independent if

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i) \quad \forall J \subseteq I$$

- If the family $\{A_i : i \in I\}$ has the property that

$$P(A_i \cap A_j) = P(A_i)P(A_j) \; \forall i \neq j$$

then it is called **pairwise independent**. **Pairwise-independent families are not necessarily independent.**

Example

- Suppose $\Omega = \{abc, acb, cab, cha, bca, bac, aaa, bbb, ccc\}$, and each of the nine elementary events in Ω occurs with equal probability 1/9. Let A_k be the event that the k^{th} letter is a. Show that the family $\{A_1, A_2, A_3\}$ is pairwise independent but not independent.

Answer :

$$P(A_1) = \frac{3}{9} = \frac{1}{3}\ , P(A_2) = \frac{1}{3}\ and\ P(A_3) = \frac{1}{3}$$

$$P(A_1 \cap A_2) = \frac{1}{9} = P(A_1)P(A_2)$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{9}$$

But, $P(A_1)P(A_2)P(A_3) = \frac{1}{3}.\frac{1}{3}.\frac{1}{3} = \frac{1}{27}$

Clearly, $P(A_1 \cap A_2 \cap A_3) \neq P(A_1)P(A_2)P(A_3)$

Difference between mutually exclusive events and independent events

Mutually exclusive events	**Independent events**
Events are mutually exclusive if the occurrence of one event excludes the occurrence of the other(s). Mutually exclusive events cannot happen at the same time.	Events are independent if the occurrence of one event does not influence (and is not influenced by) the occurrence of the other(s).
Example: when tossing a coin, the result can either be heads or tails but cannot be both.	**Example**: when tossing two coins, the result of one flip does not affect the result of the other.
$P(A \cap B) = 0$ $P(A \cup B) = P(A) + P(B)$ $P(A\|B) = 0$	$P(A \cap B) = P(A)P(B)$ $P(A \cup B) = P(A) + P(B) - P(A)P(B)$ $P(A\|B) = P(A)$

Bayes' Theorem

If

- $A_1 \ldots A_n \rightarrow$ partition of Ω
- $A_i \cap A_j = \phi$
- $A_1 \cup \cdots \cup A_n = \Omega$

Then,

$$P(A_k|B) = \frac{P(A_k)\,P(B|A_k)}{P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)}$$

B is any event with $P(B) > 0$

Example

- A rare disease X affects 1 in 10^6. Test T is 99% accurate. Person having no X and chance of T being positive is 1%. Person having X and chance of T being negative is also 1%.

Suppose a person has tested positive, what is the probability of this person having X?

Answer :

$A \rightarrow$ person has X

$B \rightarrow$ person tests positive

$$P(A) = \frac{1}{10^6}, P(B|A) = 0.99\,;\ \mathrm{P(A|B)} = \frac{\mathrm{P(B|A)P(A)}}{\mathrm{P(B)}}$$

$$P(A|B) = \frac{0.99 \times 10^{-6}}{0.99 \times 10^{-6} + 0.01 \times (1 - 10^{-6})} = 0.000099$$

Conditional Independence

- Let C be an event such that $P(C) > 0$.

- A and B are conditionally independent given C if

$$P(A \cap B|C) = P(A|C)\, P(B|C)$$